DIFFERENTIAL PROPERTIES OF THE REDUCED MITTAG-LEFFLER POLYNOMIALS

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This paper deals with the Mittag-Leffler polynomials (MLP) by extracting their essence which consists of real polynomials with fine properties. They are orthogonal on the real line instead of the imaginary axes for MLP. Beside recurrence relations and zeros, we will point to the closed form of its Fourier transform. The most important contribution consists of the new differential properties, especially the finite and infinite differential equation.

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1. Introduction

The Mittag–Leffler polynomials $\{g_n(x)\}$ are coefficients in the expansion

$$\left(\frac{1+t}{1-t}\right)^x = \sum_{n=0}^{\infty} g_n(x)t^n \qquad (|t|<1).$$

They were introduced by Mittag-Leffler in a study of the integral representations. Their main properties were discovered by H. Bateman (see [1] and [2]). He noticed that they occur as coefficients in the closed-form expressions for a several families of integrals. Also, they were used in deriving some expansions for the Euler Gamma function and the Riemann Zeta function [12]. Truncated Exponential-Based Mittag-Leffler Polynomials were examined in [14]. They were noticed in the solutions of heat diffusion equation of Fokker-Plank type in [5]. They are connected with the Sheffer polynomials in [6] and Riordan arrays in [10]. Their generalizations were considered in [13] and [11].

The article is organized as follows. In Section 2 we present the preliminaries for the Mittag-Leffler polynomials. Since the imaginary unit is present in the orthogonality relation, we have noticed that we can reduce them to the real polynomials whose examination is more obvious and much easier. They are the subject in Section 3. In the next section, we find the closed form of their Fourier transform. The main contributions of the paper are the

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differential properties exposed Section 5. Finally, the paper is concluded in Section 6 with observation on the quasi-monomial property of the reduced Mittag-Leffler polynomials.

2. Preliminaries

The Mittag–Leffler polynomials $\{g_n(x)\}$ can be represented over hypergeometric function like

$$g_n(x) = 2x\ {}_2F_1\Big(\begin{matrix} 1-n,\ 1-x \\ 2 \end{matrix}\ \Big|\ 2\Big) \qquad (n \in \mathbb{N}).$$

They can be viewed as a special case of the Meixner-Pollaczek polynomials

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!}\, \mathrm{e}^{\mathrm{i}n\phi}\ {}_2F_1\Big(\begin{matrix} -n,\ \lambda+\mathrm{i}x \\ 2\lambda \end{matrix}\ \Big|\ 1-\mathrm{e}^{-2\mathrm{i}\phi}\Big) \quad (n\in\mathbb{N};\ \mathrm{i}^2=-1) \tag{2.1}$$

for $\lambda = 1$ and $\phi = \pi/2$:

$$g_n(x) = 2\frac{\mathrm{e}^{-\mathrm{i}n\pi/2}}{n}\, xP_{n-1}^{(1)}(\mathrm{i}x;\pi/2) \qquad (n\in\mathbb{N}). \tag{2.2}$$

Also, they can be considered as a special case of Meixner polynomials

$$M_n(x;\beta,c) = {}_2F_1\Big(\begin{matrix} -n,\ -x \\ \beta \end{matrix}\ \Big|\ 1-\frac{1}{c}\Big) \qquad (n\in\mathbb{N})$$

for $\beta = 2$ and $c = -1$ (see [8]). Namely, their relation is given by

$$g_n(x) = 2xM_{n-1}(x-1, 2, -1).$$

The lack of this connection is the fact that orthogonality of the Meixner polynomials is assured only with the constraint $0 < c < 1$.

Finally, the Mittag–Leffler polynomials $g_n(x)$ are connected with Pidduck polynomials by the expression

$$P_n(x) = \frac{1}{2}\left(\mathrm{e}^D + 1\right) g_n(x),$$

where the series for the exponential function is used and D is understood as differentiation.

The Mittag–Leffler polynomials $\{g_n(x)\}$ satisfy recurrence relation

$$(n+1)g_{n+1}(x) - 2xg_n(x) + (n-1)g_{n-1}(x) = 0,$$

with initial values

$$g_0(x) = 1, \quad g_1(x) = 2x.$$

They also satisfy difference relation

$$xg_n(x+1) - 2ng_n(x) - xg_n(x-1) = 0.$$

The Mittag–Leffler polynomials $\{g_n(x)\}$ satisfy recurrence-difference relation

$$g_n(x+1) - g_{n-1}(x+1) = g_n(x) + g_{n-1}(x).$$

The orthogonality relation is given by

$$\int_{-\infty}^{+\infty} g_n(-\mathrm{i}x)g_m(\mathrm{i}x)\frac{1}{x\sinh(\pi x)}\,\mathrm{d}x = \frac{2}{n}\,\delta_{mn} \qquad (n,m\in\mathbb{N}). \tag{2.3}$$

Notice that the corresponding monic sequence is

$$\hat{g}_n(x) = \frac{(n)!}{2^n}\, g_n(x) \qquad (n\in\mathbb{N}_0).$$

Let us remind that the central difference operator is

$$\delta f(z) = f(z+1/2) - f(z-1/2).$$

Theorem 2.1. *The Mittag–Leffler polynomials* $\{g_n(x)\}$ *satisfy the Rodrigues formula*

$$g_n(x) = \frac{2}{n!}\frac{x}{w(x,1)}\delta^n w(x,n),$$

where

$$w(x,n) = \Gamma\left(\frac{n+1}{2} - x\right)\Gamma\left(\frac{n+1}{2} + x\right).$$

Proof. It is based on the connection (**2**.2) and the Rodrigues formula for the Meixner-Pollaczek polynomials in [8], pp. 37.□

3. The reduced Mittag–Leffler polynomials

In spite of the fact that the Mittag-Leffler polynomials are real, in some relations, as in (**2**.1), and especially in their orthogonality relation (**2**.3), they are considered as the complex functions. That is why we believe that is better to extract the following sequence.

Let us consider the *reduced Mittag-Leffler polynomials* defined in [13] by

$$\varphi_n(x) = \frac{g_{n+1}(\mathrm{i}x)}{\mathrm{i}^{n+1}\, x} \qquad (n \in \mathbb{N}_0).$$

The successive members of sequence $\{\varphi_n(x)\}_{n\in\mathbb{N}_0}$ satisfy the three–term recurrence relation

$$\begin{aligned}(n+2)\varphi_{n+1}(x) &= 2x\varphi_n(x) - n\,\varphi_{n-1}(x) \quad (n \in \mathbb{N})\\ \varphi_0(x) &= 2, \qquad \varphi_1(x) = 2x.\end{aligned}$$

The generating function of sequence $\{\varphi_n(x)\}_{n\in\mathbb{N}_0}$ is given by

$$\mathcal{G}(t,x) = \frac{\exp(2x\arctan t) - 1}{tx} = \sum_{n=0}^{\infty}\varphi_n(x)t^n.$$

Notice that

$$\varphi_n(-x) = (-1)^n\,\varphi_n(x) \qquad (n \in \mathbb{N}).$$

The sequence of polynomials $\{\varphi_n(x)\}_{n\in\mathbb{N}_0}$ satisfies the following orthogonality relation:

$$\int_{-\infty}^{+\infty}\varphi_n(x)\varphi_m(x)\frac{x}{\sinh(\pi x)}\,\mathrm{d}x = \frac{2}{n+1}\,\delta_{mn} \qquad (n,m \in \mathbb{N}_0).$$

Notice that

$$\int_{-\infty}^{+\infty}\frac{x^n}{\sinh x}\,\mathrm{d}x = \frac{(1-(-1)^n)(2^{n+1}-1)}{2^n}\,n!\,\zeta(n+1) \qquad (n \in \mathbb{N}),$$

where $\zeta(n)$ is the Riemann zeta function.

The reduced Mittag-Leffler polynomials $\{\varphi_n(x)\}_{n\in\mathbb{N}_0}$ are real polynomials and, because of their orthogonality on $\mathbb{R}$, they have all real zeros.

The monic sequence

$$\hat{\varphi}_n(x) = \frac{(n+1)!}{2^{n+1}}\,\varphi_n(x) \qquad (n \in \mathbb{N}_0), \tag{3.1}$$

satisfies three term recurrence relation

$$\begin{aligned} \hat{\varphi}_{n+1}(x) &= x\hat{\varphi}_n(x) - \frac{n(n+1)}{4}\,\hat{\varphi}_{n-1}(x) \quad (n \in \mathbb{N}), \\ \hat{\varphi}_0(x) &= 1, \qquad \hat{\varphi}_1(x) = x. \end{aligned} \tag{3.2}$$

Remark 3.1. Notice that $\varphi_n(x)$ is the Meixner-Pollaczeck polynomial for $\lambda = 1$ and $\phi = \pi/2$, i.e.

$$\hat{\varphi}_n(x) = \frac{n!}{2^n}\, P_n^{(1)}(x;\pi/2) \qquad (n \in \mathbb{N}_0).$$

Hence the following difference relation is valid

$$(x+\mathrm{i})\hat{\varphi}_n(x+\mathrm{i}) - 2(n+1)\mathrm{i}\hat{\varphi}_n(x) - (x-\mathrm{i})\hat{\varphi}_n(x-\mathrm{i}) = 0.$$

Example 3.1. The first members of the sequence $\{\hat{\varphi}_n(x)\}_{n\in\mathbb{N}_0}$ are:

$$\hat{\varphi}_0(x) = 1, \quad \hat{\varphi}_1(x) = x, \quad \hat{\varphi}_2(x) = x^2 - \frac{1}{2}, \quad \hat{\varphi}_3(x) = x^3 - 2x,$$

$$\hat{\varphi}_4(x) = x^4 - 5x^2 + \frac{3}{2}, \quad \hat{\varphi}_5(x) = x^5 - 10x^3 + \frac{23}{2}x.$$

The largest zeros are:

$$x_2^{(2)} \approx 0.707, \quad x_3^{(3)} \approx 1.414, \quad x_4^{(4)} \approx 2.163, \quad x_5^{(5)} \approx 2.945.$$

Using conclusions from the paper [7], we can easily prove that the zeros $\{x_k^{(n)}\}$ of the polynomial $\hat{\varphi}_n(x)$ are bordered in the next manner:

$$|x_k^{(n)}| < \sqrt{(n-1)n} \qquad (k = 1, 2, \dots, n).$$

Theorem 3.1. *The sequence* $\{\hat{\varphi}_n(x)\}_{n\in\mathbb{N}_0}$ *satisfies the Turan's inequality*

$$\mathcal{T}(\hat{\varphi}_n(x), x) = -\begin{vmatrix} \hat{\varphi}_{n-1}(x) & \hat{\varphi}_n(x) \\ \hat{\varphi}_n(x) & \hat{\varphi}_{n+1}(x) \end{vmatrix} \geq 0 \qquad (\forall x \in \mathbb{R};\ \forall n \in \mathbb{N}).$$

Proof. We will prove by the mathematical induction as in the paper [9]. Obviously, $\mathcal{T}(\hat{\varphi}_0(x), x) = 1 \geq 0$. Suppose that $\mathcal{T}(\hat{\varphi}_n(x), x) \geq 0$.

Let be $c_n = n(n+1)/4$. Consider the expression

$$\begin{aligned} &\mathcal{T}(\hat{\varphi}_{n+1}(x), x) - c_n\mathcal{T}(\hat{\varphi}_n(x), x) \\ &\quad = \hat{\varphi}_{n+1}^2(x) - \hat{\varphi}_n(x)\hat{\varphi}_{n+2}(x) - c_n\left(\hat{\varphi}_n^2(x) - \hat{\varphi}_{n-1}(x)\hat{\varphi}_{n+1}(x)\right) \\ &\quad = \hat{\varphi}_{n+1}^2(x) - \hat{\varphi}_n(x)\left(x\hat{\varphi}_{n+1}(x) - c_{n+1}\hat{\varphi}_n(x)\right) - c_n\hat{\varphi}_{n-1}(x)\hat{\varphi}_{n+1}(x) - c_n\hat{\varphi}_n^2(x). \end{aligned}$$

Applying the recurrence relation (**3**.2), the last expression reduces to

$$\mathcal{T}(\hat{\varphi}_{n+1}(x), x) - c_n\mathcal{T}(\hat{\varphi}_n(x), x) = (c_{n+1} - c_n)\,\hat{\varphi}_n^2(x) = \frac{n+1}{2}\hat{\varphi}_n^2(x) \geq 0.$$

We conclude that $\mathcal{T}(\hat{\varphi}_{n+1}(x), x) \geq c_n\mathcal{T}(\hat{\varphi}_n(x), x) \geq 0$. □

4. Fourier transform

Let us remind that the Fourier transform is defined by

$$\mathfrak{F}\left[f(t)\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\, e^{\mathrm{i}st}\, \mathrm{d}t \ = F(s). \tag{4.1}$$

The Fourier transform of the first members of the sequence $\{\hat{\varphi}_n(t)w(t)\}_{n\in\mathbb{N}_0}$, where $w(t) = \frac{t}{\sinh(\pi t)}$, are:

$$\mathfrak{F}\left[\hat{\varphi}_0 w\right] = \frac{1}{\sqrt{2\pi}}\, \frac{1}{1+\cosh s} = \sqrt{\frac{2}{\pi}}\, \frac{\sinh^2(s/2)}{\sinh^2 s}, \quad \mathfrak{F}\left[\hat{\varphi}_1 w\right] = 2\,\mathrm{i}\,\sqrt{\frac{2}{\pi}}\, \frac{\sinh^4(s/2)}{\sinh^3 s}.$$

Theorem 4.1. *The Fourier transform of the sequence* $\{\hat{\varphi}_n(t)w(t)\}_{n\in\mathbb{N}_0}$*, where* $w(t) = \frac{t}{\sinh(\pi t)}$*, is:*

$$\Phi_n(s) = \mathfrak{F}\left[\hat{\varphi}_n w\right] = \mathrm{i}^n (n+1)! \sqrt{\frac{2}{\pi}}\, \frac{\sinh^{2n+2}(s/2)}{\sinh^{n+2} s} \qquad (n\in\mathbb{N}_0). \tag{4.2}$$

Proof. We will apply the mathematical induction. It is obviously true for $n=0$ and $n=1$. Suppose that it is valid for every $k\le n$. We will multiply the recurrence relation (**3**.2) with $w(t)$ and apply the Fourier transform on it:

$$\mathfrak{F}\left[\hat{\varphi}_{n+1}(t)w(t)\right] = \mathfrak{F}\left[t\hat{\varphi}_n(t)w(t)\right] - \frac{n(n+1)}{4}\,\mathfrak{F}\left[\hat{\varphi}_{n-1}(t)w(t)\right]. \tag{4.3}$$

The Fourier transform (**4**.1) has the property

$$\mathfrak{F}\left[t^m g(t)\right](s) = (-\mathrm{i})^m \frac{d^m}{ds^m}\mathfrak{F}\left[g(t)\right](s) \qquad (m\in\mathbb{N}).$$

It able us to write

$$\mathfrak{F}\left[t\varphi_n(t)w(t)\right] = (-\mathrm{i})\frac{d}{ds}\mathfrak{F}\left[\varphi_n(t)w(t)\right](s) = -\mathrm{i}\;\Phi_n'(s).$$

Hence the relation (**4**.3) obtains the form

$$\Phi_{n+1}(s) = -\mathrm{i}\;\hat{\Phi}_n'(s) - \frac{n(n+1)}{4}\;\Phi_{n-1}(s).$$

Deriving (**4**.2), we find

$$\Phi_n'(s) = \mathrm{i}^n (n+1)! \sqrt{\frac{2}{\pi}}\;\left(n - 2\sinh^2(s/2)\right)\frac{\sinh^{2n+2}(s/2)}{\sinh^{n+3} s}\;.$$

Finally, using the assumed expression for $\Phi_{n-1}(s)$, we see that $\Phi_{n+1}(s)$ satisfies the relation (**4**.2), wherefrom the statement follows. □

Remark 4.1. Having in mind relations between hyperbolic functions, the relation (**4**.2) can be rewritten in the form

$$\Phi_n(s) = \mathfrak{F}\left[\hat{\varphi}_n w\right] = \frac{\mathrm{i}^n (n+1)!}{2^n\,\sqrt{2\pi}}\;\tanh^n(s/2)\left(1-\tanh^2(s/2)\right) \qquad (n\in\mathbb{N}_0).$$

5. Differential properties

The exponential generating function of sequence $\{\hat{\varphi}_n(x)\}_{n\in\mathbb{N}_0}$ is (see [13])

$$\hat{\mathcal{G}}(t,x) = \frac{4\exp\big(2x\ \arctan(t/2)\big)}{t^2+4} = \sum_{n=0}^{\infty} \hat{\varphi}_n(x)\frac{t^n}{n!}. \tag{5.1}$$

Theorem 5.1. *The exponential generating function $\hat{\mathcal{G}}(t,x)$ has the property*

$$\hat{\mathcal{G}}(t,x)\ \frac{\partial^2}{\partial x^2}\hat{\mathcal{G}}(t,x) = \left(\frac{\partial}{\partial x}\hat{\mathcal{G}}(t,x)\right)^2.$$

Proof. This follows directly from relations

$$\frac{\partial}{\partial x}\hat{\mathcal{G}}(t,x) = 2\arctan\frac{t}{2}\cdot\hat{\mathcal{G}}(t,x)\ ,\qquad \frac{\partial^2}{\partial x^2}\hat{\mathcal{G}}(t,x) = 4\arctan^2\frac{t}{2}\cdot\hat{\mathcal{G}}(t,x)\ . \ \square$$

Corollary 5.1. *The sequence $\{\hat{\varphi}_n(x)\}_{n\in\mathbb{N}_0}$ satisfies the recurrence-differential equation*

$$\sum_{k=0}^{n}\frac{\hat{\varphi}''_k(x)\hat{\varphi}_{n-k}(x)-\hat{\varphi}'_k(x)\hat{\varphi}'_{n-k}(x)}{k!(n-k)!} = 0 \quad (n\in\mathbb{N}).$$

Theorem 5.2. *Any polynomial $\hat{\varphi}_n(x)$ satisfies n^{th} order differential equation of the form*

$$\sum_{k=1}^{n}(\alpha_k+\beta_k x)\frac{\hat{\varphi}_n^{(k)}(x)}{k!} - n\hat{\varphi}_n(x) = 0, \tag{5.2}$$

where $\alpha_k = \cos\frac{k\pi}{2}$ and $\beta_k = \sin\frac{k\pi}{2}$.

Proof. The sequence $\{\hat{\varphi}_n(x)\}_{n\in\mathbb{N}_0}$ is a Sheffer sequence since its generating function has the form

$$\hat{\mathcal{G}}(t,x) = \frac{1}{g\left(\hat{f}(t)\right)}\mathrm{e}^{x\hat{f}(t)},$$

where $\hat{f}(t) = 2\arctan\frac{t}{2}$. It is the compositional inverse of $f(t) = 2\tan\frac{t}{2}$. Also, here is $g\left(\hat{f}(t)\right) = 1+\frac{t^2}{4}$. Hence $g(t) = \cos^2\frac{t}{2}$.

According to [15], $\{\hat{\varphi}_n(x)\}_{n\in\mathbb{N}_0}$ satisfies the differential equation of the form (5.2), where

$$\beta_k = \left(\frac{f(t)}{f'(t)}\right)^{(k)}\Big|_{t=0}, \qquad \alpha_k = \left(-\frac{f(t)}{f'(t)}\cdot\frac{g'(t)}{g(t)}\right)^{(k)}\Big|_{t=0}.$$

Hence

$$\beta_k = (\sin t)^{(k)}\Big|_{t=0} = \sin\frac{k\pi}{2}\ , \qquad \alpha_k = (1-\cos t)^{(k)}\Big|_{t=0} = \cos\frac{k\pi}{2}\ .\square$$

Example 5.1. The polynomial

$$\hat{\varphi}_4(x) = x^4-5x^2+\frac{3}{2},$$

satisfies the following differential equation:

$$\frac{1}{24}\hat{\varphi}_4^{(4)}(x) - \frac{1}{6}x\hat{\varphi}_4^{(3)}(x) - \frac{1}{2}\hat{\varphi}''_4(x) + x\hat{\varphi}'_4(x) - 4\hat{\varphi}_4(x) = 0. \tag{5.3}$$

Theorem 5.3. *Any polynomial* $\hat{\varphi}_n(x)$ *satisfies differential equation of the form*

$$(\cos D + x \sin D - (n+1)I)\,\hat{\varphi}_n(x) = 0 \qquad \left(D = \frac{\mathrm{d}}{\mathrm{d}x}\right).$$

The polynomial $\hat{\varphi}_n(x)$ *is the eigenfunction of the operator*

$$\mathcal{F} = \cos D + x \sin D - I$$

with the eigenvalue n.

Proof. Let I be the identity operator. Since

$$\alpha_k = \cos\frac{k\pi}{2} = \frac{\mathrm{i}^k + (-\mathrm{i})^k}{2}\,, \qquad \beta_k = \sin\frac{k\pi}{2} = \frac{\mathrm{i}^k - (-\mathrm{i})^k}{2\mathrm{i}}\,,$$

we can write (**5**.2) in the form

$$\left(\sum_{k=1}^{n}\left(\frac{\mathrm{i}^k + (-\mathrm{i})^k}{2} + x\frac{\mathrm{i}^k - (-\mathrm{i})^k}{2\mathrm{i}}\right)\frac{D^k}{k!} - nI\right)\hat{\varphi}_n(x) = 0. \tag{5.4}$$

Since $D^m\hat{\varphi}_n(x) \equiv 0$ for every $m > n$, we can write

$$\sum_{k=1}^{n}\mathrm{i}^k\frac{D^k}{k!}\hat{\varphi}_n(x) = \sum_{k=1}^{\infty}\frac{(\mathrm{i}D)^k}{k!}\hat{\varphi}_n(x) = \left(\mathrm{e}^{\mathrm{i}D} - I\right)\hat{\varphi}_n(x).$$

Hence the formula (**5**.4) becomes

$$\left(\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}D} + \mathrm{e}^{-\mathrm{i}D} - 2I\right) + \frac{x}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}D} - \mathrm{e}^{-\mathrm{i}D}\right) - nI\right)\hat{\varphi}_n(x) = 0.$$

The statement follows from the Euler identity for the complex functions. □

Example 5.2. Since

$$\cos D = \sum_{k=0}^{\infty}(-1)^k\frac{D^{2k}}{(2k)!}\,, \qquad \sin D = \sum_{k=0}^{\infty}(-1)^k\frac{D^{2k+1}}{(2k+1)!}\,,$$

the polynomial $\hat{\varphi}_4(x)$ satisfies

$$\left(\left(I - \frac{D^2}{2} + \frac{D^4}{4!}\right) + x\left(D - \frac{D^3}{3!}\right) - 4I\right)\hat{\varphi}_4(x) = 0,$$

what is the same as (**5**.3).

Theorem 5.4. *The sequences* $\{\hat{\varphi}_n(x)\}_{n\in\mathbb{N}_0}$ *and* $\{\varphi_n(x)\}_{n\in\mathbb{N}_0}$ *have the following differential properties:*

$$\hat{\varphi}'_{n+1}(x) = \sum_{k=0}^{[n/2]}(-1)^k\binom{n+1}{2k+1}\frac{(2k)!}{2^{2k}}\hat{\varphi}_{n-2k}(x), \tag{5.5}$$

$$\varphi'_n(x) = 2\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\varphi_{n-2k}(x). \tag{5.6}$$

Proof. By differentiation the generating function (**5**.1) over x, we get

$$\sum_{n=0}^{\infty}\hat{\varphi}'_n(x)\frac{t^n}{n!} = \frac{4\exp\bigl(2x\ \arctan(t/2)\bigr)}{t^2+4}\ 2\arctan(t/2).$$

Knowing that $\hat{\varphi}_0'(t) = 0$ and using the expansion

$$2\arctan\frac{t}{2} = \sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(2k+1)}t^{2k+1} \qquad \left(\left|\frac{t^2}{2}\right| < 1\right),$$

we have

$$\sum_{n=1}^{\infty}\hat{\varphi}_n'(x)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty}\hat{\varphi}_n(x)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(2k+1)}t^{2k+1}\right).$$

Hence

$$\sum_{n=0}^{\infty}\frac{\hat{\varphi}_{n+1}'(x)}{(n+1)!}t^{n+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\hat{\varphi}_n(x)}{n!}\frac{(-1)^k}{4^k(2k+1)}t^{n+2k+1},$$

i.e.,

$$t\sum_{n=0}^{\infty}\frac{\hat{\varphi}_{n+1}'(x)}{(n+1)!}t^n = t\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{\hat{\varphi}_{n-2k}(x)}{(n-2k)!}\frac{(-1)^k}{4^k(2k+1)}t^n.$$

Comparing the coefficients by t^n $(n \in \mathbb{N})$, we find

$$\frac{\hat{\varphi}_{n+1}'(x)}{(n+1)!} = \sum_{k=0}^{[n/2]}\frac{\hat{\varphi}_{n-2k}(x)}{(n-2k)!}\frac{(-1)^k}{4^k(2k+1)}. \qquad (\mathbf{5}.7)$$

By rearrangement of summands, we have formula (**5**.5).

Formula (**5**.6) can be obtained by (**5**.7) and (**3**.1). □

6. Quasi-monomiality

According to [4] and [3], the exponential generating function $\mathcal{G}(t,x)$ is of the Boas-Buck type if

$$\mathcal{G}(t,x) = A(t)B(xC(t)),$$

where

$$B^{(k)}(0) \neq 0 \quad (\forall k \in \mathbb{N}), \quad A(0)C'(0) \neq 0, \quad C(0) = 0.$$

Considering the exponential generating function of the reduced Mittag-Leffler polynomials $\{\hat{\varphi}_n(x)\}$, we can denote with

$$A_{\hat{\varphi}}(t) = \frac{4}{4+t^2}, \quad B_{\hat{\varphi}}(t) = \mathrm{e}^t, \quad C_{\hat{\varphi}}(t) = 2\arctan\frac{t}{2}.$$

Here, it is

$$C_{\hat{\varphi}}^{-1}(t) = 2\tan\frac{t}{2}\ .$$

Theorem 6.1. *The sequence $\hat{\varphi}_n(x)$ is quasi-monomial under the lowering operator $\mathcal{L}_x = 2\tan(D_x/2)$, i.e.*

$$\mathcal{L}_x\hat{\varphi}_n(x) = n\,\hat{\varphi}_{n-1}(x) \qquad (n \in \mathbb{N}).$$

Proof. We start with the Taylor series

$$\tan x = \sum_{k=1}^{\infty}\theta_{2k-1}x^{2k-1}, \quad \text{where} \quad \theta_{2k-1} = (-1)^{k-1}4^k(4^k-1)\frac{B_{2k}}{(2k)!}\ .$$

Here, B_n is the n-the Bernoulli number. Since

$$D_x^{2k-1}\hat{\mathcal{G}}(t,x) = \frac{4}{t^2+4}\exp\Big(2x\ \arctan\frac{t}{2}\Big)\ \Big(2\ \arctan\frac{t}{2}\Big)^{2k-1},$$

we have

$$\mathcal{L}_x \hat{\mathcal{G}}(t,x) = 2\sum_{k=1}^{\infty} \theta_{2k-1} \left(\frac{D_x}{2}\right)^{2k-1} \hat{\mathcal{G}}(t,x)$$

$$= \frac{8}{t^2+4} \exp\Big(2x \arctan\frac{t}{2}\Big) \sum_{k=1}^{\infty} \theta_{2k-1} \Big(\arctan\frac{t}{2}\Big)^{2k-1},$$

wherefrom

$$\mathcal{L}_x \hat{\mathcal{G}}(t,x) = t\, \hat{\mathcal{G}}(t,x).$$

Since

$$\mathcal{L}_x \hat{\mathcal{G}}(t,x) = \mathcal{L}_x \left(\sum_{n=0}^{\infty} \hat{\varphi}_n(x) \frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} \mathcal{L}_x \hat{\varphi}_n(x) \frac{t^n}{n!}\ ,$$

and

$$t\, \hat{\mathcal{G}}(t,x) = t \left(\sum_{n=0}^{\infty} \hat{\varphi}_n(x) \frac{t^n}{n!}\right) = \sum_{n=1}^{\infty} n\, \hat{\varphi}_{n-1}(x) \frac{t^n}{n!},$$

we have the statement proven. □

7. Conclusions

In this paper, we made some observations on the Mittag-Leffler polynomials. It is shown that it is much easier to discuss their reduced version since it is real polynomial sequence orthogonal on the real line and with fine differential and other properties.

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