FLOQUET THEORY FOR MULTITIME LINEAR DIAGONAL RECURRENCE

Cristian Ghiu[1], Raluca Tuligă[2], Constantin Udrişte[3], Ionel Ţevy[4]

Floquet theory, for periodic linear differential equations, is extended in this paper to multitime linear diagonal recurrences. We find explicitly a monodromy matrix. The Floquet point of view brings about an important simplification: the initial linear diagonal recurrence system is reduced to a linear recurrence system with constant coefficients along "diagonal lines".

Keywords: multivariate linear diagonal recurrence, diagonal periodic recurence, multitime Floquet theory.

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1. Discrete multitime recurrences

The multivariate recurrences are based on multiple sequences and come from areas like analysis of algorithms, computational biology, information theory, queueing theory, filters theory, statistical physics etc. That is why, the problem of multi-variate recurrences remains an area of active current research.

Floquet theory, first formulated for periodic linear ODEs ([1], [5], [8]) was extended to PDEs ([4]). We have extended this theory to the multitemporal first order PDEs [7] and now to multitime diagonal-periodic recurrences, borrowing mathematical ingredients from our papers [2], [3], [6]. In Floquet theory it is necessary to find explicitly the associated monodromy matrix and its eigenvalues (called Floquet multipliers).

2. Linear discrete multitime diagonal recurrence with periodic coefficients

An element $t = (t^1, \dots, t^m) \in \mathbb{N}^m$ is called *discrete multitime.* A function of the type $x : \mathbb{N}^m \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$ is called *multivariate sequence.* Also, for convenience, we denote $\mu(t) = \min\{t^1, t^2, \dots, t^m\}$ and $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^m$.

[1]Lecturer Dr., Department of Mathematical Methods and Models, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, e-mail: `crisghiu@yahoo.com`

[2]PhD student, Department of Mathematics-Informatics, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, e-mail: `ralucacoada@yahoo.com`

[3]Professor Dr., Department of Mathematics-Informatics, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, e-mail: `udriste@mathem.pub.ro`

[4]Professor Dr., Department of Mathematics-Informatics, Faculty of Applied Sciences, University "Politehnica" of Bucharest, Romania, e-mail: `vascatevy@yahoo.fr`

Let $m \geq 2$ and $A\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{R})$. Our aim is to continue the study of a linear discrete multitime diagonal recurrence system (see [2])

$$x(t+\mathbf{1}) = A(t)x(t), \quad \forall t \in \mathbb{N}^m. \tag{1}$$

In the paper [2] one proves the next result.

Theorem 2.1. *Let $m \geq 2$, $A\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{R})$, $b\colon \mathbb{N}^m \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$. We consider the $(m-1)$-sequences $f_1, f_2, \dots, f_m\colon \mathbb{N}^{m-1} \to \mathbb{R}^n$, such that*

$$f_\alpha(t^1, \dots, \widehat{t^\alpha}, \dots, t^m)\Big|_{t^\beta=0} = f_\beta(t^1, \dots, \widehat{t^\beta}, \dots, t^m)\Big|_{t^\alpha=0}, \tag{2}$$

$$\forall t^1, \dots, t^{\alpha-1}, t^{\alpha+1}, \dots, t^{\beta-1}, t^{\beta+1}, \dots, t^m \in \mathbb{N},$$

for any $\alpha, \beta \in \{1, 2, \dots, m\}$. Then the unique m-sequence $x\colon \mathbb{N}^m \to \mathbb{R}^n$ which verifies $x(t+\mathbf{1}) = A(t)x(t) + b(t)$, $\forall t \in \mathbb{N}^m$, and

$$x(t)\Big|_{t^\beta=0} = f_\beta(t^1, \dots, \widehat{t^\beta}, \dots, t^m), \forall (t^1, \dots, \widehat{t^\beta}, \dots, t^m) \in \mathbb{N}^{m-1}, \forall \beta \in \{1, \dots, m\},$$

is defined either by the formula

$$x(t) = A(t-\mathbf{1})A(t-2\cdot\mathbf{1})\cdot\ldots\cdot A(t-t^\beta\cdot\mathbf{1})\cdot f_\beta(t^1-t^\beta, \dots, \widehat{t^\beta}, \dots, t^{m-1}-t^\beta)$$
$$+ b(t-\mathbf{1}) + \sum_{k=2}^{t^\beta} A(t-\mathbf{1})A(t-2\cdot\mathbf{1})\cdot\ldots\cdot A(t-(k-1)\cdot\mathbf{1})b(t-k\cdot\mathbf{1}),$$
$$\text{if } \mu(t) = t^\beta \geq 2,$$

or by the formula

$$x(t) = A(t-\mathbf{1})\cdot f_\beta(t^1-1, \dots, \widehat{t^\beta}, \dots, t^{m-1}-1) + b(t-\mathbf{1}), \text{ if } \mu(t) = t^\beta = 1.$$

The function

$$\Phi\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{R}), \quad \Phi(t) = \begin{cases} \prod_{k=1}^{\mu(t)} A(t-k\cdot\mathbf{1}), & \text{if } \mu(t) \geq 1, \\ I_n, & \text{if } \mu(t) = 0 \end{cases}$$

is called the *fundamental matrix* (*transfer matrix*) associated to the recurrence (1). In fact, $\Phi(\cdot)$ is the unique matrix function which verifies the problem

$$\begin{cases} \Phi(t+\mathbf{1}) = A(t)\Phi(t), \quad \forall t \in \mathbb{N}^m, \\ \Phi(t)\Big|_{t^\beta=0} = I_n, \quad \forall (t^1, \dots, \widehat{t^\beta}, \dots, t^m) \in \mathbb{N}^{m-1}, \\ \qquad \forall \beta \in \{1, 2, \dots, m\}. \end{cases} \tag{3}$$

This follows applying the Theorem 2.1 for n (vector) recurrences to whom is equivalent the problem (3) (one applies the Theorem 2.1 for each column of the matrix $\Phi(\cdot)$).

If the functions $f_1, f_2, \ldots, f_m \colon \mathbb{N}^{m-1} \to \mathbb{R}^n$ verify the relations (2), then one observes (Theorem 2.1) that the unique m-sequence $x \colon \mathbb{N}^m \to \mathbb{R}^n$ which verifies the recurrence (1) and

$$x(t)\Big|_{t^\beta=0} = f_j(t^1, \ldots, \widehat{t^\beta}, \ldots, t^m), \quad \forall (t^1, \ldots, \widehat{t^\beta}, \ldots, t^m) \in \mathbb{N}^{m-1},\ \forall \beta,$$

can be written in the form

$$x(t) = \Phi(t) f_\beta(t^1 - t^\beta, \ldots, \widehat{t^\beta}, \ldots, t^{m-1} - t^\beta), \quad \text{if } \mu(t) = t^\beta.$$

If the matrix function $A(\cdot)$ is constant, i.e., $A(t) = A$, $\forall t$, then the fundamental matrix becomes $\Phi(t) = A^{\mu(t)}$.

A linear discrete multitime diagonal recurrence is called *T-diagonal-periodic* ($T \in \mathbb{N}^*$) if

$$A(t + T \cdot \mathbf{1}) = A(t), \quad \forall t \in \mathbb{N}^m. \tag{4}$$

This is the only multi-periodicity compatible to the diagonal recurrence, independently of the initial conditions.

Proposition 2.1. *Suppose the recurrence* (1) *is T-diagonal-periodic and we introduce the function*

$$\widetilde{A} \colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{R}), \quad \widetilde{A}(t) = \prod_{k=0}^{T-1} A\big(t + (T-1-k) \cdot \mathbf{1}\big), \quad \forall t \in \mathbb{N}^m.$$

Then, for any $k \in \mathbb{N}^*$, the fundamental matrix satisfies

$$\Phi\big(t + kT \cdot \mathbf{1}\big) = \Phi(t)\Big(\widetilde{A}\big(t - \mu(t) \cdot \mathbf{1}\big)\Big)^k. \tag{5}$$

Proof. If $t^\beta = \mu(t)$, then $t^\beta + kT = \min\{t^1 + kT, t^2 + kT, \ldots, t^m + kT\}$.

Induction after k. For $k = 1$:

The case $t^\beta = 0$:

$\Phi\big(t + T \cdot \mathbf{1}\big) = A\big(t + (T-1) \cdot \mathbf{1}\big) A\big(t + (T-2) \cdot \mathbf{1}\big) \cdot \ldots \cdot A\big(t + (T-T) \cdot \mathbf{1}\big)$
$= I_n \widetilde{A}(t) = \Phi(t)\widetilde{A}(t)$.

Case $t^\beta \geq 1$:

$\Phi\big(t + T \cdot \mathbf{1}\big) = A\big(t + T \cdot \mathbf{1} - \mathbf{1}\big) A\big(t + T \cdot \mathbf{1} - 2 \cdot \mathbf{1}\big) \cdot \ldots \cdot A\big(t + T \cdot \mathbf{1} - (t^\beta + T) \cdot \mathbf{1}\big)$
$= A\big(t + T \cdot \mathbf{1} - \mathbf{1}\big) A\big(t + T \cdot \mathbf{1} - 2 \cdot \mathbf{1}\big) \cdot \ldots \cdot A\big(t + T \cdot \mathbf{1} - t^\beta \cdot \mathbf{1}\big) \cdot$
$\cdot A\big(t - t^\beta \cdot \mathbf{1} + (T-1) \cdot \mathbf{1}\big) A\big(t - t^\beta \cdot \mathbf{1} + (T-2) \cdot \mathbf{1}\big) \cdot \ldots \cdot A\big(t - t^\beta \cdot \mathbf{1} + \mathbf{1}\big) A\big(t - t^\beta \cdot \mathbf{1}\big)$
$= A\big(t - \mathbf{1}\big) A\big(t - 2 \cdot \mathbf{1}\big) \cdot \ldots \cdot A\big(t - t^\beta \cdot \mathbf{1}\big) \cdot$
$\cdot A\big(t - t^\beta \cdot \mathbf{1} + (T-1) \cdot \mathbf{1}\big) A\big(t - t^\beta \cdot \mathbf{1} + (T-2) \cdot \mathbf{1}\big) \cdot \ldots \cdot A\big(t - t^\beta \cdot \mathbf{1} + \mathbf{1}\big) A\big(t - t^\beta \cdot \mathbf{1}\big)$
$= \Phi(t)\widetilde{A}\big(t - t^\beta \cdot \mathbf{1}\big)$.

Suppose the relation (5) is true for any k, and we shall verify for $k+1$:

$\Phi\big(t + (k+1)T \cdot \mathbf{1}\big) = \Phi\big(t + kT \cdot \mathbf{1} + T \cdot \mathbf{1}\big) = \Phi(t + kT \cdot \mathbf{1})\widetilde{A}\big(t + kT \cdot \mathbf{1} - (t^\beta + kT) \cdot \mathbf{1}\big)$
$= \Phi(t)\Big(\widetilde{A}\big(t - t^\beta \cdot \mathbf{1}\big)\Big)^k \widetilde{A}\big(t - t^\beta \cdot \mathbf{1}\big) = \Phi(t)\Big(\widetilde{A}\big(t - t^\beta \cdot \mathbf{1}\big)\Big)^{k+1}$. □

Suppose that we are in the conditions of the Proposition 2.1. The matrix function $D\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{R})$, $D(t) = \widetilde{A}\big(t-\mu(t)\cdot\mathbf{1}\big)$, $\forall t\in\mathbb{N}^m$, is called the *monodromy matrix* associated to the T-diagonal-periodic recurrence (1). For $k=1$, the formula (5) can be written

$$\Phi\big(t+T\cdot\mathbf{1}\big) = \Phi(t)D(t), \quad \forall t\in\mathbb{N}^m. \tag{6}$$

We observe that the relation $D\big(t+\mathbf{1}\big) = D(t)$, $\forall t\in\mathbb{N}^m$ holds.

Moreover, let us suppose that, for any $t\in\mathbb{N}^m$, the matrix $A(t)$ is invertible, hence $\widetilde{A}(t)$ is also invertible; it follows that for each $t\in\mathbb{N}^m$, there exists $\widetilde{B}(t)\in\mathcal{M}_n(\mathbb{C})$ (which is not unique), such that $\widetilde{B}(t)^T = \widetilde{A}(t)$. For each t, we fix such a matrix $\widetilde{B}(t)$; obviously the matrix $\widetilde{B}(t)$ is invertible.

Define the function

$$B\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{C}), \quad B(t) = \widetilde{B}\big(t-\mu(t)\cdot\mathbf{1}\big), \quad \forall t\in\mathbb{N}^m. \tag{7}$$

We observe that the relation

$$B\big(t+\mathbf{1}\big) = B(t), \quad \forall t\in\mathbb{N}^m \tag{8}$$

is true, i.e., the matrix B verifies a special recurrence. It follows immediately the relation

$$D(t) = B(t)^T, \quad \forall t\in\mathbb{N}^m. \tag{9}$$

Moreover, one observes that if for any t, the matrix $A(t)$ is invertible, then the matrices $\Phi(t)$ and $D(t)$ are invertible too.

Theorem 2.2. *Let $m\geq 2$ and $A\colon \mathbb{N}^m\to\mathcal{M}_n(\mathbb{R})$, with $A(t)$ invertible, for all $t\in\mathbb{N}^m$. Suppose there exists an integer $T\geq 1$ such that the relation* (4) *is true. Then there exists $P\colon\mathbb{N}^m\to\mathcal{M}_n(\mathbb{C})$, with the property that $P(t+T\cdot\mathbf{1}) = P(t)$, $\forall t$, and such that the fundamental matrix of the T-multi-periodic recurrence* (1) *is written*

$$\Phi(t) = P(t)B(t)^{\mu(t)}, \quad \forall t\in\mathbb{N}^m \tag{10}$$

$\big($*where $B(\cdot)$ is the function defined by the formula* (7)$\big)$.

Proof. Let $P\colon\mathbb{N}^m\to\mathcal{M}_n(\mathbb{C})$, $P(t) = \Phi(t)B(t)^{-\mu(t)}$, $\forall t\in\mathbb{N}^m$. It is sufficient to show $P(t+T\cdot\mathbf{1}) = P(t)$, $\forall t\in\mathbb{N}^m$.

Let $t=(t^1,t^2,\dots,t^m)\in\mathbb{N}^m$ and $t^\beta=\mu(t)$. Obviously, $t^\beta+T=\mu(t+T\cdot\mathbf{1})$ and $P(t+T\cdot\mathbf{1}) = \Phi(t+T\cdot\mathbf{1})B(t+T\cdot\mathbf{1})^{-t^\beta-T}$. From the relation (8), we deduce $B(t+k\cdot\mathbf{1}) = B(t)$, $\forall k\in\mathbb{N}$, and particularly $B(t+T\cdot\mathbf{1}) = B(t)$. From the relations (6), (9), it follows $\Phi(t+T\cdot\mathbf{1}) = \Phi(t)B(t)^T$.

Hence $P(t+T\cdot\mathbf{1}) = \Phi(t)B(t)^TB(t)^{-t^\beta-T} = \Phi(t)B(t)^{-t^\beta} = P(t)$. □

In the condition of the Theorem 2.2, from the formula (10) and the fact that the matrix $\Phi(t)$ is invertible, it follows that $P(t)$ is also invertible.

The most important result of Floquet type is

Theorem 2.3. *Let $m \geq 2$ and $A\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{R})$, with $A(t)$ invertible, for all $t \in \mathbb{N}^m$. Suppose that there exists an integer $T \geq 1$ such that the relation* (4) *is true. Let $B\colon \mathbb{N}^m \to \mathcal{M}_n(\mathbb{C})$ be the function defined by the formula* (7).

We consider the recurrences

$$x(t+\mathbf{1}) = A(t)x(t), \quad \forall t \in \mathbb{N}^m; \tag{11}$$

$$y(t+\mathbf{1}) = B(t)y(t), \quad \forall t \in \mathbb{N}^m. \tag{12}$$

If $y(t)$ is a solution of the recurrence (12), *then $x(t) := P(t)y(t)$ is a solution of the recurrence* (11). *And conversely, if $x(t)$ is a solution of the recurrence* (11), *then $y(t) := P(t)^{-1}x(t)$ is a solution of the recurrence* (12).

Proof. Let $y(t)$ be a solution of the recurrence (12) and $x(t) := P(t)y(t)$; hence $y(t) := P(t)^{-1}x(t)$. Let $t \in \mathbb{N}^m$ and $t^\beta = \mu(t)$. It follows

$y(t+\mathbf{1}) = B(t)y(t) \iff P(t+\mathbf{1})^{-1}x(t+\mathbf{1}) = B(t)P(t)^{-1}x(t)$

$\iff x(t+\mathbf{1}) = P(t+\mathbf{1})B(t)P(t)^{-1}x(t)$.

We use the formula (10) and deduce that the foregoing relations are equivalent to

$x(t+\mathbf{1}) = \Phi(t+\mathbf{1})B(t+\mathbf{1})^{-t^\beta-1}B(t)B(t)^{t^\beta}\Phi(t)^{-1}x(t)$.

This relation is equivalent to (according the formula (8))

$x(t+\mathbf{1}) = \Phi(t+\mathbf{1})B(t)^{-t^\beta-1}B(t)B(t)^{t^\beta}\Phi(t)^{-1}x(t)$

$\iff x(t+\mathbf{1}) = \Phi(t+\mathbf{1})\Phi(t)^{-1}x(t) \iff x(t+\mathbf{1}) = A(t)\Phi(t)\Phi(t)^{-1}x(t)$

$\iff x(t+\mathbf{1}) = A(t)x(t)$.

The converse is proved similarly. □

3. Conclusions

This paper presents original results regarding the multivariate recurrence equations as continuation of [2], [3]. The original results have a great potential to solve problems in various areas such as ecosystem dynamics, financial modeling, and economics.

The authors lay no claims to the paper's being a complete presentation of all current methods for investigation the linear multivariate recurrences with periodic coefficients. Indeed, the material presented here is a reflection of our scientific interests regarding Floquet theory.

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