ON THE ULAM-HYERS STABILITY OF BIHARMONIC EQUATION

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In this paper we investigate the Ulam-Hyers stability of the biharmonic equation in the class of circular symmetric functions. Biharmonic equation has many applications, for example in elasticity, fluid mechanics and many other areas. We apply our results in elasticity and civil engineering. We consider a circular plane plate. In this case the solutions will be functions with circular symmetry. In general the unknown functions are $u = u(r, \theta)$ *but in the case of the circular symmetry* $u = u(r)$. *The biharmonic equation* $\Delta^2 u = \frac{p}{D}$ *becomes* $r^4 \frac{d^4 u}{dr^4} + 2r^3 \frac{d^3 u}{dr^3} - r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} = r^4 \frac{p}{D}$, *where* p *is the normal pressure load to the plate and* D *is the flexural rigidity.*

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1. Introduction

The Ulam stability is an important concept in the theory of functional equations. The origin of Ulam stability theory was an open problem formulated by Ulam, in 1940, concerning the stability of homomorphism [14]. The first partial answer to Ulam's question came within a year, when Hyers [5] proved a stability result, for additive Cauchy equation in Banach spaces. The first result on Hyers-Ulam stability of differential equations was given by Obloza [9]. Alsina and Ger investigated the stability of differential equations $y' = y$ [1]. The result of Alsina and Ger was extended by many authors [3], [6], [7], [8], [11], [12], [13] to the stability of the first order linear differential equation and linear differential equations of higher order. In [2] Brzdek, Popa, Rasa and Xu presented a unified and systematic approach to the field. Generally, we say that a differential equation is Ulam stable if for every approximate solution of the differential equation, there exists an exact solution near it [1].

Ulam-Hyers stability has many applications in physics, economy, engineering, etc. In [4] Hegyi and Jung studied Ulam-Hyers stability for the Laplace's equation, in the class of circular symmetric functions. The solutions of Laplace ecuations, called harmonic functions, are very important in the field of electromagnetism, astronomy, thermodynamics and fluid dynamics.

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The goal of this paper is to obtain some results on generalized Ulam-Hyers stability for the biharmonic equation, in the class of circular symmetric functions. The biharmonic equation arises in areas of continuum mechanics, including linear elasticity theory. It is used in modeling of the thin structures that react elastically to external forces.

In our approach we will use some results of Popa and Pugna [10] concerning the stability of Euler's differential equations. For the sake of convenience for the reader we recall some notations from [10].

Let $(X, \|.\|)$ a Banach space over $\mathbb{C}$. Let $I = (a, b)$, $0 \leq a < b \leq \infty$.

For $c \in [a, b]$ and $\alpha \in \mathbb{C}$ define

$$\Phi_\alpha (h) (x) := e^{\Re\alpha\cdot x} \cdot \left| \int_c^x e^{-\Re\alpha\cdot t} \cdot h (t)\, dt \right|, x \in I, \tag{1}$$

for all functions $h : I \to \mathbb{C}$ with the property that the right hand of (1) exists and is finite. $\Re(\alpha)$ stands for the real part of the complex number α.

2. **Main results**

We consider the homogeneous biharmonic equation

$$\Delta^2 u = 0. \tag{2}$$

If the function $u : I \to X$, $I = (a, b)$, $0 \leq a < b \leq \infty$ is circular symmetric, that is $u = u(r)$, then the equation (2) can be written

$$r^4 \frac{d^4u}{dr^4} + 2r^3 \frac{d^3u}{dr^3} - r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} = 0.$$

This is an Euler's type equation. The inhomogeneous equation is

$$r^4 \frac{d^4u}{dr^4} + 2r^3 \frac{d^3u}{dr^3} - r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} = f(r), \tag{3}$$

where $f : I \to X$ is a continuous function. We suppose that the functions f, g, where $g : I \to X, g(r) = \frac{f(r)}{r}$ are bounded on I.

In what follows we will study generalized Ulam-Hyers stability for the equation (3).

Definition 2.1. The equation (3) is called generalized Ulam-Hyers stable if for every $\varepsilon : I \to (0, \infty)$ there exists a function $\psi : I \to (0, \infty)$ depending on ε such that for all $u \in C^4(I, X)$ satisfying

$$\left\| r^4 \frac{d^4u}{dr^4} + 2r^3 \frac{d^3u}{dr^3} - r^2 \frac{d^2u}{dr^2} + r \frac{du}{dr} - f(r) \right\| \leq \varepsilon(r), r \in I, \tag{4}$$

there exists a solution $u_1 \in C^4(I, X)$ of the equation (3) with the property

$$\|u(r) - u_1(r)\| \leq \psi(r), r \in I.$$

Let $\frac{du}{dr} = v(r)$. Hence, the equation (3) becomes

$$r^3 \frac{d^3v}{dr^3} + 2r^2 \frac{d^2v}{dr^2} - r \frac{dv}{dr} + v - \frac{1}{r} f(r) = 0. \tag{5}$$

A result on classical Ulam-Hyers stability is given in the next theorem.

Theorem 2.1. Let $\varepsilon > 0$. Then for every $v \in C^3(I, X)$ satisfying

$$\left\| r^3 \frac{d^3 v}{dr^3} + 2r^2 \frac{d^2 v}{dr^2} - r \frac{dv}{dr} + v - \frac{1}{r} f(r) \right\| \leq \varepsilon, r \in I, \tag{6}$$

there exists a solution $w \in C^3(I, X)$ of the equation (5) with the property

$$\|v(r) - w(r)\| \leq L\varepsilon, r \in I,$$

where

$$L = \begin{cases} \frac{(b-a)^3}{(b+a)^3}, \text{ if } a > 0, b \in (0, \infty) \\ 1, \text{ if } a = 0 \text{ or } b = \infty \end{cases}, \tag{7}$$

that is the equation (5) is Ulam-Hyers stable.

Proof. Let v be a solution of (6). According to Theorem 2.4 from [10] it follows that there exists a solution w of the equation (5) such that

$$\|v(r) - w(r)\| \leq L\varepsilon, r \in I,$$

where

$$L = \begin{cases} \prod_{i=1}^{3} \frac{1}{|\Re\lambda_k|} \cdot \frac{\left|b^{\Re\lambda_k} - a^{\Re\lambda_k}\right|}{b^{\Re\lambda_k} + a^{\Re\lambda_k}}, \text{ if } a > 0, b \in R \\ \frac{1}{\prod_{i=1}^{3} |\Re\lambda_k|}, \text{ if } a = 0 \text{ or } b = \infty \end{cases},$$

where$\lambda_1, \lambda_2, \lambda_3$ ar the roots of the characteristic equation

$$\lambda(\lambda - 1)(\lambda - 2) + 2\lambda(\lambda - 1) - \lambda + 1 = 0,$$

that is $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$.

We have $L = \begin{cases} \frac{1}{1} \cdot \frac{|b-a|}{b+a} \cdot \frac{1}{1} \cdot \frac{|b-a|}{b+a} \cdot \frac{1}{|-1|} \cdot \frac{\left|b^{-1} - a^{-1}\right|}{b^{-1} + a^{-1}} = \frac{(b-a)^3}{(b+a)^3}, \text{ if } a > 0, b \in (0, \infty) \\ 1, \text{ if } a = 0 \text{ or } b = \infty \end{cases}.$

□

Theorem 2.2. *Let* $\varphi : (a, b) \to (0, \infty)$ *be a given function such that* $\Phi_{-1} \circ \Phi_1 \circ \Phi_1(\varphi)(r)$ *exists and is finite for* $c = a$. *Then for every* $v \in C^3(I, X)$ *satisfying*

$$\left\| r^3 \frac{d^3 v}{dr^3} + 2r^2 \frac{d^2 v}{dr^2} - r \frac{dv}{dr} + v - \frac{1}{r} f(r) \right\| \leq \varphi(r), r \in I, \tag{8}$$

there exists a solution $w \in C^3(I, X)$ *of the equation* (5) *with the property*

$$\|v(r) - w(r)\| \leq e^{-r} \cdot \int_a^r e^{2t} \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\, ds \right) dz \right) dt, r \in I,$$

that is the equation (5) *is generalized Ulam-Hyers stable.*

Proof. Let v be a solution of (8). According to Theorem 2.3 from [10] it follows that there exists a solution w of the equation (5) such that

$$\|v(r) - w(r)\| \leq \Phi_{\lambda_3} \circ \Phi_{\lambda_2} \circ \Phi_{\lambda_1}(\varphi)(r), r \in I,$$

where $\lambda_1, \lambda_2, \lambda_3$ ar the roots of the characteristic equation

$$\lambda(\lambda - 1)(\lambda - 2) + 2\lambda(\lambda - 1) - \lambda + 1 = 0,$$

that is $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -1$.

We have

$$\Phi_{\lambda_1}(\varphi)(r) = e^{\Re\lambda_1 r} \cdot \int_a^r e^{-\Re\lambda_1 t} \cdot \varphi(t)\,dt = e^r \cdot \int_a^r e^{-t} \cdot \varphi(t)\,dt$$

$$\Phi_{\lambda_2} \circ \Phi_{\lambda_1}(\varphi)(r) = e^{\Re\lambda_2 r} \cdot \int_a^r e^{-\Re\lambda_2 t} \cdot \Phi_{\lambda_1}(\varphi)(t)\,dt =$$

$$= e^r \cdot \int_a^r e^{-t} \cdot e^t \left(\int_a^t e^{-s} \cdot \varphi(s)\,ds \right) dt = e^r \cdot \int_a^r \left(\int_a^t e^{-s} \cdot \varphi(s)\,ds \right) dt$$

$$\Phi_{\lambda_3} \circ \Phi_{\lambda_2} \circ \Phi_{\lambda_1}(\varphi)(r) = e^{\Re\lambda_3 r} \cdot \int_a^r e^{-\Re\lambda_3 t} \cdot \Phi_{\lambda_2} \circ \Phi_{\lambda_1}(\varphi)(t)\,dt =$$

$$= e^{-r} \cdot \int_a^r e^t \cdot e^t \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\,ds \right) dz \right) dt =$$

$$= e^{-r} \cdot \int_a^r e^{2t} \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\,ds \right) dz \right) dt.$$

□

If we come back to initial function we obtain the following result. Let $X = \mathbb{R}$.

Theorem 2.3. *Let $\varepsilon > 0$. For every $u \in C^4(I, X)$ satisfying*

$$\left| r^4 \frac{d^4u}{dr^4} + 2r^3 \frac{d^3u}{dr^3} - r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - f(r) \right| \leq \varepsilon r, r \in I,$$

there exists a solution $u_1 \in C^4(I, X)$ of the equation (3) *with the property*

$$|u(r) - u_1(r)| \leq \varepsilon L, r \in I,$$

where

$$L = \begin{cases} \frac{(b-a)^4}{(b+a)^3}, & \text{if } a > 0, b \in (0, \infty) \\ b, & \text{if } a = 0, b \in (0, \infty) \end{cases}.$$

If $b = \infty$ then the equation (3) *is not stable.*

Proof. Let $\frac{du}{dr} = v(r)$. Hence, we get the inequality

$$\left| r^4 \frac{d^3v}{dr^3} + 2r^3 \frac{d^2v}{dr^2} - r^2 \frac{dv}{dr} + rv - f(r) \right| \leq \varepsilon r, r \in I,$$

or

$$\left| r^3 \frac{d^3v}{dr^3} + 2r^2 \frac{d^2v}{dr^2} - r\frac{dv}{dr} + v - \frac{f(r)}{r} \right| \leq \varepsilon, r \in I. \tag{9}$$

We apply Theorem 2.1 so for every $v \in C^3(I, X)$ satisfying (9) there exists a solution $w \in C^3(I, X)$ of the equation (5) with the property

$$|v(r) - w(r)| \leq \varepsilon L, r \in I,$$

where L is given by (7). Let u_1 be a function such that $\frac{du_1}{dr} = w(r)$ and $u(a) = u_1(a)$. Then

$$\left| \frac{du}{dr}(r) - \frac{du_1}{dr}(r) \right| \leq \varepsilon L, r \in I,$$

or

$$-\varepsilon L \leq \frac{du}{dr}(r) - \frac{du_1}{dr}(r) \leq \varepsilon L, r \in I. \tag{10}$$

Firstly we suppose that $a > 0, b \in (0, \infty)$. Integrating from a to r in (10) we have

$$-\varepsilon L(r-a) \le u(r) - u_1(r) \le \varepsilon L(r-a), r \in I, \tag{11}$$

that is

$$|u(r) - u_1(r)| \le \varepsilon L(r-a) = \varepsilon \frac{(b-a)^3}{(b+a)^3}(r-a) \le \varepsilon \frac{(b-a)^4}{(b+a)^3}.$$

We suppose now that $a = 0, b \in (0, \infty)$. Integrating from a to r in (10) we have

$$|u(r) - u_1(r)| \le \varepsilon r \le \varepsilon b.$$

If $b = \infty$ then the equation (3) is not stable. □

Let $X = \mathbb{R}$.

Theorem 2.4. *Let $\varepsilon : I \to (0, \infty)$ be a bounded and continuous function on I. For every $u \in C^4(I, X)$ satisfying*

$$\left| r^4 \frac{d^4u}{dr^4} + 2r^3 \frac{d^3u}{dr^3} - r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - f(r) \right| \le \varepsilon(r), r \in I,$$

there exists a solution $u_1 \in C^4(I, X)$ of the equation (3) with the property

$$|u(r) - u_1(r)| \le \int_a^r \left(e^{-p} \cdot \int_a^p e^{2t} \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\, ds \right) dz \right) dt \right) dp, r \in I, \tag{12}$$

where $\varphi(r) = \frac{\varepsilon(r)}{r}$

Proof. Let $\frac{du}{dr} = v(r)$. Hence, we get the inequality

$$\left| r^4 \frac{d^3v}{dr^3} + 2r^3 \frac{d^2v}{dr^2} - r^2 \frac{dv}{dr} + rv - f(r) \right| \le \varepsilon(r), r \in I,$$

or

$$\left| r^3 \frac{d^3v}{dr^3} + 2r^2 \frac{d^2v}{dr^2} - r\frac{dv}{dr} + v - \frac{f(r)}{r} \right| \le \frac{\varepsilon(r)}{r}, r \in I. \tag{13}$$

We apply Theorem 2.2 for $\varphi(r) = \frac{\varepsilon(r)}{r}$, so for every $v \in C^3(I, X)$ satisfying (13) there exists a solution $w \in C^3(I, X)$ of the equation (5) with the property

$$|v(r) - w(r)| \le e^{-r} \cdot \int_a^r e^{2t} \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\, ds \right) dz \right) dt, r \in I.$$

Let u_1 be a function such that $\frac{du_1}{dr} = w(r)$ and $u(a) = u_1(a)$. Then

$$\left| \frac{du}{dr}(r) - \frac{du_1}{dr}(r) \right| \le e^{-r} \cdot \int_a^r e^{2t} \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\, ds \right) dz \right) dt, r \in I,$$

that is

$$-e^{-r} \cdot \int_a^r e^{2t} \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\, ds \right) dz \right) dt \le \frac{du}{dr}(r) - \frac{du_1}{dr}(r) \le$$

$$\le e^{-r} \cdot \int_a^r e^{2t} \left(\int_a^t \left(\int_a^z e^{-s} \cdot \varphi(s)\, ds \right) dz \right) dt. \tag{14}$$

Integrating from a to r in (14) we obtain the inequality (12). □

3. Applications

We consider, now, the problem of the bending of a uniform circular thin plate subjected to a load force p perpendicular to the plane of the plate. Let u be the vertical displacement. It is convenient to express the governing differential equation in polar coordinates r, θ. Since the plate geometry is symmetric and also the load distribution p is assumed to be axis-symmetric, the governing differential equation is

$$r^4 \frac{d^4 u}{dr^4} + 2r^3 \frac{d^3 u}{dr^3} - r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} = r^4 \frac{p(r)}{D}, \tag{15}$$

where D is the flexural rigidity factor, which includes all the elastic constants related to the material. The stress components in this case are

$$\sigma_r = \frac{1}{r} \frac{du}{dr}, \sigma_\theta = \frac{d^2 u}{dr^2}, \tau_{r\theta} = 0. \tag{16}$$

We specify that in practical examples and in general, for the biharmonic equations are imposed boundedness conditions for the solutions, at origin.

Remark 3.1. *The above results can be applied for the equation* (15) *taking* $f(r) = r^4 \frac{p(r)}{D}$ *and also for the stability of stress components* σ_r.

We suppose that the functions f, g, where $g : I \to R, g(r) = \frac{f(r)}{r}$ are bounded on I. Let $X = \mathbb{R}$.

Theorem 3.1. *Let* $\varepsilon > 0$. *For every* $u \in C^4(I, X)$ *satisfying*

$$\left| r^4 \frac{d^4 u}{dr^4} + 2r^3 \frac{d^3 u}{dr^3} - r^2 \frac{d^2 u}{dr^2} + r \frac{du}{dr} - r^4 \frac{p(r)}{D} \right| \le \varepsilon r, r \in I, \tag{17}$$

there exists a solution $u_1 \in C^4(I, X)$ *of the equation* (15) *such that:*

(1)

$$|u(r) - u_1(r)| \le \varepsilon L, r \in I, \tag{18}$$

where

$$L = \begin{cases} \frac{(b-a)^4}{(b+a)^3}, & \text{if } a > 0, b \in (0, \infty) \\ b, & \text{if } a = 0, b \in (0, \infty) \end{cases}, \tag{19}$$

that is the equation (15) *is generalized Ulam-Hyers stable. If* $b = \infty$ *then the equation* (15) *is not stable.*

(2) *the corresponding stress components* σ_r *and* σ_r' *satisfy the inequality*

$$|\sigma_r - \sigma_r'| \le \frac{1}{r} \varepsilon L, r \in I.$$

where L *is given by* (7).

Proof. (1) It follows directly from Theorem 2.3.

(2) Let $\frac{du}{dr} = v(r)$. Hence, we get the inequality

$$\left| r^4 \frac{d^3 v}{dr^3} + 2r^3 \frac{d^2 v}{dr^2} - r^2 \frac{dv}{dr} + rv - r^4 \frac{p(r)}{D} \right| \le \varepsilon r, r \in I,$$

or

$$\left| r^3 \frac{d^3 v}{dr^3} + 2r^2 \frac{d^2 v}{dr^2} - r \frac{dv}{dr} + v - r^3 \frac{p(r)}{D} \right| \le \varepsilon, r \in I. \tag{20}$$

We apply Theorem 2.1 so for every $v \in C^3(I, X)$ satisfying (20) there exists a solution $w \in C^3(I, X)$ of the equation

$$r^3 \frac{d^3 v}{dr^3} + 2r^2 \frac{d^2 v}{dr^2} - r\frac{dv}{dr} + v - r^3 \frac{p(r)}{D} = 0, r \in I,$$

with the property

$$|v(r) - w(r)| \leq \varepsilon L, r \in I,$$

where L is given by (19). Let u_1 be a function such that $\frac{du_1}{dr} = w(r)$ and $u(a) = u_1(a)$. Then

$$\left| \frac{du}{dr} - \frac{du_1}{dr} \right| \leq \varepsilon L, r \in I.$$

Let σ_r and σ_r' be the corresponding stress components. Hence

$$|\sigma_r - \sigma_r'| = \left| \frac{1}{r}\frac{du}{dr} - \frac{1}{r}\frac{du_1}{dr} \right| = \frac{1}{r}\left| \frac{du}{dr} - \frac{du_1}{dr} \right| \leq \frac{1}{r}\varepsilon L, r \in I.$$

□

4. **Conclusions**

We studied the generalized Ulam-Hyers stability of the biharmonic equation in the class of circular symmetric functions. We applied our results in elasticity in the sense that for every approximate solution u satisfying (17), there exists an exact solution u_1 of (15) near it, that is the relation (18) is satisfied. From a different perspective, approximate solution can be viewed in relation to perturbation theory, as any approximate solution of (17) is an exact solution of the perturbed equation

$$r^4 \frac{d^4 u}{dr^4} + 2r^3 \frac{d^3 u}{dr^3} - r^2 \frac{d^2 u}{dr^2} + r\frac{du}{dr} = r^4 \frac{p(r)}{D} + h(r),$$

for some perturbation h satisfying $|h(r)| \leq \varepsilon r, r \in I$.

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