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GENERATION OF THE PITCH CURVES WITH MINIMAL MEAN KINETIC ENERGY CHARACTERISTIC FOR NONCIRCULAR GEARS

Xin ZHANG[1], Shuai FAN[2]

In order to improve the stability of noncircular gears transmission, a mathematical model for the pitch curve design with minimal mean kinetic energy of noncircular gear with unspecified rotation time constraint is proposed. Meanwhile, the minimal mean kinetic energy pitch curve with specified rotation time constraint can also be obtained based on the model and the variational method. In comparison with the model of unspecified rotation time constraint, the transmission angle range for noncircular gears with the pitch curves can be expanded through the specified rotation time constraint. Particularly, the reason for the failure of unidirectional continuous rotary motion for noncircular gears of the pitch curves with specified rotation time can also be analyzed. The validity of the proposed design method is verified by the generation and modification examples of conjugate pitch curves with unspecified and specified rotation time constraints.

Keywords: noncircular gears, pitch curves design, variational method, mathematical model, transmission stability.

1. Introduction

Noncircular gear pair possesses good mechanical properties in variable transmission between the parallel axis with applications in steering mechanism, polishing mechanism, power drive mechanism, gear pumps, packaging and etc [1-5]. Additionally, noncircular gear pair was introduced to eliminate the speed and torque fluctuation that exists in rotating shafts [6]. Generations of functions and planar, helical elliptical gears were respectively developed by non-circular gears and the application of existing equipment and tools [7-9]. The base curves of involute cylindrical gears, for uniform and non-uniform transmission ratio, were synthesized by means of Aronhold's first theorem and the return circle [10]. Syntheses of elliptical gears and the tooth profiles of noncircular gears were proposed by Figliolini and Angeles [11-12]. A simple and accurate numerical method was proposed for calculating the tooth profile of a noncircular gear [13]. The generation method of tooth profile and undercutting conditions for

[1] College of Mechanical Engineering, Baoji University of Arts and Sciences, Baoji, Shaanxi, 721016, China, e-mail: zhangxin891221@163.com
[2] School of Mechanical and Electrical Engineering, University of Electronic Science and Technology of China. Chengdu, Sichuan, 611731, China, e-mail: fansuai12345@163.com

noncircular gears were developed and discussed by Chang and co-workers [14-15]. Determination method of the profile of the gear teeth on a variable radius wheel characterized by a constant pressure angle was presented by Danieli [16]. A gear hobbing machining method of higher-order elliptic gears and its machining principle were proposed by Zhang et al [17].

Additionally, conjugate pitch curves for identical *N*-lobe noncircular gears was obtained by Tong and Yang [18]. Bézier and B-spline curves were used to obtain the displacement laws of an *N*-lobe noncircular gear [19]. The generation algorithm of the conjugate pitch curves with steepest rotation characteristic for noncircular gears was presented by Zhang and Fan [20]. The Archimedes spiral was used to generate the pitch curves for *N*-lobed noncircular gears and noncircular bevel gears [21-22]. Noncircular pitch curves with concave and convex characteristics were proposed by Shi [23] and Yan [24], respectively.

In this paper, the mathematical models which contain unspecified and specified rotation time constraints are established to generate the pitch curve with minimal mean kinetic energy for noncircular gear. Moreover, the reason for the failure of unidirectional continuous rotary motion for noncircular gears with minimal mean kinetic energy pitch curves can also be analyzed. This novel design approach for conjugate noncircular gear pitch curves can be further extended to other mechanical devices with variable transmission.

2. Mathematical model of minimal mean kinetic energy pitch curve design for noncircular gear

Noncircular gear pitch curve $r(\eta)$ is shown in Fig. 1, its rotation center and boundaries are coinciding with points *o*, *a*, *b*, and the straight line *oa* is coincide with *x*-axis of Cartesian coordinate system $\Gamma(o\text{-}xy)$. Parameters η_a (η_a=0) and η_b ($0=\eta_a<\eta_b\leq 2\pi$) are the polar angles of points *a* and *b*, respectively.

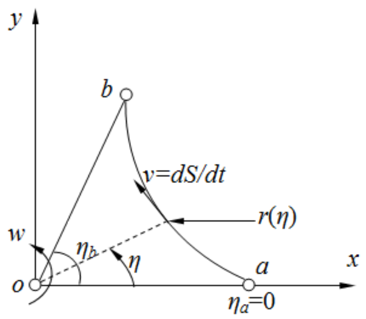

Fig. 1. Noncircular gear pitch curve $r(\eta)$

Referring to Fig. 1, the infinitesimal *dt* of rotation time that noncircular gear through the pitch curve $r(\eta)$ can be expressed as

$$dt(\eta)=\frac{dS(\eta)}{\omega r(\eta)}=\frac{\sqrt{r^2(\eta)+r'^2(\eta)}}{\omega r(\eta)}d\eta \tag{1}$$

where $dS(\eta)$ is an infinitesimal of arc S for the noncircular gear pitch curve $r(\eta)$, ω is given noncircular gear angular velocity, polar angle η is measured counterclockwise from the positive direction of x-axis.

Integrating both sides of Eq. (1) from η_a (η_a=0) to η_b, one obtains

$$T=\int_0^{\eta_b}\frac{\sqrt{r^2(\eta)+r'^2(\eta)}}{\omega r(\eta)}d\eta,\eta_b\in(0,2\pi] \tag{2}$$

where T is rotation time that noncircular gear through the pitch curve $r(\eta)$.

Assuming the mass of noncircular gear is m, then the mean kinetic energy E of noncircular gear through the pitch curve $r(\eta)$ from t_0=0 to t_1=T at the given angular velocity ω can be expressed as

$$E=\frac{1}{T}\int_0^T\frac{m}{2}[(\frac{dx}{dt})^2+(\frac{dy}{dt})^2]dt \tag{3}$$

Substituting Eq. (2) into Eq. (3), we obtain

$$E=\int_0^{\eta_b}\frac{m\omega r(\eta)\sqrt{r^2(\eta)+r'^2(\eta)}}{2T}d\eta \tag{4}$$

Referring to Eq. (4), we know that the mean kinetic energy E may be different for different pitch curve $r(\eta)$, the minimum $E_{\min}$ of mean kinetic energy E can be expressed as

$$E_{\min}=\min\int_0^{\eta_b}\frac{m\omega r(\eta)\sqrt{r^2(\eta)+r'^2(\eta)}}{2T}d\eta=\min\int_0^{\eta_b}Fd\eta \tag{5}$$

According to the variational method [25] and F of Eq. (5) doesn't contain parameter η, the integral of Eq. (5) can be obtained

$$\frac{r^3(\eta)}{\sqrt{r^2(\eta)+r'^2(\eta)}}=h_1,h_1=\frac{2hT}{m\omega} \tag{6}$$

where h_1 is an undetermined parameter.

Assuming $r'(\eta)=r(\eta)\tan\mu$, $\mu\in(-\pi/2,\pi/2)$, we can obtain

$$r^2(\eta)=h_1/\cos\mu,h_1>0 \tag{7}$$

$$d\eta=\frac{dr(\eta)}{r'(\eta)}=\frac{d\mu}{2} \tag{8}$$

The integral to both sides of Eq. (8) at the same time, one obtains

$$2\eta=\mu+h_2 \tag{9}$$

where h_2 is an undetermined parameter.

Along with Eq. (7), the pitch curve $r(\eta)$ with minimal mean kinetic energy characteristic can be obtained

$$\begin{cases} r(\eta)=\sqrt{\dfrac{h_1}{\cos(2\eta-h_2)}}, \eta\in[0,\eta_b] \\ s.t.\ h_1>0, h_2\in(2\eta_b-\pi/2,\pi/2) \end{cases} \tag{10}$$

where h_1 and h_2 are undetermined parameter and can be obtained by the boundary constraints (0, r(0)) and (η_b, $r(\eta_b)$).

Substituting the boundary constraints (0, r(0)) and (η_b, $r(\eta_b)$) of pitch curve $r(\eta)$ into Eq. (10), undetermined constant h_2 can be obtained

$$h_2=\arctan[\frac{r^2(0)-r^2(\eta_b)\cos(2\eta_b)}{r^2(\eta_b)\sin(2\eta_b)}] \tag{11}$$

In order to ensure that the parameter h_2 of Eq. (11) has a solution under the constraint $h_2\in(2\eta_b-\pi/2,\ \pi/2)$, the polar angle η_b should satisfy the following condition

$$0<\eta_b<\pi/2 \tag{12}$$

3. Minimal mean kinetic energy pitch curve with specified rotation time constraint

According to Eq. (2) and Eq. (5), the auxiliary function E_T^* with specified rotation time T based on the variational method can be obtained

$$E_T^*=\min\int_0^{\eta_b}\frac{m\omega r_T(\eta)\sqrt{r_T^2(\eta)+r_T'^2(\eta)}}{2T}+\lambda\frac{\sqrt{r_T^2(\eta)+r_T'^2(\eta)}}{\omega r_T(\eta)}d\eta=\min\int_0^{\eta_b}Hd\eta \tag{13}$$

where $r_T(\eta)$ is the minimal mean kinetic energy pitch curve with specified rotation time T, λ is Lagrange multiplier.

Referring to the solution process of Eq. (5), we can obtain

$$r_T^2(\eta)=\frac{2T}{m\omega^2}(\omega h_3\sec\mu-\lambda), h_3>0 \text{ and } h_3>\frac{\lambda}{\omega} \tag{14}$$

$$d\eta=\frac{dr_T^2(\eta)}{r_T'^2(\eta)}=\frac{\omega h_3}{2\omega h_3-2\lambda\cos\mu}d\mu \tag{15}$$

where h_3 is an undetermined parameter.

Due to the polar angle $\eta(\mu)$ should be monotonically increasing when $\mu\in(-\pi/2,\pi/2)$, so the first-order derivative $\eta'(\mu)$ should satisfy the following constraint

$$\left(\frac{\omega h_3}{2\omega h_3-2\lambda\cos\mu}\right)_{\min}>0, \mu\in(-\pi/2,\pi/2) \tag{16}$$

According to the characteristics of second-order derivative $\eta''(\mu)$

$$\eta''(\mu)=\frac{-2\omega\lambda h_3\sin\mu}{(2\omega h_3-2\lambda\cos\mu)^2}=\begin{cases}>0,\mu\in(-\pi/2,0)\\0,\mu=0\\<0,\mu\in(0,\pi/2)\end{cases}, \lambda h_3>0 \text{ or } \begin{cases}<0,\mu\in(-\pi/2,0)\\0,\mu=0\\>0,\mu\in(0,\pi/2)\end{cases}, \lambda h_3<0 \tag{17}$$

The undetermined parameter h_3 and Lagrange multiplier λ should satisfy the following constraints

$$\lambda h_3 > 0 \text{ or } \omega h_3 / (\omega h_3 - \lambda) > 0, \lambda h_3 < 0 \tag{18}$$

Along with Eq. (14), therefore, the constraints of undetermined parameter h_3 and Lagrange multiplier λ can be obtained

$$h_3 > \lambda / \omega > 0 \text{ or } h_3 > 0 > \lambda / \omega \tag{19}$$

Along with Eq. (15), we can obtain

$$\eta(\mu) = \frac{\omega h_3}{\omega h_3 - \lambda}\sqrt{\frac{\omega h_3 - \lambda}{\omega h_3 + \lambda}} \arctan\left(\sqrt{\frac{\omega h_3 + \lambda}{\omega h_3 - \lambda}} \tan\frac{\mu}{2}\right) + h_4, h_3 > \frac{\lambda}{\omega} > 0 \tag{20}$$

or

$$\eta(\mu) = \frac{\omega h_3}{2\omega h_3 - 2\lambda}\sqrt{\frac{\lambda - \omega h_3}{\lambda + \omega h_3}} \ln\left|\frac{\tan\frac{\mu}{2} + \sqrt{\frac{\lambda - \omega h_3}{\lambda + \omega h_3}}}{\tan\frac{\mu}{2} - \sqrt{\frac{\lambda - \omega h_3}{\lambda + \omega h_3}}}\right| + h_4, -\frac{\lambda}{\omega} > h_3 > 0 > \frac{\lambda}{\omega} \tag{21}$$

where h_4 is an undetermined parameter.

Along with Eq. (14), the minimal mean kinetic energy pitch curve with specified rotation time T can be obtained

$$r_T(\eta) = \begin{cases} r_T(\mu) = \sqrt{\frac{2T}{m\omega^2}(\omega h_3 \sec\mu - \lambda)} \\ \eta(\mu) = \frac{\omega h_3}{\omega h_3 - \lambda}\sqrt{\frac{\omega h_3 - \lambda}{\omega h_3 + \lambda}} \arctan\left(\sqrt{\frac{\omega h_3 + \lambda}{\omega h_3 - \lambda}} \tan\frac{\mu}{2}\right) + h_4, \mu \in [\mu_a, \mu_b] \\ s.t.\ h_3 > \frac{\lambda}{\omega} > 0 \text{ or } h_3 > -\frac{\lambda}{\omega} > 0 > \frac{\lambda}{\omega} \end{cases} \tag{22}$$

or

$$r_T(\eta) = \begin{cases} r_T(\mu) = \sqrt{\frac{2T}{m\omega^2}(\omega h_3 \sec\mu - \lambda)} \\ \eta(\mu) = \frac{\omega h_3}{2\omega h_3 - 2\lambda}\sqrt{\frac{\lambda - \omega h_3}{\lambda + \omega h_3}} \ln\left|\frac{\tan\frac{\mu}{2} + \sqrt{\frac{\lambda - \omega h_3}{\lambda + \omega h_3}}}{\tan\frac{\mu}{2} - \sqrt{\frac{\lambda - \omega h_3}{\lambda + \omega h_3}}}\right| + h_4, \mu \in [\mu_a, \mu_b] \\ s.t.\ -\frac{\lambda}{\omega} > h_3 > 0 > \frac{\lambda}{\omega} \end{cases} \tag{23}$$

where μ_a and μ_b are undetermined values of parameter μ corresponding to polar angles $\eta=\eta_a$ and $\eta=\eta_b$ of two boundaries for the pitch curve $r_T(\eta)$, parameters h_3, h_4, λ, μ_a and μ_b can be obtained by the following constraints

$$\begin{cases} \eta(\mu_a)=0, r_T(\mu_a)=r_T(0), \eta(\mu_b)=\eta_b, r_T(\mu_b)=r_T(\eta_b) \\ T=\int_0^{\eta_b} \frac{\sqrt{r_T^2(\eta)+r_T'^2(\eta)}}{\omega r_T(\eta)} d\eta \end{cases} \quad (24)$$

In order to realize unidirectional continuous rotary motion, Eqs. (22)-(23) should satisfy the following constraints

$$\eta(\mu_a)=0, \eta(\mu_b)=2\pi, r_T(\mu_a)=r_T(\mu_b) \quad (25)$$

Along with Eqs. (22)-(23), rearranging Eq. (25) to

$$\mu_a=-\mu_b, h_4=\pi, \arctan\left(\sqrt{\frac{\omega h_3+\lambda}{\omega h_3-\lambda}}\tan\frac{\mu_b}{2}\right)=\pi\sqrt{\frac{\omega^2 h_3^2-\lambda^2}{\omega^2 h_3^2}} \quad (26)$$

or

$$\mu_a=-\mu_b, h_4=\pi, \ln\left|\frac{\tan\frac{\mu_b}{2}+\sqrt{\frac{\lambda-\omega h_3}{\lambda+\omega h_3}}}{\tan\frac{\mu_b}{2}-\sqrt{\frac{\lambda-\omega h_3}{\lambda+\omega h_3}}}\right|=2\pi\sqrt{\frac{\lambda^2-\omega^2 h_3^2}{\omega^2 h_3^2}} \quad (27)$$

However, according to the assumption of $r_T'(\eta)=r_T(\eta)\tan\mu$ and $\mu_a=-\mu_b$ of Eqs. (26)-(27), one obtains

$$\begin{aligned} r_T'(\mu_a) &= \sqrt{\frac{2T}{m\omega^2}(\omega h_3 \sec\mu_a-\lambda)}\tan\mu_a \\ &= -\sqrt{\frac{2T}{m\omega^2}(\omega h_3 \sec\mu_b-\lambda)}\tan\mu_b=-r_T'(\mu_b) \end{aligned} \quad (28)$$

Namely, the pitch curve $r_T(\eta)$ satisfied the Eq. (25) possesses cusp at the polar angle $\eta=0$ or $\eta=2\pi$, and the noncircular gear with the pitch curve $r_T(\eta)$ satisfied the Eq. (25) can not realize unidirectional continuous rotary motion. Therefore, in order to ensure the smoothness of the pitch curve $r_T(\eta)$ of Eq. (22) or Eq. (23), the polar angle η_b should satisfy the following condition

$$0<\eta_b<2\pi \quad (29)$$

4. Design examples of conjugate pitch curves with minimal mean kinetic energy

According to the solution process of pitch curve with unspecified rotation time constraint, the unclosed conjugate minimal mean kinetic energy pitch curves with unspecified rotation time are, respectively, depicted in Figs. 2-3, their design parameters and pitch curve equations are listed in Table 1. For convenience, where the mass m of noncircular gear is unit mass (i.e. m=1), and the angular velocity ω is equal to 1. The units of parameters listed in Table 1 are adopted by standard international unit (SIU). Without additional explanation, the following discussion relating to parameter in this paper whose unit refers to SIU.

Referring to Figs. 2~3, $\overrightarrow{o_1a_1}$ and $\overrightarrow{o_1b_1}$ are the positioning vectors of the minimal mean kinetic energy pitch curve $r(\eta)$ (i.e. curve $\widehat{a_1b_1}$) with unspecified rotation time, $\overrightarrow{o_2a_2}$ and $\overrightarrow{o_2b_2}$ are the positioning vectors of external meshing noncircular gear pitch curves $r_e(\eta_e)$ (i.e. curve $\widehat{a_2b_2}$) conjugated with the pitch curve $r(\eta)$.

Table 1

Design parameters and pitch curve equations of Figs. 2-3

Boundary constraints	Undetermined parameters	Pitch curve $r(\eta)$ with unspecified rotation time	External meshing maximum polar angle η_{emax}	Solved center distance a_e	Conjugate external meshing pitch curve $r_e(\eta_e)$
$\begin{cases}(0,1)\\(\frac{\pi}{6},\frac{\sqrt{2}}{2})\end{cases}$	$\begin{cases}h_1=\frac{1}{2}\\h_2=\frac{\pi}{3}\end{cases}$	$\begin{cases}r(\eta)=\sqrt{\frac{1/2}{\cos(2\eta-\frac{\pi}{3})}}\\\eta\in[0,\frac{\pi}{6}]\end{cases}$ (Fig. 2(a))	$\eta_{emax}=\frac{\pi}{6}=\int_0^{\frac{\pi}{6}}\frac{\sqrt{\frac{1/2}{\cos(2\eta-\pi/3)}}}{\alpha_e-\sqrt{\frac{1/2}{\cos(2\eta-\pi/3)}}}d\eta$	1.5958	$\begin{cases}r_e(\eta_e)=1.5958-r(\eta)\\\eta_e=\int_0^{\frac{\pi}{6}}\frac{r(\eta)}{1.5958-r(\eta)}d\eta\end{cases}$ $\eta_e\in[0,\frac{\pi}{6}]$ (Fig. 2(b))
$\begin{cases}(0,1)\\(\frac{\pi}{4},1)\end{cases}$	$\begin{cases}h_1=\frac{\sqrt{2}}{2}\\h_2=\frac{\pi}{4}\end{cases}$	$\begin{cases}r(\eta)=\sqrt{\frac{\sqrt{2}/2}{\cos(2\eta-\frac{\pi}{4})}}\\\eta\in[0,\frac{\pi}{4}]\end{cases}$ (Fig. 3(a))	$\eta_{emax}=\frac{\pi}{4}=\int_0^{\frac{\pi}{4}}\frac{\sqrt{\frac{\sqrt{2}/2}{\cos(2\eta-\pi/4)}}}{\alpha_e-\sqrt{\frac{\sqrt{2}/2}{\cos(2\eta-\pi/4)}}}d\eta$	1.7841	$\begin{cases}r_e(\eta_e)=1.7841-r(\eta)\\\eta_e=\int_0^{\frac{\pi}{4}}\frac{r(\eta)}{1.7841-r(\eta)}d\eta\end{cases}$ $\eta_e\in[0,\frac{\pi}{4}]$ (Fig. 3(b))

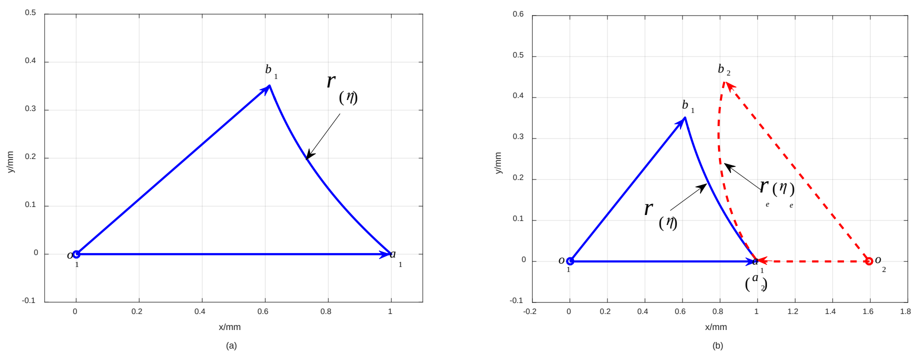

Fig. 2. Conjugate minimal mean kinetic energy pitch curves with unequal boundaries

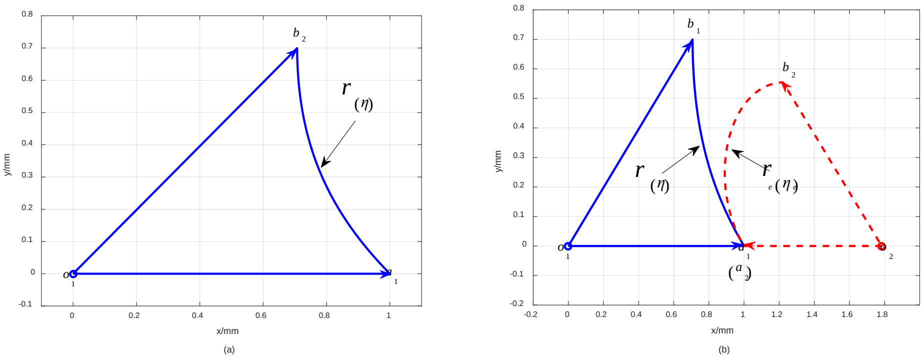

Fig. 3. Conjugate minimal mean kinetic energy pitch curves with equal boundaries

According to the solution process of pitch curve with specified rotation time constraint, the unclosed conjugate minimal mean kinetic energy pitch curves with specified rotation time are depicted in Fig. 4, their design parameters and pitch curve equations are listed in Table 2. Similarly, referring to Fig. 4, vector group $(\overrightarrow{o_1a_1}, \overrightarrow{o_1b_1})$ and $(\overrightarrow{o_2a_2}, \overrightarrow{o_2b_2})$ are, respectively, the positioning vectors of the pitch curve $r_T(\eta)$ (i.e. curve $\overset{\frown}{a_1b_1}$) with specified rotation time constraint and its conjugated external meshing noncircular gear pitch curve $r_e(\eta_e)$ (i.e. curve $\overset{\frown}{a_2b_2}$).

Table 2

Design parameters and pitch curve equations of Fig. 4

Boundary and specified rotation time constraints	Undetermined parameters	Pitch curve equation $r(\eta)$ with specified rotation time	External meshing maximum polar angle $\eta_{e\max}$	Solved center distance a_e	Conjugate external meshing pitch curve $r_e(\eta_e)$
$\begin{cases}(0,1)\\(\frac{5\pi}{3},1)\\T=5.5\end{cases}$	$\begin{cases}h_3=0.2900\\h_4=5\pi/6\\\mu_a=-0.6275\\\mu_b=0.6275\\\lambda=0.2673\end{cases}$	$\begin{cases}r_T(\mu)=\\\sqrt{3.1896\sec\mu-2.9403}\\\eta(\mu)=\\2.5801\arctan(4.9585\tan\frac{\mu}{2})+\frac{5\pi}{6}\end{cases}$ $\mu\in[-0.6275,0.6275]$ (Fig. 4(a))	$\eta_{e\max}=\frac{5\pi}{3}=\int_0^{\frac{5\pi}{3}}\frac{r_T(\eta)}{\alpha_e-r_T(\eta)}d\eta$	1.1266	$\begin{cases}r_e(\eta_e)=\\1.1266-r_T(\eta)\\\eta_e=\\\int_0^{\frac{5\pi}{3}}\frac{r_T(\eta)}{1.1266-r_T(\eta)}d\eta\end{cases}$ $\eta_e\in[0,\frac{5\pi}{3}]$ (Fig. 4(b))

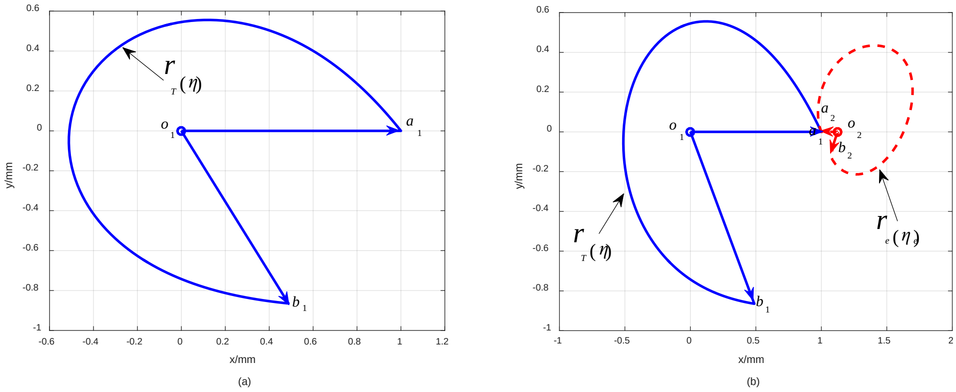

Fig. 4. Conjugate minimal mean kinetic energy pitch curves with specified rotation time

5. Conclusions

The mathematical models for the generation of pitch curve with minimal mean kinetic energy characteristic for noncircular gear were proposed, which contain unspecified and specified rotation time constraints. The transmission angle ranges for the pitch curves with unspecified and specified rotation time constraint were given by the analysis of corresponding constraints and undetermined parameters. Results of this research should be helpful in the application of this type of gearing due to the noncircular gears with minimal mean kinetic energy pitch curves possess good stability in variable transmission.

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