U.P.B. Sci. Bull., Series A, Vol. 81, Iss. 4, 2019 ISSN 1223-7027

INEQUALITIES ON UNIVALENT FUNCTIONS ASSOCIATED WITH BESSEL GENERALIZED FUNCTION

Shahram NAJAFZADEH[1]

Making use of a convolution structure and familiar Bessel function, we introduce two new subclasses of univalent analytic functions. The result presented in this paper includes some inequalities and integral representation for an operator considered on these functions.

Keywords: Univalent function, Convolution, Bessel function.

MSC2010: 30C45, 30C50.

1. Introduction

The particular solution of the second-order differential equation:

$$z^2w''(z)+bzw'(z)+\Big[cz^2-u^2+(1-b)u\Big]w(z)=0, \qquad (u,b,c\in\mathbb{C}), \tag{1}$$

is the generalized Bessel function of order u which is defined as follows:

$$w_{u,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-c)^n}{n!\Gamma(u+n+\frac{b+1}{2})}\left[\frac{z}{2}\right]^{2n+u}, \qquad (z\in\mathbb{C}), \tag{2}$$

where $\Gamma(x)$ is the Euler gamma function.

The function $\varphi_{u,b,c}$ in terms of $w_{u,b,c}(z)$ is defined by:

$$\varphi_{u,b,c}(z)=2^u\Gamma\Big(u+\frac{b+1}{2}\Big)z^{1-\frac{u}{2}}w_{u,b,c}(\sqrt{z}). \tag{3}$$

By applying the well known Pochhammer symbol (or the shifted factorial) $(x)_\mu$ given by:

$$(x)_\mu=\frac{\Gamma(x+\mu)}{\Gamma(x)}=\begin{cases}1 & , \quad \mu=0,\\ x(x+1)\cdots(x+\mu-1) & , \quad \mu\in\mathbb{N},\end{cases} \tag{4}$$

we conclude the following series representation for $\varphi_{u,b,c}(z)$ given by (3):

$$\varphi_{u,b,c}(z)=z+\sum_{n=1}^{\infty}\frac{(-c)^n}{4^n(u+\frac{b+1}{2})_n n!}z^{n+1}, \tag{5}$$

where $u+\frac{b+1}{2}\notin\mathbb{Z}_0^-=\{0,-1,-2,\dots\}$.

Let $\mathcal{A}$ be the class of all analytic functions in the open unit disk $\mathbb{D}=\{z:|z|<1\}$ as follows:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{6}$$

and S be the subclass of $\mathcal{A}$ consisting of univalent functions and normalized with $f(0)=f'(0)-1=0$. For $f(z)\in\mathcal{A}$ given by (6) and $g(z)=z+\sum\limits_{n=1}^{\infty}b_nz^n\in\mathcal{A}$, the Hadamard product (or convolution) of

[1]Department of Mathematics, Payame Noor University, Tehran, Iran, E-mail: najafzadeh1234@yahoo.ie (Corresponding author)

f and g denoted by $(f*g)(z)$ is defined by:

$$(f*g)(z) = z + \sum_{n=1}^{\infty} a_n b_n z^n = (g*f)(z), \qquad (z \in \mathbb{D}). \tag{7}$$

Now, we consider the operator $\mathcal{B}_{u,b}^c : S \to S$ which is defined by:

$$\mathcal{B}_{u,b}^c f(z) = \varphi_{u,b,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1}(u+\frac{b+1}{2})_{n-1}(n-1)!} a_n z^n. \tag{8}$$

For more details see [1, 3, 2, 4] and [8].

A function $f(z) \in \mathcal{A}$ is said to be in the class $X_{u,b}^c$ if it satisfies the inequality:

$$\left| \frac{\mathcal{B}_{u,b}^c f(z)}{z} - 1 \right| < 1. \tag{9}$$

Also, $f(z) \in \mathcal{A}$ is said to be a member of the class $Y_{u,b}^c$ if it satisfies the condition:

$$\left| \frac{z(\mathcal{B}_{u,b}^c f(z))'}{\mathcal{B}_{u,b}^c f(z)} - 1 \right| < 1. \tag{10}$$

To prove the main theorem, we need the following lemma due to Jack [5], see also [7].

Lemma 1.1. *Let $Q(z)$ be non-constant in $\mathbb{D}$ and $Q(0)=0$. If $|Q|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then $z_0 Q'(z_0) = qQ(z_0)$, where $q \geqslant 1$ is a real number.*

For more details about univalent functions, one may refer to [6] and [9].

2. **Main Result**

In this section we obtain some important inequalities and verify integral representation for the operator $\mathcal{B}_{u,b}^c$.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies the inequality:*

$$\mathrm{Re}\left\{ \frac{z(\mathcal{B}_{u,b}^c f(z))'}{\mathcal{B}_{u,b}^c f(z)} - 1 \right\} < \frac{1}{2}, \tag{11}$$

then $f(z) \in X_{u,b}^c$.

Proof. Let $f(z) \in \mathcal{A}$, we define the function $Q(z)$ by:

$$\frac{\mathcal{B}_{u,b}^c f(z)}{z} = 1 + Q(z), \qquad (z \in \mathbb{D}). \tag{12}$$

With a simple calculation we get $Q(0) = 0$ (in $\mathbb{D}$)).

From (12) we conclude:

$$\mathcal{B}_{u,b}^c f(z) = z + zQ(z),$$

or

$$(\mathcal{B}_{u,b}^c f(z))' = 1 + Q(z) + zQ'(z).$$

Thus

$$(\mathcal{B}_{u,b}^c f(z))' = \frac{\mathcal{B}_{u,b}^c f(z)}{z} + zQ'(z),$$

so

$$\frac{z(\mathcal{B}_{u,b}^c f(z))'}{\mathcal{B}_{u,b}^c f(z)} - 1 = \frac{zQ'(z)}{1+Q(z)}. \tag{13}$$

Now, let for $z_0 \in \mathbb{D}$, $\max\limits_{|z|\leqslant|z_0|} |Q(z)| = |Q(z_0)| = 1$, then by using Jack's lemma and putting $Q(z_0) = e^{i\theta} \neq -1$ in (13), we have:

$$\begin{aligned}\operatorname{Re}\left\{\frac{z(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)} - 1\right\} &= \operatorname{Re}\left\{\frac{z_0Q'(z_0)}{1+Q(z_0)}\right\}\\ &= \operatorname{Re}\left\{\frac{qQ(z_0)}{1+Q(z_0)}\right\}\\ &= \operatorname{Re}\left\{\frac{qe^{i\theta}}{1+e^{i\theta}}\right\} = \frac{q}{2} \geqslant \frac{1}{2},\end{aligned}$$

which is a contradiction with (11). Thus we have $|Q(z)| < 1$ for all $z \in \mathbb{D}$. So from (12) we get:

$$\left|\frac{\mathcal{B}^c_{u,b}f(z)}{z} - 1\right| = |Q(z)| < 1,$$

and this gives the result. □

By considering c, u and b in (8), such that:

$$\frac{(-c)^{n-1}}{4^{n-1}(u+\frac{b+1}{2})_{n-1}(n-1)!} = 1, \tag{14}$$

we have the following corollary.

Corollary 2.1. *If $f(z) \in \mathcal{A}$ satisfies:*

$$\operatorname{Re}\left\{\frac{zf'}{f} - 1\right\} < \frac{1}{2}, \qquad (z \in \mathbb{D}),$$

and (14) *holds, then* $|\frac{f(z)}{z} - 1| < 1$.

Theorem 2.2. *If $f(z) \in \mathcal{A}$ satisfies:*

$$\operatorname{Re}\left\{1 + z\left[\frac{(\mathcal{B}^c_{u,b}f(z))''}{(\mathcal{B}^c_{u,b}f(z))'} - \frac{(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)}\right]\right\} < \frac{1}{2}, \qquad (z \in \mathbb{D}), \tag{15}$$

then $f(z) \in Y^c_{u,b}$.

Proof. Let $f(z) \in \mathcal{A}$, we define the function $Q(z)$ by:

$$\frac{z(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)} = 1 + Q(z), \qquad (z \in \mathbb{D}). \tag{16}$$

It is easy to verify that $Q(z)$ is analytic in $\mathbb{D}$ and $Q(0) = 0$. By (16), we have:

$$z(\mathcal{B}^c_{u,b}f(z))' = \mathcal{B}^c_{u,b}f(z) + [\mathcal{B}^c_{u,b}f(z)]Q(z),$$

or

$$(\mathcal{B}^c_{u,b}f(z))' + z(\mathcal{B}^c_{u,b}f(z))'' = (\mathcal{B}^c_{u,b}f(z))' + (\mathcal{B}^c_{u,b}f(z))'Q(z) + Q'(z)(\mathcal{B}^c_{u,b}f(z)),$$

or

$$1 + \frac{z(\mathcal{B}^c_{u,b}f(z))''}{(\mathcal{B}^c_{u,b}f(z))'} = 1 + Q(z) + Q'(z)\frac{\mathcal{B}^c_{u,b}f(z)}{(\mathcal{B}^c_{u,b}f(z))'}.$$

By using (16) we get:

$$1 + \frac{z(\mathcal{B}^c_{u,b}f(z))''}{(\mathcal{B}^c_{u,b}f(z))'} = 1 + Q(z) + \frac{zQ'(z)}{1+Q(z)}.$$

Now, let for a point $z_0 \in \mathbb{D}$, $\max\limits_{|z|\leqslant|z_0|} |Q(z)| = |Q(z_0)| = 1$. By Jack's lemma and putting $Q(z_0) = e^{i\theta}$ we have:

$$\mathrm{Re}\left\{1+z\Big[\frac{(\mathcal{B}^c_{u,b}f(z))''}{(\mathcal{B}^c_{u,b}f(z))'} - \frac{(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)}\Big]\right\} = \mathrm{Re}\left\{\frac{z_0Q'(z_0)}{1+Q(z_0)}\right\}$$
$$= \mathrm{Re}\left\{\frac{qQ(z_0)}{1+Q(z_0)}\right\} = q\,\mathrm{Re}\left\{\frac{e^{i\theta}}{1+e^{i\theta}}\right\} = \frac{q}{2} \geqslant \frac{1}{2},$$

which is a contradiction with (15). Thus, for all $z \in \mathbb{D}$, $|Q(z)| < 1$ and so from (16) we have $\left|\frac{z(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)} - 1\right| < 1$, thus the proof is complete. □

Corollary 2.2. *If* $f(z) \in \mathcal{A}$ *satisfies* $\mathrm{Re}\left\{1+\left(\frac{f''}{f'}-\frac{f'}{f}\right)\right\} < \frac{1}{2}$, $(z \in \mathbb{D})$, *and* (14) *holds, then:*

$$\left|\frac{zf'}{f} - 1\right| < 1.$$

In the last theorem, we find integral representation for $\mathcal{B}^c_{u,b}f(z)$, *where* $f(z) \in Y^c_{u,b}$.

Theorem 2.3. *Let* $f(z) \in Y^c_{u,b}$. *Then* $\mathcal{B}^c_{u,b}f(z) = \exp\left[\int_0^z \frac{1+W(t)}{t}dt\right]$, $(|W(z)| < 1)$.

Proof. For $f(z) \in Y^c_{u,b}$, by (10) we have $\left|\frac{z(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)} - 1\right| < 1$. Thus $\frac{z(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)} - 1 = W(z)$, where $|W(z)| < 1$, so we obtain: $\frac{(\mathcal{B}^c_{u,b}f(z))'}{\mathcal{B}^c_{u,b}f(z)} = \frac{1+W(z)}{z}$, which is equivalent by integration, to $\log\left(\mathcal{B}^c_{u,b}f(z)\right) = \int_0^z \frac{1+W(t)}{t}dt$, and this gives the required result. □

3. Conclusion

By applying the Bessel function, we defined two new subclasses of analytic functions. We introduced various inequalities relate to these subclasses. The study will lead possibility to define new univalent functions by applying q-derivative and q-analogue of well-known operators or even conclude some geometric structures such as starlike, convex and close-to-convex functions. This idea could be topics for our next research.

REFERENCES

[1] *K. Ahmad, S. Mustafa, M. U. Din, M. Raza, M. Arif, et al.* On geometric properties of normalized hyper-bessel functions. *Mathematics*, **7**(4):316, 2019.

[2] *Á. Baricz.* Geometric properties of generalized Bessel functions. *Publicationes Mathematicae*, **73**(1-2):155–178, 2008.

[3] *Á. Baricz. Generalized Bessel functions of the first kind*. Springer, 2010.

[4] *E. Deniz.* Convexity of integral operators involving generalized Bessel functions. *Integral Transforms and Special Functions*, **24**(3):201–216, 2013.

[5] *I. Jack.* Functions starlike and convex of order α. *Journal of the London Mathematical Society*, **2**(3):469–474, 1971.

[6] *H. Kadakal.* New inequalities for strongly r-convex functions. *Journal of Function Spaces*, 2019, (DOI 10.1155/2019/1219237).

[7] *S. S. Miller and P. T. Mocanu.* Second order differential inequalities in the complex plane. *Journal of Mathematical Analysis and Applications*, **65**(2):289–305, 1978.

[8] *C. Ramachandran, K. Dhanalakshmi, and L. Vanitha.* Certain aspects of univalent function with negative coefficients defined by Bessel function. *Brazilian Archives of Biology and Technology*, **59**(SPE2), 2016.

[9] *S. Sivasubramanian, M. Govindaraj, and K. Piejko.* On certain class of univalent functions with conic domains involving Sokol-Nunokawa class. *University Politehnica Of Bucharest Scientific Bulletin-Series A-Applied Mathematics And Physics*, **80**(1):123–134, 2018.