SCALARIZATION AND OPTIMALITY CONDITIONS FOR GENERALIZED VECTOR EQUILIBRIUM PROBLEMS UNDER IMPROVEMENT SETS IN REAL LINEAR SPACES

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In this paper, we investigate linear scalarization results for weakly efficient solutions and Benson proper efficient solutions for generalized vector equilibrium problems under improvement sets in real linear spaces. Meanwhile, using standard separation theorem for convex sets, we establish optimality conditions for Henig proper efficient solutions for constrained generalized vector equilibrium problems. Our results extend several results of some literature.

Keywords: Generalized vector equilibrium problems; Improvement sets; Proper efficient solutions; Scalarization; Optimality conditions

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1. Introduction

In recent years, many researchers paid attention to unify different kinds of solution notions of vector optimization problems, such as the efficiency, weak efficiency, proper efficiency and ε-efficiency. Chicco et al. [1] proposed a new concept of improvement set E and defined E-optimality in finite dimensional spaces. E-optimality unifies some known concepts of exact and approximate solutions of vector optimization problems. Gutiérrez et al. [2] extended the notions of improvement set and E-optimal solution to a general topological linear space. Many follow-on works about the aspect one can look up [3, 4, 5, 6, 7, 8, 9]. Chen et al. [10] introduced a new vector equilibrium problem by virtue of an improvement set E, scalarization characterizations have been established and some stability conclusions of parametric vector equilibrium problems under improvement sets were obtained. Solution concepts, approximate solution concepts and their characterizations of vector optimization problems have been generalized to real linear spaces not equipped with any topology in [11, 12, 13, 14, 15, 16, 17]. Gutiérrez et al. [18] obtained some

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characterizations of E-optimal solutions, weak E-optimal solutions and proper E-optimal solutions of constrained vector optimization problems in terms of linear scalarizations and Lagrange multiplier rules in real linear spaces.

Vector equilibrium problems (shortly, VEP) provided a unified model of many problems (see [19, 20, 21]). An important topic of the optimality conditions for VEP with constraints have been investigated in [22, 23, 24, 25, 26]. Through nolinear scalarization, Gutiérrez et al.[27] characterized weak efficient solutions for a class of VEP in real linear spaces. Gutiérrez et al.[28] obtained existence result of weak efficient solutions for VEP by means of algebraic version of the Gerstewitz's functional in real linear spaces.

As far as we know, there are few papers consider generalized VEP under improvement sets in real linear spaces. Inspired by the preceding work [10, 18, 22, 23, 24, 25, 26, 27], firstly, we give linear scalarization results of some efficiency for unified generalized vector equilibrium problems (UGVEP) with generalized convexity assumptions in real linear spaces, these results include the corresponding ones in [10, 18] as particular cases. Secondly, we establish optimality conditions of Henig proper efficient solutions for unified generalized vector equilibrium problems with constrains (UGVEPC) in real linear spaces.

The remaining of this article is organized as follows. Section 2 provides some basic definitions we need in the paper. In Section 3, we give linear scalarization results for weakly efficient solutions and Benson proper efficient solutions for (UGVEP). Then, by using standard separation theorem for convex sets, we provide optimality conditions for Henig proper efficient solutions for (UGVEPC) in Section 4.

2. Preliminaries

Throughout the paper, let X, Y and Z be real linear spaces, C and D be nontrivial pointed convex cones in Y and Z, respectively. Let K be a nonempty subset in Y, 0_Y denotes the zero element of Y. We denote by

$$\operatorname{cone}(K) := \{\lambda k | k \in K, \lambda \geq 0\},$$

$$\operatorname{span}(K) := \{\sum_{i=1}^{n} \lambda_i k_i | \forall i \in \{1, \ldots, n\}, k_i \in K,\ \lambda_i \in \mathbb{R}\},$$

$$\mathrm{L}(K) := \operatorname{span}(K - K),$$

the generated cone, linear hull and linear subspace of K, respectively. K is called a cone if $\lambda K \subseteq K$ for any $\lambda \geq 0$. A cone K is said to be pointed if $K \cap (-K) = \{0_Y\}$. A cone $K \subseteq Y$ is said to be nontrivial if $\{0_Y\} \neq K \neq Y$. Let Y^* and Z^* stand for the algebraic dual spaces of Y and Z, respectively. The algebraic dual cone C^+ and strictly algebraic dual cone C^{+i} of C are defined as

$$C^+ := \{y^* \in Y^* \mid y^*(y) \geq 0,\ \forall y \in C\},$$

and

$$C^{+i} := \{y^* \in Y^* \mid y^*(y) > 0,\ \forall y \in C \setminus \{0_Y\}\}.$$

Definition 2.1 ([30]). Let K be a nonempty subset in Y.
(i)The algebraic interior of K is the set

$$\operatorname{cor}(K) := \{k \in K \mid \forall v \in Y, \exists \lambda' > 0, \forall \lambda \in [0, \lambda'], k + \lambda v \in K\}.$$

(ii)The relative algebraic interior of K is the set

$$\operatorname{icr}(K) := \{k \in K \mid \forall v \in L(K), \exists \lambda' > 0, \forall \lambda \in [0, \lambda'], k + \lambda v \in K\}.$$

We say that K is solid (relatively solid) if $\operatorname{cor}(K) \neq \emptyset$ ($\operatorname{icr}(K) \neq \emptyset$). Clearly, $\operatorname{cor}(K) \subseteq \operatorname{icr}(K)$.

Definition 2.2 ([11]). Let K be a nonempty subset in Y. The vector closure of K is the set

$$\operatorname{vcl}(K) := \{k \in Y \mid \exists v \in Y, \forall \lambda' > 0, \exists \lambda \in [0, \lambda'], k + \lambda v \in K\}.$$

It is clear that $k \in \operatorname{vcl}(K)$ if and only if there exist $v \in Y$ and a sequence $\lambda_n \to 0^+$ such that $k + \lambda_n v \in K$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

The set K is called vectorially closed (v-closed) if $K = \operatorname{vcl}(K)$.

Lemma 2.1 ([11]). Let $K \subset Y$ be convex. Then $\operatorname{cor}(K)$ and $\operatorname{vcl}(K)$ are convex. Moreover, if K is relatively solid, then $\operatorname{vcl}(K)$ is v-closed and $\operatorname{icr}(K) = \operatorname{icr}(\operatorname{vcl}(K))$.

Lemma 2.2 ([11]). Let $K \subset Y$ and $C \subset Y$ be a nontrivial convex cone. Then,
(i)$\operatorname{vcl}(\operatorname{cone}(K) + C) = \operatorname{vcl}(\operatorname{cone}(K + C))$.
(ii) If $\operatorname{icr}(C) \neq \emptyset$, then $\operatorname{vcl}(K + C) = \operatorname{vcl}(K + \operatorname{icr}(C))$.
(iii) If C is solid, then $\operatorname{cor}(\mathrm{K}+\mathrm{C}) = \operatorname{cor}(\operatorname{vcl}(\mathrm{K}+\mathrm{C})) = \mathrm{K} + \operatorname{cor}(\mathrm{C}) = \operatorname{cor}(\mathrm{K} + \operatorname{cor}(\mathrm{C}))$.

Lemma 2.3 ([18]). Let $\emptyset \neq A, B \subset Y$ and suppose that B is solid and convex. Then, $A \cap \operatorname{cor}(B) = \emptyset$ if and only if $\operatorname{vcl}(A) \cap \operatorname{cor}(B) = \emptyset$.

Definition 2.3 ([2]). A nonempty set $E \subset Y$ is said to be an improvement set with respect to C if $0_Y \notin E$ and $E + C = E$. The family of the improvement sets in Y is denoted by $\Im_Y$.

Definition 2.4 ([18]). Let $\emptyset \neq E \subset Y$. The mapping $F : X \rightrightarrows Y$ is said to be v-nearly E-subconvexlike on a nonempty set $A \subset X$ if $\operatorname{vcl}(\operatorname{cone}(F(A)+E))$ is a convex set.

Definition 2.5 ([18]). Let $\emptyset \neq E \subset Y$ and $\operatorname{icr}(C) \neq \emptyset$. The mapping $F : X \rightrightarrows Y$ is said to be E-subconvexlike (respectively, generalized E-subconvexlike) on a nonempty set $A \subset X$ (with respect to C) if $F(A) + E + \operatorname{icr}(C)$ (respectively, $\operatorname{cone}(F(A) + E) + \operatorname{icr}(C)$) is a convex set.

Definition 2.6 ([18]). Let $\emptyset \neq E \subset Y$ and $\operatorname{icr}(C) \neq \emptyset$. The mapping $F : X \rightrightarrows Y$ is said to be relatively solid E-subconvexlike (respectively, relatively solid generalized E-subconvexlike) on a nonempty set $A \subset X$(with respect to C) if F is E-subconvexlike (respectively, generalized E-subconvexlike) on A (with respect to C) and $F(A)+E+\text{icr}(C)$ (respectively, $\text{cone}(F(A)+E)+\text{icr}(C)$) is relatively solid.

Lemma 2.4 ([18])**.** Let M, K be two v-closed convex cones in Y such that M is relatively solid and K^+ is solid. If $M \cap K = \{0_Y\}$, then there exists a linear functional $l \in Y^* \setminus \{0_{Y^*}\}$ such that $\forall k \in K, m \in M,\ l(k) \geq 0 \geq l(m)$ and furthermore, $\forall k \in K \setminus \{0_Y\},\ l(k) > 0,\ i.e.,\ l \in K^{+i}$.

Lemma 2.5. Let $E \in \Im_Y$ and $C \subset Y$ be a solid convex cone. Then $\text{cor}(E) = E + \text{cor}(C)$.

Proof. It follows directly from that $E \in \Im_Y$ and Lemma 2.2 (iii). □

Lemma 2.6 ([29])**.** Let S and T be nonempty convex subsets of a real linear space X with $\text{cor}(S) \neq \emptyset$. Then $\text{cor}(S) \cap T = \emptyset$ if and only if there are a linear functional $l \in X^* \setminus \{0_{X^*}\}$ and a real number α with $l(s) \leq \alpha \leq l(t)$ for all $s \in S$ and all $t \in T$, and $l(s) < \alpha$ for all $s \in \text{cor}(S)$.

From now on, we assume that $A \subset X$ is nonempty, $F : A \times A \rightrightarrows Y$ is a set-valued mapping and $E \in \Im_Y$.

We consider the following unified generalized vector equilibrium problem (for short, (UGVEP)) of finding $\bar{x} \in A$ such that

$$\text{(UGVEP)}\ F(\bar{x}, x) \cap (-E) = \emptyset,\ \forall x \in A.$$

For $x \in A$, we define

$$F(x, A) := \bigcup_{y \in A} F(x, y).$$

Definition 2.7. An element $\bar{x} \in A$ is a weakly efficient solution of the (UGVEP) iff,

$$F(\bar{x}, x) \cap (-\text{cor}(E)) = \emptyset,\ \forall x \in A.$$

We denote by WE(F,A;E) the set of weakly efficient solutions of the (UGVEP).

Definition 2.8. An element $\bar{x} \in A$ is an E-Benson proper efficient solution of the (UGVEP) iff,

$$\text{vcl}(\text{cone}(F(\bar{x}, A) + E)) \cap (-C) = \{0_Y\}.$$

We denote by BE(F,A,C;E) the set of weakly efficient solutions of the (UGVEP).

3. **Linear scalarization**

In this section, we will discuss linear scalarization results of the weak efficiency and the Benson proper efficiency for (UGVEP), respectively. For $\varphi \in Y^*$, we denote $\sigma_E(\varphi) := \inf_{e \in E} \varphi(e)$. It is easy to check that if $\varphi \in E^+$, then $\sigma_E(\varphi) \geq 0$.

For each $\varphi \in E^+ \setminus \{0_{Y^*}\}$, let $S_\varphi(F,A;E)$ denote the set of $\sigma_E(\varphi)$-efficient solutions to (UGVEP), i.e.,

$$S_\varphi(F,A;E) := \{x \in A \mid \inf_{z\in F(x,y)} \varphi(z) \geq -\sigma_E(\varphi),\ \forall y \in A\}.$$

Lemma 3.1 ([18])**.** Let $K \subset Y$ be solid, $\varphi \in Y^*$, $k \in \mathrm{cor}(K)$, if $\min_{y\in K} \varphi(y) = \varphi(k)$, then $\varphi = 0_{Y^*}$.

Theorem 3.1. Assume that for each $x \in A$, $F(x,\cdot)$ is v-nearly E-subconvexlike on A and C is solid. If $0_Y \in F(x,x)$, $\forall x \in A$. Then

$$WE(F,A;E) = \bigcup_{\varphi\in E^+\setminus\{0_{Y^*}\}} S_\varphi(F,A;E).$$

Proof. Necessity. Let $x_0 \in$ WE(F,A;E), then $x_0 \in A$ and $F(x_0,x)\cap(-\mathrm{cor}(E)) = \emptyset$, $\forall x \in A$. Thus, $F(x_0,A) \cap (-\mathrm{cor}(E)) = \emptyset$. It follows from Lemma 2.5 that

$$(F(x_0,A) + E) \cap (-\mathrm{cor}(C)) = \emptyset. \tag{1}$$

We assert that

$$\mathrm{cone}(F(x_0,A) + E) \cap (-\mathrm{cor}(C)) = \emptyset. \tag{2}$$

Otherwise, there exist $\alpha > 0$, $x \in A$, $z \in F(x_0,x)$, $e \in E$, $c \in \mathrm{cor}(C)$ such that

$$\alpha(z + e) = -c,$$

that is,

$$z + e = -\frac{c}{\alpha} \in -\mathrm{cor}(C).$$

Which contradicts (1). Therefore, (2) holds. From Lemma 2.3, we note that

$$\mathrm{vcl}(\mathrm{cone}(F(x_0,A) + E)) \cap (-\mathrm{cor}(C)) = \emptyset. \tag{3}$$

It follows from $F(x,\cdot)$ is v-nearly E-subconvexlike on A that vcl(cone($F(x_0,A)+E$)) is a convex set. By Lemma 2.6, there exist $\bar\varphi \in Y^* \setminus \{0_{Y^*}\}$ and a real number β such that

$$\bar\varphi(z + e) \geq \beta \geq -\bar\varphi(c),\ \forall x \in A,\ z \in F(x_0,x),\ c \in C,\ e \in E. \tag{4}$$

Taking $x = x_0$, $z = 0_Y \in F(x_0,x_0)$ and $c = 0_Y$ in (4), this yields $\bar\varphi(e) \geq 0$ for all $e \in E$. Hence $\bar\varphi \in E^+ \setminus \{0_{Y^*}\}$.

Letting $c = 0_Y$ in (4), it results $\bar\varphi(e) \geq -\bar\varphi(z)$, $\forall x \in A$, $z \in F(x_0,x)$, $e \in E$. Then,

$$\sigma_E(\bar\varphi) \geq -\bar\varphi(z),\ \forall x \in A,\ \forall z \in F(x_0,x).$$

In consequence, $\bar\varphi(z) \geq -\sigma_E(\bar\varphi), \forall x \in A$, $z \in F(x_0,x)$. Thus,

$$\inf_{z\in F(x_0,x)} \bar\varphi(z) \geq -\sigma_E(\bar\varphi), \forall x \in A.$$

Therefore, $x_0 \in S_{\bar\varphi}(F,A;E) \subset \bigcup_{\varphi\in E^+\setminus\{0_{Y^*}\}} S_\varphi(F,A;E)$.

Sufficiency. Let $x_0 \in \bigcup_{\varphi \in E^+ \setminus \{0_{Y^*}\}} S_\varphi(F, A; E)$, then there exists $\hat{\varphi} \in E^+ \setminus \{0_{Y^*}\}$ such that $x_0 \in S_{\hat{\varphi}}(F, A; E)$. Thus $x_0 \in A$ and

$$\inf_{z \in F(x_0, x)} \hat{\varphi}(z) \geq -\sigma_E(\hat{\varphi}), \forall x \in A. \tag{5}$$

Suppose that $x_0 \notin$WE(F,A;E), then there exist $\hat{x} \in A$ and $\hat{e} \in \mathrm{cor}(E)$ such that $-\hat{e} \in F(x_0, \hat{x})$. By applying (5) to $x = \hat{x}$ it results

$$-\hat{\varphi}(\hat{e}) \geq \inf_{z \in F(x_0, \hat{x})} \hat{\varphi}(z) \geq -\sigma_E(\hat{\varphi}). \tag{6}$$

namely, $\sigma_E(\hat{\varphi}) \geq \hat{\varphi}(\hat{e})$, i.e., $\inf_{e \in E} \hat{\varphi}(e) \geq \hat{\varphi}(\hat{e})$ and $\min_{e \in E} \hat{\varphi}(e) = \hat{\varphi}(\hat{e})$. By Lemma 3.1, it means that $\hat{\varphi} = 0_{Y^*}$, a contradiction. Hence, $x_0 \in$WE(F,A;E). □

Remark 3.1. (i) If $F : A \times A \to Y$ is a vector-valued mapping and we choose $F(x, y) = g(y) - g(x)$, where $g : X \to Y$, then Theorem 3.1 reduces to the corresponding result stated in Corollary 4.6 of [18].

(ii)By Proposition 2.9 of [18], when $\mathrm{cor}(C) \neq \emptyset$, the assumption of v-nearly E-subconvexlikeness is weaker than v-closely C-convexlikeness. So, if $F : A \times A \to Y$ is a vector-valued mapping, Theorem 3.1 extends Theorem 3.1 of [10] which was stated in the topological setting. Moreover, if $E = \varepsilon q + C$, $\varepsilon \geq 0$, $q \in \mathrm{cor}(C)$, Theorem 3.1 reduces to Theorem 3.2 of [26] which was done in the framework of topological linear spaces.

Theorem 3.2. Let $E \in \Im_Y$, C be v-closed and C^+ be solid. Assume that for $\forall x \in A,\ F(x, \cdot)$ is relatively solid generalized E-subconvexlike on A. Then

$$BE(F, A, C; E) = \bigcup_{\varphi \in C^{+i}} S_\varphi(F, A; E).$$

Proof. Necessity. Let $x_0 \in$ BE(F,A,C;E), then

$$\mathrm{vcl}(\mathrm{cone}(F(x_0, A) + E)) \cap (-C) = \{0_Y\}. \tag{7}$$

Since $\forall x \in A,\ F(x, \cdot)$ is relatively solid generalized E-subconvexlike on A, then $\mathrm{cone}(F(x_0, A) + E) + \mathrm{icr}(C)$ is a relatively solid, convex set in Y. It follows from Lemma 2.1 that $\mathrm{vcl}(\mathrm{cone}(F(x_0, A) + E) + \mathrm{icr}(C))$ is v-closed, relatively solid and convex. From Lemma 2.2 (i), (ii) and $E \in \Im_Y$ that

$$\mathrm{vcl}(\mathrm{cone}(F(x_0, A)+E))+\mathrm{icr}(C)) = \mathrm{vcl}(\mathrm{cone}(F(x_0, A)+E)+C)) = \mathrm{vcl}(\mathrm{cone}(F(x_0, A)+E)).$$

As C^+ be solid, according to Proposition 2.3 of [12], C is relatively solid. By Lemma 2.4, there exists a linear functional $\bar{\varphi} \in C^{+i}$ such that

$$\bar{\varphi}(y) \geq 0,\ \forall y \in \mathrm{vcl}(\mathrm{cone}(F(x_0, A) + E)).$$

Since $F(x_0, A) + E \subset \mathrm{vcl}(\mathrm{cone}(F(x_0, A) + E))$, we get

$$\bar{\varphi}(z + e) \geq 0,\ \forall y \in A,\ z \in F(x_0, y),\ e \in E. \tag{8}$$

(8) is equivalent to

$$\bar{\varphi}(e) \geq -\bar{\varphi}(z),\ \forall y \in A,\ z \in F(x_0, y),\ e \in E,$$

that is,

$$\sigma_E(\bar{\varphi}) \geq -\bar{\varphi}(z),\ \forall y \in A,\ z \in F(x_0, y),$$

i.e.,

$$\bar{\varphi}(z) \geq -\sigma_E(\bar{\varphi}),\ \forall y \in A,\ z \in F(x_0, y).$$

Thus,

$$\inf_{z \in F(x_0,y)} \bar{\varphi}(z) \geq -\sigma_E(\bar{\varphi}), \forall x \in A.$$

Therefore, $x_0 \in S_{\bar{\varphi}}(F, A; E) \subset \bigcup_{\varphi \in C^{+i}} S_{\varphi}(F, A; E)$.

Sufficiency. Let $x_0 \in \bigcup_{\varphi \in C^{+i}} S_{\varphi}(F, A; E)$, then there exists $\hat{\varphi} \in C^{+i}$ such that $x_0 \in S_{\hat{\varphi}}(F, A; E)$. Thus $x_0 \in A$ and

$$\inf_{z \in F(x_0,y)} \hat{\varphi}(z) \geq -\sigma_E(\hat{\varphi}),\ \forall y \in A. \tag{9}$$

In what follows, we show that $x_0 \in$ BE(F,A,C;E).

Let $c \in \mathrm{vcl}(\mathrm{cone}(F(x_0, A)+E)) \cap (-C)$, then $c \in \mathrm{vcl}(\mathrm{cone}(F(x_0, A)+E))$. Because of $E \in \Im_Y$, then

$$c \in \mathrm{vcl}(\mathrm{cone}(F(x_0, A) + E + C))$$

By Definition 2.2, there exist $v \in Y$ and a sequence $\lambda_n \to 0^+$ such that $c+\lambda_n v \in \mathrm{cone}(F(x_0, A)+E+C)$ for all $n \in \mathbb{N}$. Therefore, there exist sequences $\{\mu_n\} \subset \mathbb{R}_+,\ \{c_n\} \subset C,\ \{e_n\} \subset E$ and $\{y_n\} \subset A$ such that

$$c + \lambda_n v = \mu_n(z_n + e_n + c_n),\ \forall z_n \in F(x_0, y_n).$$

Hence,

$$\hat{\varphi}(c) + \lambda_n(\hat{\varphi}(v)) = \mu_n(\hat{\varphi}(z_n) + \hat{\varphi}(e_n) + \hat{\varphi}(c_n)),\ \forall z_n \in F(x_0, y_n). \tag{10}$$

By (9), $\hat{\varphi}(z_n) \geq -\sigma_E(\hat{\varphi}) \geq -\hat{\varphi}(e_n)$ and $\hat{\varphi}(z_n) + \hat{\varphi}(e_n) \geq 0$. As $\hat{\varphi}(c_n) \geq 0$ for all $n \in \mathbb{N}$, then the right-hand side of (10) is nonnegative and

$$\hat{\varphi}(c) + \lambda_n(\hat{\varphi}(v)) \geq 0. \tag{11}$$

Taking $\lambda_n \to 0^+$ in (11), we have

$$\hat{\varphi}(c) \geq 0.$$

On the other hand, since $c \in -C$ and $\hat{\varphi} \in C^{+i}$, we obtain

$$\hat{\varphi}(c) \leq 0.$$

Thus, $\hat{\varphi}(c) = 0$, $c = 0_Y$ and

$$\mathrm{vcl}(\mathrm{cone}(F(x_0, A) + E)) \cap (-C) = \{0_Y\}.$$

Hence, $x_0 \in$ BE(F,A,C;E). □

4. **Optimality conditions for** (UGVEPC)

Let $A \subset X$, $G : A \rightrightarrows Z$ and $F : A \times A \rightrightarrows Y$ be two set-valued mappings.

We consider the following unified generalized vector equilibrium problem with constraints (for short, (UGVEPC)) of finding $\bar{x} \in S$ such that

$$\text{(UGVEPC)}\ F(\bar{x}, x) \cap (-E) = \emptyset,\ \forall x \in S.$$

The constraint set is defined by $S = \{x \in A \mid G(x) \cap (-D) \neq \emptyset\}$.

Definition 4.1 ([30])**.** Let K be a nonempty subset in Y. K is called balanced iff $\forall k \in K$, $\forall \lambda \in [-1, 1]$, $\lambda k \in K$. K is called absorbent iff $0_Y \in \mathrm{cor}(K)$.

Definition 4.2 ([15])**.** Let B be a nonempty convex subset in Y. B is a base of C iff $C = \mathrm{cone}(B)$ and there exists a balanced, absorbent and convex set V such that $0_Y \notin B + V$ in Y.

Let B be a base of C and write $B^{st} := \{y^* \in Y^* \mid \text{there exists } t > 0 \text{ such that } y^*(b) \geq t,\ \forall b \in B\}$. Let $V \subseteq Y$ be a balanced, absorbent and convex set with $0_Y \notin B + V$. Write $C_V(B) := \mathrm{cone}(B + V)$.

Remark 4.1. We observe that $C_V(B)$ is a nontrivial, pointed and convex cone in Y. Moreover, $0_Y \notin \mathrm{cor}(C_V(B))$ and $C \setminus \{0_Y\} \subseteq \mathrm{cor}(C_V(B))$.

Lemma 4.1. Let B be a base of C and $y^* \in Y^* \backslash \{0_{Y^*}\}$. Then $y^* \in B^{st}$ if and only if there exists a balanced, absorbent and convex set V such that $y^*(v - b) \leq 0,\ \forall v \in V,\ \forall b \in B$.

Proof. Sufficiency. Let there exists a balanced, absorbent and convex set V such that $y^*(v - b) \leq 0,\ \forall v \in V,\ \forall b \in B$, by the absorption of V, there exists $v_0 \in V$ such that $t = y^*(v_0) > 0$. Hence $y^*(b) \geq y^*(v_0) = t > 0,\ \forall b \in B$, which means that $y^* \in B^{st}$.

Necessity. Assume that $y^* \in B^{st}$, let $V = \{v \in Y \mid y^*(v) \leq t\}$, then V is a balanced, absorbent and convex set. For all $y \in V - B$, there exist $v \in V,\ b \in B$ such that $y = v - b,\ y^*(y) = y^*(v - b) = y^*(v) - y^*(b) \leq t - t = 0$. □

Definition 4.3. Let B be a base of C and $E \in \Im_Y$. An element $\bar{x} \in S$ is said to be an E-Henig proper efficient solution of the (UGVEPC) with respect to B iff, there exists a balanced, absorbent and convex set V with $0_Y \notin B + V$ such that

$$\mathrm{cone}(F(\bar{x}, S) + E) \cap (-C_V(B)) = \{0_Y\}.$$

We denote by HE(F,S,B;E) the set of all E-Henig proper efficient solutions of the (UGVEPC).

Theorem 4.1. Assume that the following conditions hold:

(i)B is a base of C and $E \in \Im_Y$.

(ii) $\bar{x} \in S$ and there exists $x_0 \in A$ such that $G(x_0) \cap (-\mathrm{cor}(D)) \neq \emptyset$.

(iii) $H(y) = (F(\bar{x}, y), G(y))$ is v-nearly $E \times D$-subconvexlike on A.

Then $\bar{x} \in$ HE(F,S,B; E) if and only if there exists $\varphi \in B^{st}$ and $\psi \in D^+$ such that

$$\inf_{y \in A}\{\varphi(F(\bar{x}, y)) + \psi(G(y))\} \geq -\sigma_E(\varphi). \tag{12}$$

Proof. Necessity. Let $\bar{x} \in$HE(F,S,B;E), by Definition 4.3, there exists a balanced, absorbent and convex set V with $0_Y \notin B + V$ such that

$$\mathrm{cone}(F(\bar{x}, S) + E) \cap (-C_V(B)) = \{0_Y\}.$$

By Remark 4.1, $\mathrm{cone}(F(\bar{x}, S) + E) \cap (-\mathrm{cor}(C_V(B))) = \emptyset$.

Then,

$$(F(\bar{x}, S) + E) \cap (-\mathrm{cor}(C_V(B))) = \emptyset. \tag{13}$$

Note that $H(y) = (F(\bar{x}, y), G(y)),\ \forall y \in A$. By (13), we get

$$(H(A) + E \times D) \cap (-\mathrm{cor}(C_V(B))) \times (-\mathrm{cor}(D)) = \emptyset. \tag{14}$$

Otherwise, there exists $\bar{y} \in A$ such that

$$(F(\bar{x}, \bar{y}) + E, G(\bar{y}) + D) \cap (-\mathrm{cor}(C_V(B))) \times (-\mathrm{cor}(D)) \neq \emptyset.$$

Then,

$$(F(\bar{x}, \bar{y}) + E) \cap (-\mathrm{cor}(C_V(B))) \neq \emptyset, \tag{15}$$

and

$$(G(\bar{y}) + D) \cap (-\mathrm{cor}(D)) \neq \emptyset. \tag{16}$$

According to (16), $G(\bar{y}) \in -D - \mathrm{cor}(D) = -\mathrm{cor}(D) \subset -D$, that is, $\bar{y} \in S$. By (15), we obtain $(F(\bar{x}, S) + E) \cap (-\mathrm{cor}(C_V(B))) \neq \emptyset$. Which contradicts (13). Hence, (14) holds.

Since $\mathrm{cor}(C_V(B)),\ \mathrm{cor}(D)$ are both algebraic open convex sets, then

$$\mathrm{cone}(H(A) + E \times D) \cap (-\mathrm{cor}(C_V(B))) \times (-\mathrm{cor}(D)) = \emptyset.$$

From Lemma 2.3, it follows that

$$\mathrm{vcl}(\mathrm{cone}(H(A) + E \times D)) \cap (-\mathrm{cor}(C_V(B))) \times (-\mathrm{cor}(D)) = \emptyset.$$

Moreover, by condition (iii), $\mathrm{vcl}(\mathrm{cone}(H(A) + E \times D))$ is a nonempty convex set in $Y \times Z$. By Lemma 2.6, there exists $(\varphi, \psi) \in (Y^* \times Z^*) \setminus \{0_{Y^*} \times 0_{Z^*}\}$ and $\alpha \in \mathbb{R}$ such that

$$(\varphi, \psi), (\mathrm{vcl}(\mathrm{cone}(H(A) + E \times D))) \geq \alpha \geq \varphi(-C_V(B)) + \psi(-D). \tag{17}$$

Since $\mathrm{vcl}(\mathrm{cone}(H(A)+E\times D))$ is a cone, then $\forall z \in \mathrm{vcl}(\mathrm{cone}(H(A)+E\times D))$ and $\lambda > 0$, furthermore, $\lambda z \in \mathrm{vcl}(\mathrm{cone}(H(A)+E\times D))$. By (17), $(\varphi, \psi)(z) \geq \frac{\alpha}{\lambda}$. Letting $\lambda \to \infty$, we obtain $(\varphi, \psi)(z) \geq 0$. Therefore,

$$(\varphi, \psi)(\mathrm{vcl}(\mathrm{cone}(H(A) + E \times D))) \geq 0.$$

Since $H(A) + E \times D \subset \mathrm{vcl}(\mathrm{cone}(H(A) + E \times D))$ and $0_Y \in D$, then,

$$\varphi(z_1 + e) + \psi(z_2) \geq 0,\ \forall y \in A,\ z_1 \in F(\bar{x}, y),\ z_2 \in G(y),\ \forall e \in E,$$

implying,

$$\varphi(e) \geq -(\varphi(z_1) + \psi(z_2)),\ \forall y \in A,\ z_1 \in F(\bar{x}, y),\ z_2 \in G(y),\ \forall e \in E. \quad (18)$$

Thus, we have

$$\sigma_E(\varphi) \geq -(\varphi(z_1) + \psi(z_2)),\ \forall y \in A,\ z_1 \in F(\bar{x}, y),\ z_2 \in G(y),$$

then,

$$\varphi(z_1) + \psi(z_2) \geq -\sigma_E(\varphi),\ \forall y \in A,\ z_1 \in F(\bar{x}, y),\ z_2 \in G(y),$$

i.e.,

$$\inf_{y \in A} \{\varphi(F(\bar{x}, y)) + \psi(G(y))\} \geq -\sigma_E(\varphi).$$

Therefore, (12) holds.

On the other hand, it follows from $(0_Y, 0_Z) \in \text{vcl}(\text{cone}(h(A) + E \times D))$ and (17) that,

$$\varphi(-C_V(B)) + \psi(-D) \leq 0. \quad (19)$$

For $\forall y \in C_V(B)$, $\forall \lambda > 0$, we have $\lambda y \in C_V(B)$, by (19), one obtains

$$\varphi(y) \geq \frac{1}{\lambda}\psi(-d),\ \forall y \in C_V(B),\ \forall \lambda > 0,\ \forall d \in D. \quad (20)$$

Letting $\lambda \to \infty$ in (20), we obtain

$$\varphi(y) \geq 0,\ \forall y \in C_V(B).$$

Then,

$$\varphi(v + b) \geq 0,\ \forall b \in B,\ \forall v \in V,$$

that is,

$$\varphi(b - v) \geq 0,\ \forall b \in B,\ \forall v \in V.$$

By Lemma 4.1, $\varphi \in B^{st}$. A similar proof to $\varphi \in B^{st}$, we can prove that $\in D^+$.

Sufficiency. If $\bar{x} \notin$ HE(F,S,B; E), then there exists a balanced, absorbent and convex set V with $0_Y \notin B + V$ such that

$$\text{cone}(F(\bar{x}, S) + E) \cap (-C_V(B)) \neq \{0_Y\}.$$

Let $0_Y \neq z \in \text{cone}(F(\bar{x}, S) + E) \cap (-C_V(B))$, then $z \in -C_V(B)$, so there exist $\lambda > 0$, $v \in V,\ b \in B$ such that

$$z = -\lambda(-v + b),$$

i.e.,

$$z = \lambda(v - b).$$

By $\varphi \in B^{st}$, then

$$\varphi(z) < 0. \quad (21)$$

Since $0_Y \neq z \in \text{cone}(F(\bar{x}, S) + E)$, then there exists $t > 0$, $\bar{y} \in S$, $\bar{z} \in F(\bar{x}, \bar{y})$, $e \in E$ such that $z = t(\bar{z} + e)$. By (21),

$$t(\varphi(\bar{z} + e)) < 0,$$

then,

$$\varphi(e) < -\varphi(\bar{z}).$$

Since $\sigma_E(\varphi) = \inf_{e \in E} \varphi(e) \leq \varphi(e)$, then,

$$\varphi(\bar{z}) < -\sigma_E(\varphi). \tag{22}$$

Moreover, note that $\bar{y} \in S$ and $\quad \in D^+$, then there exists $\hat{z} \in G(\bar{y}) \cap (-D)$ such that

$$\psi(\hat{z}) \leq 0. \tag{23}$$

By (22) and (23), we have $\varphi(\bar{z}) + \psi(\hat{z}) < -\sigma_E(\varphi)$. Which contradicts (12). Therefore, $\bar{x} \in HE(F, S, B; E)$. □

Remark 4.2. Theorem 4.1 obtains a scalar Lagrange multiplier rule, similar results for vector-valued Lagrange multiplier rule are given in Theorem 5.1 of [14]. Meanwhile, Theorem 4.1 extends Theorem 3.1 in [24] for the exact notion of Henig solution in the setting of locally convex spaces.

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