APPROXIMATE FIXED POINTS OF SOME SET-VALUED CONTRACTIONS

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By using and mixing some idea of some recent papers, we provide some results about approximate fixed point and fixed point results of some set-valued contractions.

Keywords: Approximate fixed point, fixed point, multifunction, set-valued contraction.

1. Introduction

Let (X, d) be a metric space, $C_b(X)$ the set of closed and bounded subsets of X, T a multifunction on X with closed and bounded values and H the Hausdorff metric with respect to d, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for all closed and bounded subsets A and B of X. We say that T has approximate fixed points whenever $\inf_{x\in X} d(x, Tx) = 0$. By using the idea of iterative scheme method which have been used in [1], [4], [9] and [10] and by mixing the method with the main idea of [2], [3] and [5], we provide some results about approximate fixed point and fixed point results of some set-valued contractions. For related results, please see [6, 7, 8, 11].

2. Main results

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let (X, d) be a metric space and $T : X \to C_b(X)$ a multifunction. Suppose that there exists $r \in [0, 1)$ such that $\frac{1}{1+r} d(x, Tx) \le d(x, y)$ implies $H(Tx, Ty) \le rd(x, y)$ for all $x, y \in X$. Assume that $x_0 \in X$, $\{\varepsilon_i\}_{i=0}^{\infty}$ is a sequence of positive numbers with $\Sigma_{i=0}^{\infty}\varepsilon_i < \infty$ and there exists $x_{i+1} \in Tx_i$ such that $d(x_i, x_{i+1}) \le d(x_i, Tx_i) + \varepsilon_i$ for all $i \ge 0$. Then T has approximate fixed points.*

Proof. Since $\frac{1}{1+r} d(x_i, Tx_i) \le d(x_i, x_{i+1})$ for all $i \ge 0$, we have

$$H(Tx_i, Tx_{i+1}) \le rd(x_i, x_{i+1})$$

for all $i \ge 0$. Note that,

$$d(x_{i+1}, x_{i+2}) \le d(x_{i+1}, Tx_{i+1}) + \varepsilon_{i+1} \le H(Tx_i, Tx_{i+1}) + \varepsilon_{i+1} \le rd(x_i, x_{i+1}) + \varepsilon_{i+1}$$

for all $i \ge 0$. Hence, $d(x_n, x_{n+1}) \le r^n d(x_0, x_1) + \Sigma_{i=0}^{n-1} r^i \varepsilon_{n-i}$ for all n. Hence,

$$\Sigma_{n=1}^{\infty} d(x_n, x_{n+1}) \le \Sigma_{n=1}^{\infty}(r^n d(x_0, x_1) + \Sigma_{i=0}^{n-1} r^i \varepsilon_{n-i})$$

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$$\leq d(x_0, x_1)\Sigma_{n=1}^{\infty} r^n + \Sigma_{i=1}^{\infty}(\Sigma_{j=0}^{\infty} r^j)\varepsilon_i < \infty.$$

Thus, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and so $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, and this implies that $\inf_{x\in X} d(x, Tx) = 0$. Therefore, T has approximate fixed points. □

Corollary 2.1. *Let (X, d) be a complete metric space and $T\colon X \to C_b(X)$ a multifunction. Suppose that there exists $r \in [0, 1)$ such that $\frac{1}{1+r} d(x, Tx) \leq d(x, y)$ implies $H(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Assume that $x_0 \in X$, $\{\varepsilon_i\}_{i=0}^{\infty}$ is a sequence of positive numbers with $\Sigma_{i=0}^{\infty}\varepsilon_i < \infty$ and there exists $x_{i+1} \in Tx_i$ such that $d(x_i, x_{i+1}) \leq d(x_i, Tx_i) + \varepsilon_i$ for all $i \geq 0$. Then the sequence $\{x_i\}_{i\geq 0}$ converges to a fixed point of T.*

Proof. By following the proof of Theorem 2.1, we observe that

$$\Sigma_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \Sigma_{n=1}^{\infty}(r^n d(x_0, x_1) + \Sigma_{i=0}^{n-1} r^i \varepsilon_{n-i})$$

$$\leq d(x_0, x_1)\Sigma_{n=1}^{\infty} r^n + \Sigma_{i=1}^{\infty}(\Sigma_{j=0}^{\infty} r^j)\varepsilon_i \leq (\Sigma_{n=0}^{\infty} r^n)[d(x_0, x_1) + \Sigma_{n=1}^{\infty}\varepsilon_n] < \infty$$

and so it is easy to get that $\{x_n\}$ is a Cauchy sequence. Note that, $d(x_n, x_{n+1}) \to 0$ and so $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Choose $z \in X$ such that $x_n \to z$. First, we show that $d(z, Tx) \leq rd(z, x)$ for all $x \in X\backslash z$. Let $x \in X\backslash z$ be given. Choose a natural number n_0 such that $d(z, x_n) \leq 1/3d(z, x)$ for all $n \geq n_0$. Thus,

$$\frac{1}{1+r} d(x_n, Tx_n) \leq d(x_n, x_{n+1}) \leq d(z, x_n) + d(z, x_{n+1}) \leq 2/3d(z, x)$$

$$= d(z, x) - 1/3d(z, x) \leq d(z, x) - d(z, x_n) \leq d(x_n, x)$$

and so $H(Tx_n, Tx) \leq rd(x_n, x)$ for all $n \geq n_0$. Hence,

$$d(x_n, Tx) \leq d(x_n, Tx_n) + H(Tx_n, Tz) \leq d(x_n, Tx_n) + rd(x_n, x)$$

for all $n \geq n_0$. This implies that $d(z, Tx) \leq rd(z, x)$ for all $x \in X\backslash z$. Also, we have

$$d(x, Tx) \leq d(x, z) + d(z, Tx) \leq d(x, z) + rd(x, z)$$

and so $\frac{1}{1+r} d(x, Tx) \leq d(x, z)$ for all $x \in X\backslash z$. Thus, $H(Tx, Tz) \leq rd(x, z)$ for all x. Since $d(z, Tz) = \lim_{n\to\infty} d(x_{n+1}, Tz) \leq \lim_{n\to\infty} d(x_n, Tx_n) + \lim_{n\to\infty} H(Tx_n, Tz)$, we get $d(z, Tz) \leq \lim_{n\to\infty} rd(x_n, z) = 0$. Since Tz is closed, $z \in Tz$. □

Let (X, d) be a metric space and G a graph such that $V(G) = X$. We say that X has the condition (C) whenever for each sequence $\{x_n\}_{n\geq 1}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $(x_{n_k}, x) \in E(G)$ for all k.

Lemma 2.1. *Let (X, d) be a complete metric space, G a graph such that $V(G) = X$ and $T : X \to C_b(X)$ a multifunction on X via $graphT = \{(x, y) \mid y \in Tx\}$. Suppose that there exists $0 \leq c < 1$ such that $H(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$ with $(x, y) \in E(G)$, $graphT \subseteq E(G)$ and X has the condition (C). Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in X$ with $d(x, Tx) < \delta$ there exists $x^\star \in X$ such that $x^\star \in Tx^\star$ and $d(x, x^\star) < \varepsilon$.*

Proof. Let $\varepsilon > 0$ is given. Choose $\delta > 0$ such that $\frac{4\delta}{1-c} < \varepsilon$. Let $x \in X$ be such that $d(x, Tx) < \delta$. Put $x_0 = x$ and choose $x_1 \in Tx_0$ such that $d(x_0, x_1) < \delta$. Then, $(x_0, x_1) \in E(G)$. If $x_1 \in Tx_1$, then $d(x_0, x_1) < \delta < \varepsilon$. Assume that $x_1 \notin Tx_1$. Put $q = \frac{1+c}{2}$. Then $c < q < 1$. Since $q/c > 1$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < d(x_1, Tx_1)q/c \leq H(Tx_0, Tx_1)q/c \leq d(x_0, x_1)q.$$

If $x_2 \in Tx_2$, then

$$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq (1+q)\delta \leq \delta \Sigma_{i=0}^{\infty} q^i = \frac{\delta}{1-q} = \frac{2\delta}{1-c} < \varepsilon.$$

Assume that $x_2 \notin Tx_2$. Choose $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < d(x_2, Tx_2)q/c \leq H(Tx_1, Tx_2)q/c \leq d(x_1, x_2)q \leq d(x_0, x_1)q^2.$$

By continuing this process, we obtain a sequence $\{x_n\}_{n\geq 1}$ in X such that $x_n \in Tx_{n-1}$, $(x_{n-1}, x_n) \in E(G)$, $x_n \notin Tx_n$ and $d(x_n, x_{n+1}) \leq q^n d(x_0, x_1)$ for all n. Now, note that $d(x_n, x_m) \leq \Sigma_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \Sigma_{i=n}^{m-1} q^i d(x_0, x_1)$ for all m and n. This implies that $\{x_n\}$ is a Cauchy sequence in X. Choose $x^\star \in X$ such that $x_n \to x^\star$. Since X has the condition (C), there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $(x_{n_k}, x) \in E(G)$ for all k. Thus,

$$d(x^\star, Tx^\star) = \lim_{k\to\infty} d(x_{n_k+1}, Tx^\star) \leq \lim_{k\to\infty} H(Tx_{n_k}, Tx^\star) \leq \lim_{k\to\infty} c(d(x_{n_k}, x^\star)) = 0$$

and so $x^\star \in Tx^\star$. Since $d(x_0, x^\star) = \lim_{n\to\infty} d(x_0, x_{n+1}) \leq \lim_{n\to\infty} \sum_{i=0}^{n} d(x_i, x_{i+1})$, we get $d(x_0, x^\star) \leq \sum_{i=0}^{\infty} q^i d(x_0, x_1) < \frac{\delta}{1-q} = \frac{2\delta}{1-c} < \varepsilon$. This completes the proof. □

Next example shows that the multifunction in last result is not a contraction necessarily.

Example 2.1. Let $X = [0,1] \cup \{5/4\}$ and $d(x,y) = |x-y|$. Define the multifunction $T : X \to C_b(X)$ by

$$T(x) = \begin{cases} [0, x/2] & x \in [0,1], \\ \{5/9\} & x = 5/4. \end{cases}$$

Put $x = 1$ and $y = \frac{5}{4}$. Then $H(Tx, Ty) = H([0, \frac{1}{2}], \{\frac{5}{9}\}) = \frac{5}{9} > \frac{1}{4} = d(1, \frac{5}{4})$. Hence, T is not a contraction. Define the graph G by $E(G) = \{(\frac{5}{4}, \frac{5}{9})\} \cup \bigcup_{x\in[0,1]} (\{x\} \times [0, \frac{x}{2}])$. Note that, $graphT = E(G)$ and

$$H(T\frac{5}{4}, T\frac{5}{9}) = H(\{\frac{5}{9}\}, [0, \frac{5}{18}]) = \frac{5}{9} = (\frac{4}{5})(\frac{25}{36}) = \frac{4}{5} d(\frac{5}{4}, \frac{5}{9}).$$

Let $x \in [0,1]$ and $y \in Tx = [0, \frac{x}{2}]$. Then,

$$H(Tx, Ty) = H([0, \frac{x}{2}], [0, \frac{y}{2}]) = \frac{|x-y|}{2} \leq \frac{4|x-y|}{5}.$$

Hence, $H(Tx, Ty) \leq \frac{4|x-y|}{5}$ for all $x, y \in X$ with $(x,y) \in E(G)$. Also, it is easy to show that X has the condition (C). Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in X$ with $d(x, Tx) < \delta$ there exists $x^\star \in X$ such that $x^\star \in Tx^\star$ and $d(x, x^\star) < \varepsilon$. In fact, we choose $\delta < \min\{\frac{25}{36}, \frac{\varepsilon}{2}\}$ for $\varepsilon > 0$. If $d(x, Tx) < \delta$, then $x \in [0,1]$. Put $x^\star = 0 \in T0$. Then, $d(x^\star, x) = x = 2d(x, [0, \frac{x}{2}]) < 2\delta < \varepsilon$.

Theorem 2.2. *Let (X,d) be a metric space, $x_0, \theta \in X$, G a graph such that $V(G) = X$ and $T : X \to C_b(X)$ a multifunction on X via $graphT = \{(x,y) \mid y \in Tx\}$. Suppose that there exists $0 \leq c < 1$ such that $H(Tx, Ty) \leq cd(x,y)$ for all $x, y \in X$ with $(x,y) \in E(G)$, $graphT \subseteq E(G)$, $\varepsilon > 0$, $d(x_0, \theta) = M > 0$ and $(\theta, x_0) \in E(G)$. Suppose that there exist $\delta \in (0, \min\{\frac{(1-c)\varepsilon}{2}, 1\})$ and a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) \leq d(x_n, Tx_n) + \delta$ for all $n \geq 0$. Then T has approximate fixed points.*

Proof. It is sufficient we show that there exists a natural number n_0 such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq n_0$. Choose a natural number $n_0 \geq 1$ such that $c^{n_0}(2M + 1 + d(\theta, T\theta)) < \frac{\varepsilon}{2}$. Since $(\theta, x_0) \in E(G)$, we get

$$d(x_0, Tx_0) \leq d(x_0, \theta) + d(\theta, T\theta) + H(T\theta, Tx_0)$$
$$\leq 2d(x_0, \theta) + d(\theta, T\theta) = 2M + d(\theta, T\theta)$$

and $d(x_0, x_1) \leq d(x_0, Tx_0) + \delta \leq 2M + d(\theta, T\theta) + 1$. Since $x_{n+1} \in Tx_n$ for all $n \geq 0$, it is easy to see that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$. Thus,

$$d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, Tx_{n+1}) + \delta \leq H(Tx_n, Tx_{n+1}) + \delta \leq cd(x_n, x_{n+1}) + \delta$$

for all n. Hence, $d(x_{n+1}, x_{n+2}) \leq c^{n+1}d(x_0, x_1) + (\sum_{i=0}^{n} c_i)\delta$ for all $n \geq 0$. Thus, $d(x_n, x_{n+1}) \leq c^n d(x_0, x_1) + (\sum_{i=0}^{n-1} c_i)\delta \leq c^n(2M + d(\theta, T\theta) + 1) + \frac{1}{1-c}\delta$ for all n. If $n \geq n_0$, then $c^n \leq c^{n_0}$ and so $d(x_n, x_{n+1}) \leq c^{n_0}(2M + d(\theta, T\theta) + 1) + \frac{1}{1-c}\delta < \varepsilon$. This implies that $d(x_n, Tx_n) \to 0$ and so T has approximate fixed points. □

Corollary 2.2. *Let (X, d) be a metric space, $x_0, \theta \in X$, $\alpha : X \times X \to [0, \infty)$ a map G and $T : X \to C_b(X)$ a multifunction on X such that $graphT \subseteq \{(x, y) : \alpha(x, y) \geq 1\}$. Suppose that there exists $0 \leq c < 1$ such that $\alpha(x, y)H(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$, $\varepsilon > 0$, $d(x_0, \theta) = M > 0$ and $\alpha(\theta, x_0) \geq 1$. Suppose that there exist $\delta \in (0, \min\{\frac{(1-c)\varepsilon}{2}, 1\})$ and a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $x_{n+1} \in Tx_n$ and $d(x_n, x_{n+1}) \leq d(x_n, Tx_n) + \delta$ for all $n \geq 0$. Then T has approximate fixed points.*

Proof. It is sufficient we define the graph G by $E(G) = \{(x, y) : \alpha(x, y) \geq 1\}$ and $V(G) = X$. Then by using Theorem 2.2 the proof is completed. □

Theorem 2.3. *Let (X, d) be a complete metric space, G a graph such that $V(G) = X$ and $T : X \to C_b(X)$ a multifunction on X. Suppose that there exists $0 \leq c < 1$ such that $H(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$ with $(x, y) \in E(G)$, X has the condition (C), $\{\varepsilon_i\}_{i=0}^{\infty}$ and $\{\delta_i\}_{i=0}^{\infty}$ are two sequences of positive numbers such that $\sum_{i=0}^{\infty} \varepsilon_i < \infty$ and $\sum_{i=0}^{\infty} \delta_i < \infty$ and $\{T_i\}_{i=0}^{\infty}$ is a sequence of closed and bounded valued multifunctions on X such that $H(T_i x, Tx) \leq \varepsilon_i$ for all $x \in X$ and $i \geq 0$. If there exists a sequence $\{x_i\}_{i=0}^{\infty}$ in X such that $(x_i, x_{i+1}) \in E(G)$, $x_{i+1} \in T_i x_i$ and $d(x_i, x_{i+1}) \leq d(x_i, T_i x_i) + \delta_i$ for all $i \geq 0$, then $\{x_i\}_{i=0}^{\infty}$ converges to a fixed point of T.*

Proof. Note that, $H(Tx_i, Tx_{i+1}) \leq cd(x_i, x_{i+1})$ for all $i \geq 0$. Hence,

$$d(x_{i+1}, x_{i+2}) \leq d(x_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1} \leq d(x_{i+1}, Tx_{i+1}) + H(Tx_{i+1}, T_{i+1}x_{i+1}) + \delta_{i+1}$$
$$\leq d(x_{i+1}, Tx_{i+1}) + \varepsilon_{i+1} + \delta_{i+1} \leq H(T_i x_i, Tx_{i+1}) + \varepsilon_{i+1} + \delta_{i+1}$$
$$\leq H(Tx_i, T_i x_i) + H(Tx_i, Tx_{i+1}) + \varepsilon_{i+1} + \delta_{i+1} \leq cd(x_i, x_{i+1}) + \varepsilon_i + \varepsilon_{i+1} + \delta_{i+1}$$

for all $i \geq 0$. On the other hand, we have $d(x_1, x_2) \leq cd(x_0, x_1) + \varepsilon_0 + \varepsilon_1 + \delta_1$ and so $d(x_2, x_3) \leq cd(x_1, x_2) + \varepsilon_1 + \varepsilon_2 + \delta_2 \leq c^2 d(x_0, x_1) + c(\varepsilon_0 + \varepsilon_1 + \delta_1) + (\varepsilon_1 + \varepsilon_2 + \delta_2)$. By following this process, it is easy to show that

$$d(x_n, x_{n+1}) \leq c^n d(x_0, x_1) + \Sigma_{i=0}^{n-1} c^i(\varepsilon_{n-i-1} + \varepsilon_{n-i} + \delta_{n-i})$$

for all $n \geq 1$. Hence,

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq (\sum_{n=1}^{\infty} c^n)d(x_0, x_1) + \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(\varepsilon_{n-i-1} + \varepsilon_{n-i} + \delta_{n-i})$$
$$\leq (\sum_{n=1}^{\infty} c^n)[d(x_0, x_1) + \sum_{n=1}^{\infty}(\varepsilon_{n-i-1} + \varepsilon_{n-i} + \delta_{n-i})] < \infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence. Choose $x^\star \in X$ such that $x_n \to x^\star$. Since X has the condition (C), there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ such that $(x_{n_k}, x) \in E(G)$ for all k. Hence,

$$\begin{aligned} d(x^\star, Tx^\star) &= \lim_{k\to\infty} d(x_{n_k+1}, Tx^\star) \leq \lim_{k\to\infty} (d(x_{n_k+1}, Tx_{n_k}) + H(Tx_{n_k}, Tx^\star)) \\ &\leq \lim_{k\to\infty} H(T_{n_k}x_{n_k}, Tx_{n_k}) + \lim_{k\to\infty} cd(x_{n_k}, x^\star) = 0. \end{aligned}$$

Thus, $x^\star \in Tx^\star$ and so $\{x_i\}_{i=0}^\infty$ converges to a fixed point of T. □

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to C_b(X)$ a multifunction on X. Suppose that there exists $0 < r < 1$ such that $\frac{1}{1+r}d(x, Tx) \leq d(x, y)$ implies $H(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$, $\{\varepsilon_i\}_{i=0}^\infty$ and $\{\delta_i\}_{i=0}^\infty$ are two sequences of positive numbers such that $\sum_{i=0}^\infty \varepsilon_i < \infty$ and $\sum_{i=0}^\infty \delta_i < \infty$ and $\{T_i\}_{i=0}^\infty$ is a sequence of closed and bounded valued multifunctions on X such that $H(T_i x, Tx) \leq \varepsilon_i$ for all $x \in X$ and $i \geq 0$. If there exists a sequence $\{x_i\}_{i=0}^\infty$ in X such that $x_{i+1} \in T_i x_i$ and $\frac{\varepsilon_i}{r} \leq d(x_i, x_{i+1}) \leq d(x_i, T_i x_i) + \delta_i$ for all $i \geq 0$, then $\{x_i\}_{i=0}^\infty$ converges to a fixed point of T.*

Proof. Define the graph G on X by $V(G) = X$ and

$$E(G) = \{(x, y) : \frac{1}{1+r}d(x, Tx) \leq d(x, y)\} \cup \{(x, x) : x \in X\}.$$

Note that,

$$d(x_i, Tx_i) \leq d(x_i, T_i x_i) + \varepsilon_i \leq d(x_i, x_{i+1}) + rd(x_i, x_{i+1}) = (1 + r)d(x_i, x_{i+1})$$

for all $i \geq 0$. Hence, $(x_i, x_{i+1}) \in E(G)$ for all $i \geq 0$. Now, we show that X has the condition (C). Let $\{x_n\}_{n=0}^\infty$ be a sequence in X, $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$ and $x_n \to z$. If there exists a natural number n_0 such that $x_n = x_{n+1}$ for all $n \geq n_0$, then we have nothing to prove. Suppose that there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 0}$ such that $x_{n_k} \neq x_{n_k+1}$ for all k. Then, we have

$$\frac{1}{1+r}d(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, x_{n_k+1})$$

for all k. Let $x \neq z$. Choose a natural number k_0 such that $d(z, x_n) \leq \frac{1}{3}d(z, x)$ for all $n \geq n_{k_0}$. Since $n_k \geq n_{k_0}$ for all $k \geq k_0$, $d(z, x_{n_k}) \leq \frac{1}{3}d(z, x)$ for all $k \geq k_0$. Thus,

$$\begin{aligned} \alpha d(x_{n_k}, Tx_{n_k}) &\leq d(x_{n_k}, x_{n_k+1}) \leq d(z, x_{n_k}) + d(z, x_{n_k+1}) \leq 2/3d(z, x) \\ &= d(z, x) - 1/3d(z, x) \leq d(z, x) - d(z, x_{n_k}) \leq d(x_{n_k}, x) \end{aligned}$$

and so $H(Tx_{n_k}, Tx) \leq rd(x_{n_k}, x)$ for all $k \geq k_0$. Since

$$\begin{aligned} d(x_{n_k+1}, Tx) &\leq d(x_{n_k+1}, Tx_{n_k}) + H(Tx_{n_k}, Tx) \\ &\leq d(x_{n_k+1}, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + \beta d(x_n, x) \end{aligned}$$

for all $k \geq k_0$, we get $d(z, Tx) \leq rd(z, x)$. If $x_{n_k} = z$ for some k, then $(x_{n_k}, z) \in E(G)$. If $x_{n_k} \neq z$, then

$$d(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, z) + d(z, Tx_{n_k}) \leq d(x_{n_k}, z) + rd(x_{n_k}, z) = (1 + r)d(x_{n_k}, z)$$

and so $\frac{1}{1+r}d(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, z)$. This implies that $(x_{n_k}, z) \in E(G)$ for all k. Thus, shows that X has the condition (C). Now by using Theorem 2.3, the sequence $\{x_i\}_{i=0}^\infty$ converges to a fixed point of T. □

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