END POINT OF SOME GENERALIZED WEAKLY CONTRACTIVE MULTIVALUED MAPPINGS

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In this paper, we prove the existence of a common end point for a pair of multivalued mappings satisfying a new generalized weakly contractive condition in a complete metric space. Our result generalizes some results of Dutta and Choudhury (2008), and Zhang, Song (2010).

Keywords: : end point, weakly contractive mapping, multivalued mapping
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1. Introduction

Banach contraction principle is a remarkable result in metric fixed point theory. Over the years, it has been generalized in different directions and spaces by mathematicians.

In 1997, Alber and Guerre-Delabriere [1] introduced the concept of weak contraction in the following way:

Definition 1.1. *Let (E, d) be a metric space. A mapping $T : E \longrightarrow E$ is said to be weakly contractive provided that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

where $x, y \in E$ and $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

Using the concept of weakly contractiveness, they succeeded to establish the existence of fixed points for such mappings in Hilbert spaces. Later on Rhoades [10] proved that the result of [1] is also valid in complete metric spaces. Rhoades [10] also proved the following fixed point theorem which is a generalization of the Banach contraction principle, because it contains contractions as a special case when we assume that $\varphi(t) = (1 - k)t$ for some $0 < k < 1$.

Theorem 1.1. *Let (E, d) be a complete metric space and let $T : E \longrightarrow E$ be a weakly contractive mapping. Then T has a fixed point.*

Since then, fixed point theory for single valued, as well as for multivalued weakly contractive type mappings was studied by many authors; see [2-9], and [11-13].

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Let (E, d) be a metric space, and let $B(E)$ denote the family of all nonempty bounded subsets of E. For $A, B \in B(E)$, define the distance between A and B by

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

and the diameter of A and B by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

Let $T : E \longrightarrow B(E)$ be a multivalued operator, then an element $x \in E$ is said to be a *fixed point* of T provided that $x \in T(x)$ and it is called an *end point* of T if $T(x) = \{x\}$. The purpose of this paper is to prove the existence of a common end point for a pair of multivalued mappings satisfying a new generalized weakly contractive condition in a complete metric space. Our result generalizes and extends some results of Dutta and Choudhury [6], and Zhang, Song [13].

2. **Main Results**

In this sequel, we denote by Φ the class of all mappings $\varphi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

(i) $\varphi(t) = 0$ if and only if $t = 0$;
(ii) φ is a lower semi continuous function ;
(iii) for any sequence $\{t_n\}$ with $\lim_{n\longrightarrow\infty} t_n = 0$, there exist $k \in (0, 1)$ and $n_0 \in \mathbb{N}$, such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$.

Examples of such mappings are $\varphi(x) = kx$ for $0 < k < 1$ and $\varphi(x) = \ln(x + 1)$ (see also [9]). Let Ω denote the class of all mappings $f : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

(i) $f(t) = 0$ if and only if $t = 0$;
(ii) f is non-decreasing;
(iii) f is continuous;
(iv) $f(x + y) \leq f(x) + f(y)$.

Finally, let Ψ denote the class of mappings $\psi : [0, \infty) \longrightarrow [0, \infty)$ which are continuous and non-decreasing with $\psi(t) = 0$ if and only if $t = 0$.

Let (E, d) be a metric space, and let $T, S : E \longrightarrow B(E)$ be two multivalued mappings, we define

$$M(x, y) = \max\left\{d(x, y), \delta(Tx, x), \delta(y, Sy), \frac{D(y, Tx) + D(x, Sy)}{2}\right\},$$

and

$$N(x, y) = \min\{D(y, Tx), D(x, Sy)\}.$$

We now state the main result of this paper.

Theorem 2.1. *Let (E, d) be a complete metric space, and let $T, S : E \longrightarrow B(E)$ be two mappings such that for all $x, y \in E$*

$$f(\delta(Tx, Sy) \leq f(M(x, y)) - \varphi(f(M(x, y))) + \psi(N(x, y)) \quad (2.1)$$

where $\varphi \in \Phi$, $\psi \in \Psi$ and $f \in \Omega$. Then S and T have a common end point $z \in E$, i.e, $Sz = Tz = \{z\}$.

Proof. We construct a sequence $\{x_n\}$ as follows. Take $x_0 \in E$ and for $n \geq 1$ we choose $x_{2n+1} \in Tx_{2n} := A_{2n}$ and $x_{2n+2} \in Sx_{2n+1} := A_{2n+1}$. Now we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) & \\ &= \max\{d(x_{2n}, x_{2n+1}), \delta(Tx_{2n}, x_{2n}), \delta(Sx_{2n+1}, x_{2n+1}), \\ &\qquad \frac{D(Tx_{2n}, x_{2n+1}) + D(Sx_{2n+1}, x_{2n})}{2}\} \\ &\leq \max\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n-1}, A_{2n}), \delta(A_{2n+1}, A_{2n}), \\ &\qquad \frac{D(Tx_{2n}, x_{2n+1}) + D(Sx_{2n+1}, x_{2n})}{2}\} \\ &\leq \max\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n+1}, A_{2n}), \frac{\delta(A_{2n+1}, A_{2n-1})}{2}\} \\ &\leq \max\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n+1}, A_{2n}), \\ &\qquad \frac{\delta(A_{2n}, A_{2n-1}) + \delta(A_{2n}, A_{2n+1})}{2}\} \\ &= \max\{\delta(A_{2n-1}, A_{2n}), \delta(A_{2n+1}, A_{2n})\} \end{aligned}$$

and

$$N(x_{2n}, x_{2n+1}) = \min\{D(Tx_{2n}, x_{2n+1}), D(Sx_{2n+1}, x_{2n})\} = 0.$$

By assumption

$$\begin{aligned} f(\delta(A_{2n}, A_{2n+1})) &= f(\delta(Tx_{2n}, Sx_{2n+1})) \\ &\leq f(M(x_{2n}, x_{2n+1})) - \varphi(f(M(x_{2n}, x_{2n+1}))) + \psi(N(x_{2n}, x_{2n+1})) \\ &= f(M(x_{2n}, x_{2n+1})) - \varphi(f(M(x_{2n}, x_{2n+1})) \\ &\leq f(M(x_{2n}, x_{2n+1}). \end{aligned}$$

Since f is non-decreasing, we have

$$\delta(A_{2n}, A_{2n+1}) \leq M(x_{2n}, x_{2n+1}).$$

Now, if $\delta(A_{2n-1}, A_{2n}) < \delta(A_{2n+1}, A_{2n})$ then

$$M(x_{2n}, x_{2n+1}) \leq \delta(A_{2n+1}, A_{2n}),$$

from which we obtain

$$M(x_{2n}, x_{2n+1}) = \delta(A_{2n+1}, A_{2n}) > \delta(A_{2n-1}, A_{2n}) \geq 0,$$

and

$$\begin{aligned} f(\delta(A_{2n}, A_{2n+1})) &= f(\delta(Tx_{2n}, Sx_{2n+1})) \\ &\leq f(M(x_{2n}, x_{2n+1})) - \varphi(f(M(x_{2n}, x_{2n+1}))) + \psi(N(x_{2n}, x_{2n+1})) \\ &= f(M(x_{2n}, x_{2n+1})) - \varphi(f(M(x_{2n}, x_{2n+1})) \\ &< f(M(x_{2n}, x_{2n+1}) = f(\delta(A_{2n+1}, A_{2n})) \end{aligned}$$

which is a contradiction. So we have

$$\delta(A_{2n+1}, A_{2n}) \leq M(x_{2n}, x_{2n+1}) \leq \delta(A_{2n}, A_{2n-1}).$$

Similarly we obtain

$$\delta(A_{2n+1}, A_{2n+2}) \leq M(x_{2n+1}, x_{2n+2}) \leq \delta(A_{2n+1}, A_{2n+2}).$$

Therefore the sequence $\{\delta(A_n, A_{n+1})\}$ is monotone decreasing and bounded below. So there exists $r \geq 0$ such that

$$\lim_{n\longrightarrow\infty} \delta(A_n, A_{n+1}) = \lim_{n\longrightarrow\infty} M(x_n, x_{n+1}) = r.$$

We now claim that $r = 0$. In fact taking upper limits as $n \longrightarrow \infty$ on either sides of the inequality

$$\begin{aligned} f(\delta(A_{2n}, A_{2n+1})) &= f(\delta(Tx_{2n}, Sx_{2n+1})) \\ &\leq f(M(x_{2n}, x_{2n+1})) - \varphi(f(M(x_{2n}, x_{2n+1}))) + \psi(N(x_{2n}, x_{2n+1})) \\ &= f(M(x_{2n}, x_{2n+1})) - \varphi(f(M(x_{2n}, x_{2n+1}))) \end{aligned}$$

we have

$$f(r) \leq f(r) - \varphi f(r)$$

which is a contradiction unless $r = 0$. Thus $\lim_{n\longrightarrow\infty} \delta(A_n, A_{n+1}) = 0$ and hence $\lim_{n\longrightarrow\infty} d(x_n, x_{n+1}) = 0$. Now we shall prove that $\{x_n\}$ is a Cauchy sequence. Indeed, Since $\lim_{n\longrightarrow\infty} f(M(x_n, x_{n+1})) = 0$ by the property of φ there exist $0 < k < 1$ and $n_0 \in \mathbb{N}$, such that $\varphi(f(M(x_n, x_{n+1}))) \geq kf(M(x_n, x_{n+1}))$ for all $n > n_0$. On the other hand, for any given $\varepsilon > 0$, we can choose $\eta > 0$ in such a way that $f(\eta) \leq \frac{k}{1-k} f(\varepsilon)$. Moreover, there exists n_0 such that $\delta(A_n, A_{n-1}) \leq \eta$ for each $n > n_0$. For any natural number $m > n > n_0$ if n is even, we have

$$\begin{aligned} f(\delta(A_n, A_{n+1})) &\leq f(\delta(Tx_n, Sx_{n+1})) \\ &\leq f(M(x_n, x_{n+1})) - \varphi(f(M(x_n, x_{n+1})) + \psi(N(x_n, x_{n+1})) \\ &\leq (1-k)f(M(x_n, x_{n+1})) \leq (1-k)f(\delta(A_n, A_{n-1})). \end{aligned}$$

By this inequality, we get for $l > n$

$$f(\delta(A_l, A_{l-1})) \leq (1-k)f(\delta(A_{l-1}, A_{l-2})) \leq \cdots \leq (1-k)^{l-n} f(\delta(A_n, A_{n-1}))$$

Therefore we have

$$\begin{aligned} f(\delta(A_n, A_m)) &\leq f(\delta(A_n, A_{n+1}) + \delta(A_{n+1}, A_{n+2}) + \cdots + \delta(A_{m-1}, A_m)) \\ &\leq f(\delta(A_n, A_{n+1})) + f(\delta(A_{n+1}, A_{n+2})) + \cdots + f(\delta(A_{m-1}, A_m)) \\ &\leq (1-k)f(\delta(A_n, A_{n-1})) + \cdots \\ &+ (1-k)^{m-n-1} f(\delta(A_n, A_{n-1})) + (1-k)^{m-n} f(\delta(A_n, A_{n-1})) \\ &= \frac{(1-k) - (1-k)^{m-n+1}}{1-(1-k)} f(\delta(A_n, A_{n-1})) \\ &< \frac{1-k}{k} f(\delta((A_n, A_{n-1}))) \leq \frac{1-k}{k} f(\eta) < f(\varepsilon). \end{aligned}$$

Now, by the nondecreasingness of f we obtain $\delta(A_n, A_m) < \varepsilon$. From the construction of the sequence $\{x_n\}$, it follows that the same conclusion holds for $\{x_n\}$, i.e. for each $\varepsilon > 0$ there exist n_0 such that for any natural numbers $m > n > n_0$, $d(x_n, x_m) < \varepsilon$. This shows that $\{x_n\}$ is a Cauchy sequence. Notice that E is complete, hence $\{x_n\}$ is convergent. Let us denote its limit by $\lim_{n\longrightarrow\infty} x_n = z$ for some $z \in E$. Now we prove that $\delta(Tz, z) = 0$. Suppose that this is not true, then $\delta(Tz, z) > 0$. For large enough n, we claim that the following equations hold true:

$$M(z, x_{2n+1}) = \max\{d(z, x_{2n+1}), \delta(z, Tz), \delta(Sx_{2n+1}, x_{2n+1}), \frac{D(Tz, x_{2n+1}) + D(Sx_{2n+1}, z)}{2}\} = \delta(z, Tz).$$

Indeed, since

$$\delta(Sx_{2n+1}, x_{2n+1}) \leq \delta(A_{2n+1}, A_{2n}) \longrightarrow 0,$$

and

$$\lim_{n\longrightarrow\infty} \frac{D(Tz, x_{2n+1}) + D(Sx_{2n+1}, z)}{2} \leq \lim_{n\longrightarrow\infty} \frac{\delta(Tz, z) + d(z, x_{2n+1}) + \delta(Sx_{2n+1}, x_{2n+1}) + d(x_{2n+1}, z)}{2} = \frac{\delta(Tz, z)}{2},$$

it follows that there exists $k \in \mathbb{N}$ such that $M(z, x_{2n+1}) = \delta(z, Tz)$ for $n > k$. Note that

$$f(\delta(Tz, x_{2n+2})) \leq f(\delta(Tz, Sx_{2n+1})) \leq f(M(z, x_{2n+1})) - \varphi(f((M(z, x_{2n+1})) - \psi(N(z, x_{2n+1})).$$

Letting $n \longrightarrow \infty$, we have

$$f(\delta(Tz, z)) \leq f(\delta(Tz, z)) - \varphi(f(\delta(Tz, z)))$$

i.e, $\varphi(f(\delta(Tz, z))) \leq 0$. This is a contradiction, therefore $\delta(Tz, z) = 0$ i.e., $Tz = \{z\}$. And since

$$M(z, z) = \max\{d(z, z), \delta(Tz, z), \delta(z, Sz), \frac{D(Tz, z) + D(Sz, z)}{2}\} = \max\{\delta(Sz, z), \frac{D(Sz, z)}{2}\} = \delta(Sz, z)$$

and

$$N(z, z) = \min\{D(z, Tz), D(z, Sz)\} = 0$$

we conclude that

$$f(\delta(z, Sz)) \leq f(\delta(Tz, Sz)) \leq f(M(z, z)) - \varphi(f(M(z, z))) + \psi(N(z, z)) \leq f(\delta(z, Sz)) - \varphi(f(\delta(Sz, z))),$$

which in turn implies that $Sz = \{z\}$. Hence the point z is a common end point of S and T.

□

Theorem 2.2. *Let (E, d) be a complete metric space, and let $T, S : E \longrightarrow B(E)$ be two mappings such that for all $x, y \in E$*

$$f(\delta(Tx, Sy) \leq f(M(x, y)) - \varphi(f(M(x, y))) \tag{2.2}$$

where $\phi \in \Phi$ *and* $f \in \Omega$. *Then* S *and* T *have a unique common end point* $z \in E$.*i.e,* $Sz = Tz = \{z\}$.

Proof. By theorem 2.1, T and S have a common end point z. Now let $y \in E$ be another common end point of S and T. Notice that

$$M(y,y) = \max\{d(y,y), \delta(Ty,y), \delta(y,Sy), \frac{D(Ty,y)+D(Sy,y)}{2}\}$$
$$= \max\{\delta(Sy,y), \delta(y,Ty)\}.$$

Hence

$$f(\delta(y,Ty)) \leq f(\delta(Sy,Ty)) \leq f(M(y,y)) - \varphi(f(M(y,y)))$$
$$\leq f(max\{\delta(y,Sy), \delta(y,Ty)\}) - \varphi(f(\max\{\delta(y,Sy), \delta(y,Ty)\})).$$

Similarly, we have

$$f(\delta(y,Sy)) \leq f(\delta(Ty,Sy)) \leq f(M(y,y)) - \varphi(f(M(y,y)))$$
$$\leq f(\max\{\delta(y,Sy), \delta(y,Ty)\} - \varphi(f(\max(\delta(y,Sy), \delta(y,Ty)\})).$$

Therefore

$$f(\max\{\delta(y,Sy), \delta(y,Ty)\}) \leq$$
$$f(\max\{\delta(y,Sy), \delta(y,Ty)\}) - \phi(f(\max\{\delta(y,Sy), \delta(y,Ty)\})))$$

which implies that $\max\{\delta(y,Sy), \delta(y,Ty)\} = 0$, hence $\delta(Ty,y) = \delta(Sy,y) = 0$. Now we have

$$M(z,y) = \max\{d(z,y), \delta(z,Tz), \delta(y,Sy), \frac{D(y,Tz)+D(z,Sy)}{2}\}$$

and

$$f(d(z,y)) = f(\delta(Sz,Ty)) \leq f(M(z,y)) - \varphi(f(M(z,y)))$$
$$= f(d(z,y)) - \varphi(f(d(z,y)))$$

that imply $d(z,y) = 0$ i.e, $z = y$. Hence z is the unique common end point of S and T. □

If in Theorem 2.1 we put $f(t) = t$ and $\varphi(t) = (1-k)t$, for some $0 < k < 1$, then we obtain the following result.

Theorem 2.3. *Let* (E,d) *be a complete metric space, and let* $T, S : E \longrightarrow B(E)$ *be two mappings such that for all* $x, y \in E$

$$\delta(Tx,Sy) \leq k(M(x,y)) + \psi(N(x,y)) \tag{2.3}$$

where $\psi \in \Psi$. *Then* S *and* T *have a common end point* $z \in E$, *i.e,* $Sz = Tz = \{z\}$.

Let T and S be two single valued mappings, the we obtain the following theorem:

Theorem 2.4. *Let* (E,d) *be a complete metric space, and let* $T, S : E \longrightarrow E$ *be two mappings such that for all* $x, y \in E$

$$f(d(Tx,Sy) \leq f(M(x,y)) - \varphi(f(M(x,y))) + \psi(N(x,y)) \tag{2.4}$$

where $\varphi \in \Phi$, $\psi \in \Psi$, $f \in \Omega$ *and*

$$M(x,y) = \max\left\{d(x,y), d(Tx,x), d(y,Sy), \frac{d(y,Tx)+d(x,Sy)}{2}\right\},$$

$$N(x,y) = \min\{d(Tx,y), d(x,Sy)\}.$$

Then S *and* T *have a common fixed point* $z \in E$, *i.e,* $Sz = Tz = z$.

Example 2.1. *Let* $E = [0,1]$ *and* $d(x,y) = |x-y|$. *For each* $x \in E$ *define* $S, T : E \longrightarrow B(E)$ *by*

$$Tx = [\frac{x}{4}, \frac{x}{2}], \qquad Sx = [0, \frac{x}{5}].$$

Then

$$\delta(Tx,Sy) = \begin{cases} \frac{x}{2} & 0 \le \frac{y}{5} \le \frac{x}{2} \\ \max\{\frac{y}{5} - \frac{x}{4}, \frac{x}{2}\} & \frac{x}{2} \le \frac{y}{5} \le 1. \end{cases}$$

and

$$\delta(x,Tx) = \frac{3x}{4}, \qquad \delta(y,Sy) = y.$$

We also consider $f(t) = 2t$ *and* $\phi(t) = \frac{t}{4}$. *We note that if* $\delta(Tx,Sy) = \frac{x}{2}$ *then*

$$\begin{aligned} f(\delta(Tx,Sy) = x \le \frac{9x}{8} = \frac{3}{2}\delta(x,Tx) \\ \le \frac{3}{2}(M(x,y)) = f(M(x,y)) - \varphi(f(M(x,y))) \end{aligned}$$

and if $\delta(Tx,Sy) = \frac{y}{5} - \frac{x}{4}$ *then*

$$\begin{aligned} \psi(\delta(Tx,Sy) = 2(\frac{y}{5} - \frac{x}{4}) \le \frac{2y}{5} \le \frac{3y}{2} \\ = \frac{3}{2}\delta(y,Sy) \le \frac{3}{2}(M(x,y)) = f(M(x,y)) - \varphi(f(M(x,y))). \end{aligned}$$

This arguments show that the mappings T *and* S *satisfy the conditions of Theorem 2.2. Now it is easy to see that* 0 *is the only common end point of this two mappings.*

In the following we shall see that Theorem 2.1 is a real generalization of Theorem 2.2. We note that by Theorem 2.2, T *and* S *have a unique common end point.*

Example 2.2. *Let* $E = [0,1]$ *and* $d(x,y) = |x-y|$. *For each* $x \in E$ *define* $T, S : E \longrightarrow B(E)$ *by*

$$Tx = Sx = \begin{cases} [\frac{x}{3}, \frac{x}{2}] & x \ne 1 \\ 1 & 1. \end{cases}$$

We also consider $f(t) = t$, $\phi(t) = \frac{t}{5}$, *and* $\psi(t) = 2t^2$. *We note that if* $x, y \ne 1$ *and* $x \le y$ *then* $\delta(y,Ty) = \frac{2y}{3}$ *hence*

$$\begin{aligned} f(\delta(Tx,Ty) = \frac{y}{2} - \frac{x}{3} \le \frac{8}{15}\delta(y,Ty) \\ \le \frac{8}{15}M(x,y) = f(M(x,y)) - \varphi(f(M(x,y))) \end{aligned}$$

Similar result holds if $x, y \neq 1$ *and* $y \leq x$. *Now if* $y = 1$ *and* $x \neq 1$, *then* $\delta(Tx, Ty) = 1 - \frac{x}{3}$, $D(y, Tx) = 1 - \frac{x}{2}$, $D(x, Ty) = 1 - x$, $d(x, y) = 1 - x$, $\delta(y, Ty) = 0$ *and* $\delta(x, Tx) = \frac{2x}{3}$. *Hence*

$$M(x,y)) = \begin{cases} 1 - \frac{3x}{4}, & x \leq \frac{12}{17} \\ \frac{2x}{3}, & x \geq \frac{12}{17}. \end{cases}$$

and

$$N(x, y) = 1 - x$$

therefore we have

$$f(\delta(Tx, Ty)) = 1 - \frac{x}{3} \leq$$
$$2(1-x)^2 = \psi(N(x, y)) \leq f(M(x, y)) - \varphi(f(M(x, y))) + \psi(N(x, y)).$$

Similar result holds if $x = 1$ *and* $y \neq 1$ *This arguments show that the mappings* T *and* S *satisfy the conditions of Theorem 2.1. We observe that* 0 *and* 1 *are two end points for* T *and* S.

REFERENCES

1. Ya.I. Alber, S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces*, in: I. Gohberg, Yu Lyubich (Eds.), New Results in Operator Theory, in: Advances and Appl., vol. **98**, Birkhuser, Basel, 1997, pp. 7-22.
2. H. Aydi, W. Shatanawi, M. Postolache, Z. Mustafa and N. Tahat: Theorems for Boyd-Wong type contractions in ordered metric spaces, Abstr. Appl. Anal., 2012, ID 359054, 14 p.
3. J. S. Bae, *Fixed point theorems for weakly contractive multivalued maps*. J. Math. Anal. Appl. **284**(2003), 690-697.
4. B.S. Choudhury, P. Konar, B.E. Rhoades, N. Metiya, *Fixed point theorems for generalized weakly contractive mappings*, Nonlinear Anal.**74** (2011) 2116-2126.
5. D. Doric, *Common fixed point theorem for generalized* (ψ, ϕ)*-weak contraction*, Appl. Math. Lett. **22** (2009) 1896-1900.
6. P.N. Dutta and B.S. Choudhury, *A generalization of contraction principle in metric space*, Fixed Point Theory and Applications, (2008), Article ID-406368,8 pages
7. M. Eslamian, A. Abkar, *Fixed point theory for generalized weakly contractive mappings in complete metric spaces*, Italian J. Pure and Appl. Math., (to appear).
8. J. Harjani, K. Sadarangani *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. **71**(2009) 3403-3410.
9. S. Hong, *Fixed points of multivalued operators in ordered metric spaces with applications* Nonlinear Anal., **72** (2010), 3929-3942.
10. B.E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., 47 (2001) 2683-2693.
11. H. K. Nashine, B. Samet, *Fixed point results for mappings satisfying* (φ, ψ)*-weakly contractive condition in partially ordered metric spaces*, Nonlinear Anal. **74** (2011), 2201-2209.
12. X. Zhang, *Common fixed point theorems for some new generalized contractive type mappings*, J. Math. Anal. Appl., **3**33(2007) , 780-786.
13. Q. Zhang, Y. Song, *Fixed point theory for* ϕ*-weak contractions*, Appl. Math. Lett. 22 (1) (2009) 75-78.