SOME APPROXIMATE FIXED POINT RESULTS FOR GENERALIZED α-CONTRACTIVE MAPPINGS

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Recently, Samet, Vetro and Vetro [Nonlinear Anal. 75 (2012) 2154-2165*] introduced α-ψ-contraction maps and gave some results on the mappings on complete metric spaces. On the other hand, by introducing a type of generalized contractive mappings, Aleomraninejad, Rezapour and Shahzad [Appl. Math. Lett.* 24 (2011) 1037-1040*] generalized some results about the Suzuki's method. In this paper, by using the main idea of these works, we introduce the concept of generalized α-contractive mapping and give two results about approximate fixed points and fixed points of the mappings on metric spaces. We show that these results generalize some related classical results in the literature.*

Keywords: Metric space, α-contractive mapping, approximate fixed point, fixed point.

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1. Introduction

Let (X, d) be a metric space and T a selfmap on X. If $\alpha\colon X \times X \to [0, \infty)$ is a mapping and ε a positive number, then, according to [17], we say that T is α-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$. An element $x_0 \in X$ is called ε-fixed point of T whenever $d(T(x_0), x_0) < \varepsilon$. We say that T has the approximate fixed point property whenever T has an ε-fixed point for all $\varepsilon > 0$.

As we know, there are selfmaps which have approximate fixed points but have no fixed points. The interest is the study of approximate fixed points, associated to several classes of mappings, is given by the important number of works in this direction; please, see [2], [4], [10], [14] and [16], for illustrative examples.

To develop our new results, we appeal the following lemma [5].

Lemma 1.1. *Let (X, d) be a metric space, and $T\colon X \to X$ a mapping such that T is asymptotic regular, that is, $d(T^n x, T^{n+1} x) \to 0$ for all $x \in X$. Then T has the approximate fixed point property.*

Now, denote by $\mathcal{R}$ the set of all continuous mappings $g\colon [0, \infty)^5 \longrightarrow [0, \infty)$ satisfying the following conditions, for more details see [1]:

a) $g(1, 1, 1, 2, 0) = g(1, 1, 1, 0, 2) = h \in (0, 1)$,

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b) $g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha g(x_1, x_2, x_3, x_4, x_5)$, for all $(x_1, x_2, x_3, x_4, x_5)$ in $[0, \infty)^5$ and $\alpha \geq 0$,

c) if $x_i, y_i \in [0, \infty)$ and $x_i < y_i$ for $i = 1, \dots, 4$, then

$$g(x_1, x_2, x_3, x_4, 0) < g(y_1, y_2, y_3, y_4, 0), \qquad g(x_1, x_2, x_3, 0, x_4) < g(y_1, y_2, y_3, 0, y_4).$$

This class of mappings has the property in Proposition 1.1, see [1] and Lemma 1.3 in [9], which is necessary for developing our new results.

Proposition 1.1. *If $g \in \mathcal{R}$ and $u, v \in [0, \infty)$ are such that*

$$u \leq \max\{g(v, v, u, v + u, 0), g(v, v, u, 0, v + u), g(v, u, v, v + u, 0), g(v, u, v, 0, v + u)\},$$

then $u \leq hv$.

Now, we are ready to state and prove our main results.

2. Main Results

First, by using the main idea of [1] and [17], we introduce the notion of generalized α-contractive mapping. In this respect, we emphasize that throughout this paper we suppose that (X, d) is a metric space, T is a selfmap on X and $\alpha\colon X \times X \to [0, \infty)$ is a mapping.

We say that the selfmap T of X is a *generalized α-contractive mapping* whenever there exists $g \in \mathcal{R}$ such that

$$\alpha(x, y)d(Tx, Ty) \leq g(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)),$$

for all $x, y \in X$.

Theorem 2.1. *Let (X, d) be a metric space and T a generalized α-contractive and an α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has an approximate fixed point.*

Proof. Fix $1 > r > h$ and $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ by $x_{n+1} = T^{n+1}x_0$ for all $n \geq 0$.

If $x_n = x_{n+1}$, for some n, then we have nothing to prove.

Assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is α-admissible, it is easy to check that $\alpha(x_n, x_{n+1}) \geq 1$ for all n.

Since

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \leq \alpha(x_0, x_1)d(Tx_0, Tx_1) \\ &\leq g(d(x_0, x_1), d(x_1, Tx_1), d(x_0, Tx_0), d(x_0, Tx_1), d(x_1, Tx_0)) \\ &\leq g(d(x_0, x_1), d(x_1, x_2), d(x_0, x_1), d(x_0, x_1) + d(x_1, x_2), 0), \end{aligned}$$

by using Proposition 1.1, we obtain $d(x_1, x_2) \leq hd(x_0, x_1) < rd(x_0, x_1)$.

Since

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \leq \alpha(x_1, x_2)d(Tx_1, Tx_2) \\ &\leq g(d(x_1, x_2), d(x_2, Tx_2), d(x_1, Tx_1), d(x_1, Tx_2), d(x_2, Tx_1)) \\ &\leq g(d(x_1, x_2), d(x_2, x_3), d(x_1, x_2), d(x_1, x_2) + d(x_2, x_3), 0), \end{aligned}$$

again by using Proposition 1.1, we obtain

$$d(x_2, x_3) \leq hd(x_1, x_2) < rd(x_1, x_2) < r^2 d(x_0, x_1).$$

By continuing this process, we obtain $d(x_n, x_{n+1}) < r^n d(x_0, x_1)$ for all n. Hence, $d(T^{n+1}x_0, T^n x_0) \to 0$, and from Lemma 1.1, we conclude that T has the approximate fixed point property. □

The following example shows that there are generalized α-contractive selfmaps on non-complete metric spaces, satisfying Theorem 2.1, which have no fixed point while have approximate fixed points.

Example 2.1. Let $X = (0,1)$, $d(x,y) = |x-y|$ and $\alpha(x,y) = 1$ whenever $x^2 = y$ and $\alpha(x,y) = \frac{1}{20}$ otherwise. Define the selfmap T on X by $Tx = x^2$ for all $x \in X$. Also, define $g \in \mathcal{R}$ by $g(x_1,x_2,x_3,x_4,x_5) = \frac{9}{20}(x_2+x_3)$. Then, it is easy to check that T is generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for $x_0 = \frac{1}{2}$.

The following corollaries show us that there are different types of generalized α-contractive mappings satisfying Theorem 2.1. One can provide many examples for each type of such mappings. We shall provide some examples for each type of the maps but one could provide more examples.

Corollary 2.1. *Let (X,d) be a complete metric space and T a continuous, generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has a fixed point.*

Proof. By using a similar argument in proof of Theorem 2.1, we obtain

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) < rd(x_{n-1}, x_n) < r^n d(x_0, x_1),$$

for all n. If suppose $m < n$, then it is easy to see that

$$d(x_m, x_n) \leq (r^m + r^{m+1} + \cdots + r^{n-1})d(x_0, x_1) < \frac{r^m}{1-r} d(x_0, x_1).$$

Thus, $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Moreover, T is continuous, $Tx_n \to Tx^*$ and so $Tx^* = x^*$. □

As we know, Banach proved his contraction principle result in 1922; please, see [3]. Following this direction, we say that the selfmap T is α-contractive whenever exists $\lambda \in (0,1)$ such that $\alpha(x,y)d(Tx,Ty) \leq \lambda d(x,y)$ for all $x, y \in X$.

Corollary 2.2. *Let (X,d) be a metric space and T an α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Then T has the approximate fixed point property.*

Proof. Consider $g \in \mathcal{R}$ given by $g(x_1,x_2,x_3,x_4,x_5) = \lambda x_1$. Then, T is a generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Hence by Theorem 2.1, T has the approximate fixed point property. □

In 2011, Haghi, Rezapour and Shahzad proved that some fixed point generalizations are not real generalizations [11]. The following example shows that these results are real ones.

Example 2.2. Let $X = [0,\infty)$ and $d(x,y) = |x-y|$. Define the selfmap T on X by $Tx = \frac{4}{3}x$ for all $x \in X$ and put $\lambda = \frac{1}{2}$. Then, the selfmap T is not contractive. Let $\lambda \in [0,1)$. Define $\alpha(x,y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{3\lambda}{4}$. It is easy to see that T is α-contractive.

In 1968, the notion of Kannan-contraction is introduced by Kannan [13]. Now, we say that the selfmap T on a metric space is α-Kannan mapping whenever there exists $\beta \in (0, \frac{1}{2})$ such that $\alpha(x,y)d(Tx,Ty) \leq \beta(d(x,Tx)+d(y,Ty))$ for all $x, y \in X$.

Corollary 2.3. *Let (X,d) be a metric space and T an α-Kannan and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has the approximate fixed point property.*

Proof. Consider $g \in \mathcal{R}$ by the formula $g(x_1,x_2,x_3,x_4,x_5) = \beta(x_2+x_3)$. Then, T is a generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0,Tx_0) \geq 1$ for some $x_0 \in X$. By Theorem 2.1, T has the approximate fixed point property. □

Corollary 2.4. *Let (X,d) be a complete metric space and T a continuous, α-Kannan and α-admissible selfmap on X such that $\alpha(x_0,Tx_0) \geq 1$, for some $x_0 \in X$. Then T has a fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1,x_2,x_3,x_4,x_5) = \beta(x_2 + x_3)$. Then, by using Corollary 2.1, it follows that T has a fixed point. □

The following example shows that there exist α-Kannan mappings which are not Kannan maps.

Example 2.3. Let $X = [0,\infty)$ and $d(x,y) = |x-y|$. Define the selfmap T on X by $Tx = \frac{5}{4}x$ for all $x \in X$. Put $\beta = \frac{1}{4}$ and $\alpha(x,y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{1}{20}$. Then, it is easy to check that the selfmap T is not Kannan map whereas T is an α-Kannan map.

In 1972, the notion of Chatterjea-contraction is introduced by Chatterjea in [8]. Now, we say that the selfmap T is α-Chatterjea mapping whenever exists $\beta \in (0, \frac{1}{2})$ such that $\alpha(x,y)d(Tx,Ty) \leq \beta(d(x,Ty) + d(y,Tx))$ for all $x, y \in X$.

Corollary 2.5. *Let (X,d) be a metric space, T an α-Chatterjea and α-admissible selfmap on X such that $\alpha(x_0,Tx_0) \geq 1$, for some $x_0 \in X$. Then T has the approximate fixed point property.*

Proof. Let $g \in \mathcal{R}$ given by the formula $g(x_1,x_2,x_3,x_4,x_5) = \beta(x_4 + x_5)$. Then, T is a generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0,Tx_0) \geq 1$ for some $x_0 \in X$. Hence by using Theorem 2.1, T has the approximate fixed point property. □

Next corollary generalizes the results in the literature about fixed point of Chatterjea selfmaps.

Corollary 2.6. *Let (X,d) be a complete metric space and T a continuous, α-Chatterjea and α-admissible selfmap on X such that $\alpha(x_0,Tx_0) \geq 1$, for some $x_0 \in X$. Then T has a fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1,x_2,x_3,x_4,x_5) = \beta(x_4 + x_5)$. Then by using Corollary 2.1, T has a fixed point. □

The following example shows that there exist α-Chatterjea mappings which are not Chatterjea selfmaps.

Example 2.4. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define the selfmap T on X by $Tx = \frac{4}{3}x$ for all $x \in X$. For each $\beta \in (0, \frac{1}{2})$, put $x = \beta$ and $y = \frac{3}{4}\beta$. Then, it is easy to see that T is not a Chatterjea mapping. If we put $\beta = \frac{1}{4}$ and define $\alpha(x, y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{3}{80}$, then one can easily check that T is an α-Chatterjea selfmap.

In 1972, the notion of Zamfirescu-contraction is introduced by Zamfirescu in [18]. Now, we say that the selfmap T is α-Zamfirescu mapping whenever exists $\beta \in (0, 1)$ such that $\alpha(x, y)d(Tx, Ty) \leq \beta M_T(x, y)$ for all $x, y \in X$, where

$$M_T(x, y) = \max\{d(x, y), \frac{1}{2}[d(x, Ty) + d(y, Tx)], \frac{1}{2}[d(x, Tx) + d(y, Ty)]\}.$$

Corollary 2.7. *Let (X, d) be a metric space and T an α-Zamfirescu and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has the approximate fixed point property.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = \beta \max\{x_1, \frac{1}{2}[x_4 + x_5], \frac{1}{2}[x_2 + x_3]\}$. Then, T is a generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Hence by using Theorem 2.1, T has the approximate fixed point property. □

Next corollary generalizes the results in the literature about fixed point of Zamfirescu selfmaps.

Corollary 2.8. *Consider (X, d) be a complete metric space and T a continuous, α-Zamfirescu and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has a fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = \beta \max\{x_1, \frac{1}{2}[x_4 + x_5], \frac{1}{2}[x_2 + x_3]\}$. Then by using Corollary 2.1, T has a fixed point. □

The following example shows that there exist α-Zamfirescu mappings which are not Zamfirescu selfmaps.

Example 2.5. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define the selfmap T on X by $Tx = \frac{4}{3}x$ for all $x \in X$. For each $\beta \in (0, 1)$, put $x = \beta$ and $y = \frac{3}{4}\beta$. Then, it is easy to see that T is not a Zamfirescu mapping. If we put $\beta = \frac{1}{4}$ and define $\alpha(x, y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{3}{16}$, then one can easily check that T is an α-Zamfirescu selfmap.

In 1971, the notion of Reich-contraction is introduced by Reich ([15]). Following this idea, we say that the selfmap T is α-Reich mapping whenever there exists nonnegative real numbers α, β, γ with $\alpha + \beta + \gamma < 1$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty), \quad \forall\, x, y \in X.$$

Corollary 2.9. *Let (X, d) be a metric space and T an α-Reich and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has an approximate fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = \alpha x_1 + \beta x_2 + \gamma x_3$. Then, T is a generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Hence by using Theorem 2.1, T has the approximate fixed point property. □

Corollary 2.10. *Let (X, d) be a complete metric space and T a continuous, α-Reich and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has a fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = \alpha x_1 + \beta x_2 + \gamma x_3$. Then by using Corollary 2.1, we get that T has a fixed point. □

The following example shows that there exist α-Reich mappings which are not Reich selfmaps.

Example 2.6. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define the selfmap T on X by $Tx = 2x$ for all $x \in X$. For each $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, put $x = 0$ and $y = 1$. Then, it is easy to see that T is not a Reich mapping. If we put $\alpha = \beta = \gamma = \frac{1}{4}$ and define $\alpha(x, y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{1}{8}$, then one can easily check that T is an α-Reich selfmap.

In 1972, the notion of Ćirić-contraction is introduced by Ćirić in [7]. Now, we say that the selfmap T is α-Ćirić mapping whenever there exists $\lambda \in (0, 1)$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \lambda M_T(x, y), \quad \forall\, x, y \in X,$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$.

Corollary 2.11. *Let (X, d) be a metric space and T an α-Ćirić and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has the approximate fixed point property.*

Proof. Let $g \in \mathcal{R}$ given by $g(x_1, x_2, x_3, x_4, x_5) = \lambda \max\{x_1, x_2, x_3, \frac{1}{2}[x_4 + x_5]\}$. Then, T is a generalized α-contractive and α-admissible selfmap on X with $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Hence by using Theorem 2.1, T has the approximate fixed point property. □

Corollary 2.12. *Let (X, d) be a complete metric space and T a continuous, α-Ćirić and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has a fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = \beta \max\{x_1, x_2, x_3, \frac{1}{2}[x_4 + x_5]\}$. Then by using Corollary 2.1, T has a fixed point. □

The following example shows that there exist α-Ćirić mappings which are not Ćirić selfmaps.

Example 2.7. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define the selfmap T on X by $Tx = \frac{3}{2}x$ for all $x \in X$. For each $\lambda \in (0, 1)$, put $x = \lambda$ and $y = \frac{2}{3}\lambda$. Then, it is easy to see that T is not a Ćirić mapping. If $\lambda \in (0, 1)$ and we define $\alpha(x, y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{2}{3}\lambda$, then one can easily check that T is an α-Ćirić selfmap.

In 1972, the notion of Bianchini-contraction is introduced by Bianchini in [6]. Now, we say that the selfmap T is α-Bianchini mapping whenever there exists $h \in (0, 1)$ such that $\alpha(x, y)d(Tx, Ty) \leq h \max\{d(x, Tx), d(y, Ty)\}$ for all $x, y \in X$.

Corollary 2.13. *Let (X, d) be a metric space and T an α-Bianchini and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$, for some $x_0 \in X$. Then T has the approximate fixed point property.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = h \max\{x_2, x_3\}$. Then, T is a generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Hence by using Theorem 2.1, T has the approximate fixed point property. □

Next corollary generalizes the results in the literature about fixed point of Bianchini selfmaps.

Corollary 2.14. *Let (X, d) be a complete metric space, T a continuous, α-Bianchini and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Then T has a fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = h \max\{x_2, x_3\}$. Then by using Corollary 2.1, T has a fixed point. □

The following example shows that there exist α-Bianchini mappings which are not Bianchini selfmaps.

Example 2.8. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define the selfmap T on X by $Tx = \frac{8}{7}x$ for all $x \in X$. For each $h \in (0, 1)$, put $x = h$ and $y = \frac{7}{8}h$. Then, it is easy to see that T is not a Bianchini mapping. If $h = \frac{1}{4}$ and we define $\alpha(x, y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{1}{64}$, then one can easily check that T is an α-Bianchini selfmap.

In 1973, the notion of Hardy-Rogers-contraction is introduced by Hardy and Rogers [12]. Now, we say that the selfmap T is α-Hardy-Rogers mapping whenever there exist nonnegative real numbers a, b, c, e, f with $a + b + c + 2\max\{e, f\} < 1$ such that

$$\alpha(x, y)d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(y, Tx) + fd(x, Ty)$$

for all $x, y \in X$.

Corollary 2.15. *Let (X, d) be a metric space and T an α-Hardy-Rogers and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Then T has the approximate fixed point property.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = ax_1 + bx_2 + cx_3 + ex_4 + fx_5$. Then, T is a generalized α-contractive and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Hence by using Theorem 2.1, T has the approximate fixed point property. □

The following corollary generalizes the results in the literature about fixed point of Hardy-Rogers selfmaps.

Corollary 2.16. *Let (X, d) be a complete metric space and T a continuous, α-Hardy-Rogers and α-admissible selfmap on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Then T has a fixed point.*

Proof. Define $g \in \mathcal{R}$ by $g(x_1, x_2, x_3, x_4, x_5) = ax_1 + bx_2 + cx_3 + ex_4 + fx_5$. Then by using Corollary 2.1, T has a fixed point. □

The following example shows that there exist α-Hardy-Rogers mappings which are not Hardy-Rogers selfmaps.

Example 2.9. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define the selfmap T on X by $Tx = 3x$ for all $x \in X$. For each nonnegative real numbers a, b, c, e, f with $a + b + c + 2\max\{e, f\} < 1$, put $x = 0$ and $y = 1$. Then, it is easy to see that T is not a Hardy-Rogers mapping. If we put $a = b = c = e = f = \frac{1}{8}$ and define $\alpha(x, y) = \mu$ for all $x, y \in X$, where $\mu \leq \frac{1}{24}$, then one can easily check that T is an α-Hardy-Rogers selfmap.

3. **Concluding remarks**

In this article, we introduced the concept of generalized α-contractive mapping and gave results about approximate fixed points and fixed points of mappings on metric spaces. We proved that these results generalize some classical results in the literature. The present work follows the direction of previous research articles, such as Samet, Vetro and Vetro [17], Aleomraninejad, Rezapour and Shahzad [1].

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