IMPLICIT ITERATIVE SCHEMES OF STRICTLY PSEUDOCONTRACTIVE OPERATORS IN UNIFORMLY CONVEX BANACH SPACES

Liping Yang[1], Zhaohui Li[2], Jinxin Wang[3]

We study the strong convergence of an implicit iteration scheme for a strictly pseudo-contractive operator in a real uniformly convex Banach space E whose norm is Fréchet differentiable with mild conditions. Additionally, it proves a weak convergence theorem for a strictly pseudo-contractive operator without the Opial property. Results proved in this paper represent an extension and refinement of the previously known results in this area.

Keywords: Strictly pseudo-contractive operators, Implicit iteration approaches, Uniformly convex Banach spaces, Fixed points

1. Introduction

As the theoretical basis and basic tool of nonlinear science, nonlinear operator theory has become a basic branch of modern mathematics. Fixed point theory is also a fascinating emerging field in modern mathematics and can be deemed to a heart subject in studying nonlinear analysis. In particular, it has several important applications in connection with numerical analysis, physics, engineering, economics, biology, game theory, optimization theory. Best proximity theory has attracted the attention of many mathematicians working in game theory and optimization theory. Wang et al. [18] introduced an adaptive fixed-point proximity algorithm to solve a class of unconstrained optimization problem. In fuzzy game theory, because fuzzy fixed point theory or fuzzy common fixed point theory can be used to prove the existence of equilibrium solution, Tian et al. [13] introduced a new FMS (fuzzy metric space)-tripled fuzzy metric space, and introduced topological properties, Cauchy sequences and completeness of the tripled fuzzy metric space. By Using Meir-Keeler type contraction condition, Lakzian and Rhoades [6] mainly obtained two new fixed point theorems defined in complete metric spaces with $w-$distance.

In the sequel, let $J : E \to 2^{E^*}$ be the duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E,$$

where E^* denotes the topological dual of a Banach space E with norm $\|\cdot\|$ and $\langle x, v\rangle$ is the duality pairing $v(x)$ of $x \in E$ and $v \in E^*$.

Let $K \subset E$ be a nonempty subset and $f : K \to K$ be said to be a contraction operator if there exists a constant $\alpha \in [0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \tag{1.1}$$

for all x, $y \in K$. A self-operator $T : K \to K$ is said to be nonexpansive if inequality (1.1) holds for $\alpha = 1$.

[1]School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510520, P. R. China, e-mail: yanglp2019@gdut.edu.cn

[2]School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510520, P. R. China, e-mail: 1298557505@qq.com

[3]School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510520, P. R. China, e-mail: gamgamwong@163.com

A self-operator $T : K \to K$ is said to be a pseudo-contraction if we have

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$$

holds for some $j(x-y) \in J(x-y)$ and each $x, y \in K$.

Let every $x, y \in K$ and $j(x-y) \in J(x-y)$. If we have

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2 \tag{1.2}$$

holds for every constant $\lambda > 0$, then self-operator $T : K \to K$ is said to be a $\lambda-$strictly pseudo-contraction [2]. We may assume $0 < \lambda < 1$.

An operator $T : K \to K$ is said to be a strong pseudo-contraction, if we have

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2$$

holds for every $k \in (0,1)$ and $\forall x, y \in K$.

Indeed, (1.2) can be rewritten in the following form

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda\|(I-T)x - (I-T)y\|^2, \tag{1.3}$$

where I is the identity operator. And if T is $\lambda-$strictly pseudo-contraction, it can prove from (1.3) that T is Lipschitzian continuous with $L \geq (1+\lambda)/\lambda$. Meanwhile, we note that the class of $\lambda-$strictly pseudo-contraction mappings is not dependent of the class of strong pseudo-contractive operators.

Example 1.1. *Let $\mathbb{R}$ denote the set of real numbers with the usual norm. Let $T : \mathbb{R} \to \mathbb{R}$ be an operator defined by $Tx = -3x$. Then we have*

$$|x - Tx - (y - Ty)|^2 = (1+3)^2|x-y|^2,$$

and

$$\langle x - Tx - (y - Ty), x - y\rangle = (1+3)|x-y|^2.$$

Hence T is a strictly pseudo-contractive operator with unique fixed point $x^* = 0$ but not a strong pseudo-contraction and not a nonexpansive operator.

In 1967, Browder and Petryshyn [2] introduced $\lambda-$strictly pseudo-contractive operators, however iterative methods of $\lambda-$strictly pseudo-contractive operators are far less developed than those for nonexpansive operators, the reason is maybe that the second item to the right of equation (1.2) is obstruct of the convergence analysis for iterative algorithms used to find a fixed point of the $\lambda-$strictly pseudo-contraction. Indeed, in solving inverse problems (see, for example, Scherzer [9]), $\lambda-$strictly pseudo-contractive operators have more widely used than nonexpansive operators do. Hence, it is more interesting to study the iterative manner of $\lambda-$strictly pseudo-contractive operators. Yang [20] established the strong convergence of the Ishikawa iterative process for a $\lambda-$strictly pseudo-contractive operator in Banach spaces under mild assumptions. Yao et al. [22] studied the split feasibility problem and the fixed point problem involved in the pseudo-contractive mappings.

Construction algorithms for fixed points problems is an important subject in nonlinear operator theory and its applications to convex programming, feasibility problems, image processing, and much more. In this respect, please see: Sahu et al. [8], Usurelu et al. [14–17], Wang et al. [19]. We point out that an implicit process is generally desirable when no explicit scheme is available. Such a process is generally used as a "tool" to establish the convergence of an explicit scheme. Suzuki [12] introduced the following implicit iteration scheme

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad u \in K, \quad n \geq 1$$

for the nonexpansive semigroup case $\mathfrak{T} = \{T(t) : 0 \leq t < \infty\}$ in Hilbert spaces. Since then construction of fixed points of mappings via the implicit iterative algorithm has been investigated extensively by many authors (see, e.g., [7, 11, 21] and the references therein).

It is important to know whether implicit iterative procedures can be extended to the $\lambda-$strictly pseudo-contractive operators.

Motivated and inspired by the above described works, the main objective of this paper is to study the convergence of implicit iteration scheme for $\lambda-$strictly pseudo-contractive operators in Banach spaces. We shall prove that implicit iterative processes converge strongly to a fixed point of strictly pseudo-contractive operators in the framework of uniformly smooth and strictly convex Banach spaces. Additionally, we shall prove a weak convergence theorem for the strictly pseudo-contractive operators without the Opial property.

2. **Preliminaries**

We shall need the following basic concepts, notations and lemmas throughout this paper.

Let $(E, \|\cdot\|)$ be Banach spaces with dimension $E \geq 2$. The function $\delta_E : (0,2] \to [0,1]$ is the modulus of convexity of E defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}$$

for every $\varepsilon > 0$. A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A Banach space E is said to be strictly convex if $\|x+y\|/2 < 1$ holds for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

Let $S(E) = \{z \in E : \|z\| = 1\}$. The space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. The norm of E is said to be Fréchet differentiable, if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. The norm of E is said to be uniformly Fréchet differentiable if the limit (2.1) is attained for $x, y \in S(E)$. We known that the Sobolev spaces $W^{k,p}$ $(1 < p < \infty)$ and the spaces L^p $(1 < p < \infty)$ are both uniformly convex and uniformly Fréchet differentiable.

The function $\rho_E : [0,\infty) \to [0,\infty)$ is the modulus of smoothness of E defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| \leq \tau\right\}$$

for every $\tau > 0$. A Banach space E is said to be uniformly smooth if $\rho_E(\tau)/\tau \to 0$ as $\tau \to 0^+$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. We know that if E is said to be smooth, then the normalized duality mapping J is single-valued and continuous from the strong topology to the weak star topology.

Recall that the mapping $T : K \to K$ is semi-compact if any sequence $\{x_n\} \subseteq K$ satisfying $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ has a convergent subsequence.

Lemma 2.1. *(see [4]) Let E be a real Banach space with the Fréchet chet differentiable norm. For each $x \in E$, let $\rho^*(t)$ be defined for $t \in (0,\infty)$ by*

$$\rho^*(t) = \sup\left\{\frac{\|x+ty\|^2 - \|x\|^2}{t} - 2\langle y, J(x)\rangle : y \in S(E)\right\}.$$

Then $\rho^*(t) > 0$, $\rho^*(t)$ is an increasing for $t > 0$ and $\lim_{t\to t^+} \rho^*(t) = 0$ such that

$$\|x+h\|^2 \leq \|x\|^2 + 2\langle h, J(x)\rangle + \|h\|\rho^*(\|h\|) \tag{2.2}$$

for all $h \in E\backslash\{0\}$.

Zhou [23] pointed out that $\rho^*(t)$ depends on $x \in E$. For $x \in E\backslash\{0\}$, $\rho^*(t) \geq t$ for $t > 0$. For $x = 0$, $\rho^*(t) = t$ for $t > 0$. If one defines $\rho^*(0) = 0$, then Equation (2.2) holds for all $x, h \in E$.

In the sequel, we will assume that

$$\rho^*(t) \leq st, \tag{2.3}$$

where $s \geq 1$ is some fixed constant.

Combining (2.2) and (2.3), we have

$$\|x+h\|^2 \le \|x\|^2 + 2\langle h, J(x)\rangle + s\|h\|^2 \tag{2.4}$$

for all x, $h \in E$, where J is the normalized duality map from E to E^*.

Lemma 2.2. *(see [10]) Let $(E, \|\cdot\|)$ be a uniformly convex Banach space, and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{\alpha_n\}$ is a real sequence in $[a,b]$ and that $\{x_n\}$, $\{y_n\} \subset E$ such that*

$$\limsup_{n\to\infty}\|x_n\| \le d, \quad \limsup_{n\to\infty}\|y_n\| \le d, \quad \lim_{n\to\infty}\|\alpha_n x_n + (1-\alpha_n)y_n\| = d,$$

for some constant $d \ge 0$. Then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 2.3. *(see [3, Lemma 1.12]) For every normed space E. Then $\frac{\delta_E(s)}{s}$ is a nondecreasing function on $(0,2]$.*

A Banach space is said to have the Kadec-Klee property if, whenever $x \in \varpi_w(\{x_n\})$ with $\lim_{n\to\infty}\|x_n\| = \|x\|$, it follows that $\lim_{n\to\infty} x_n = x$ strongly, where $\varpi_w(\{x_n\}) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak limit set of $\{x_n\}$ and $\{x_{n_j}\} \subset \{x_n\}$.

Lemma 2.4. *(see [5, Lemma 2]) Assume that E is a real reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E, with p_1, $p_2 \in \varpi_w(\{x_n\})$. Suppose that $\lim_{n\to\infty}\|\alpha x_n + (1-\alpha)p_1 - p_2\|$ exists for all $\alpha \in [0,1]$. Then $p_1 = p_2$.*

Lemma 2.5. *(see [1]) Let K be a nonempty closed convex subset of a real uniformly smooth Banach space E. Suppose that T is a nonexpansive mapping of K into itself such that $F(T) \neq \emptyset$. Let $\{x_n\} \subset K$ be a sequence such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $\{x_n\}$ converge weakly to z. Then z is fixed point of T.*

Lemma 2.6. *Let K be a nonempty convex subset of a real uniformly convex Banach space E whose norm is Fréchet differentiable and $T : K \to K$ be $\lambda-$strictly pseudo-contractive mappings. For $x \in K$, we define $T_\alpha x = (1-\alpha)x + \alpha Tx$, where $0 \le \alpha < \mu$, and $\mu = \frac{\lambda}{s}$. Then $T_\alpha : K \to K$ is nonexpansive mapping such that $F(T_\alpha) = F(T)$.*

Proof. For any x, $y \in K$, it follows from Equations (1.3) and (2.4) that

$$\begin{aligned}
\|T_\alpha x - T_\alpha y\|^2 &= \|(1-\alpha)x + \alpha Tx - (1-\alpha)y - \alpha Ty\|^2 \\
&= \|(x-y) + \alpha(Tx - Ty - (x-y))\|^2 \\
&\le \|x-y\|^2 + 2\alpha\langle Tx - Ty - (x-y), j(x-y)\rangle \\
&\quad + s\alpha^2\|Tx - Ty - (x-y)\|^2 \\
&\le \|x-y\|^2 - 2\lambda\alpha\|Tx - Ty - (x-y)\|^2 \\
&\quad + s\alpha^2\|Tx - Ty - (x-y)\|^2 \\
&\le \|x-y\|^2 - \lambda\alpha\|Tx - Ty - (x-y)\|^2 \\
&\le \|x-y\|^2,
\end{aligned}$$

which implies that T_α is a nonexpansive mapping.

It is easy to see that $T_\alpha x = x \Longleftrightarrow Tx = x$. This completes the proof. □

Remark 2.7. *Lemma 2.6 of this paper extends Lemma 1.2 of [24] from 2-uniformly smooth Banach spaces to uniformly smooth Banach spaces.*

Let K be a closed convex subset of a uniformly smooth Banach space E and $T : K \to K$ be a $\lambda-$strictly pseudo-contractive mapping. For $x_0 \in K$, $n \ge 1$, compute the implicit iteration process $\{x_n\}$ defined by the following formula:

$$x_n = (1-c_n)x_{n-1} + c_n T_\alpha x_n, \tag{2.5}$$

where $\{c_n\} \subset (0,\mu)$ and $\mu = \frac{\lambda}{s}$.

For u, $v \in K$, and every $n \geq 1$, we define

$$T_n^v(u) = (1-c_n)v + c_n T_\alpha u. \tag{2.6}$$

Since each $T_n^v : K \to K$ is a contraction, it follows from the Banach Contraction Principle that each x_n in (2.5) is uniquely defined.

The next result deal with the general behavior of the implicit iterative processes of (2.5).

Lemma 2.8. *Let E be a uniformly convex Banach space whose norm is Fréchet differentiable, let K be a nonempty convex subset of E, and let b be a constant with $0 < b < 1$. Let $T : K \to K$ be a $\lambda-$strictly pseudo-contractive mapping , and $F(T) \neq \emptyset$, Assume that $\{x_n\}$ is defined by (2.5) with $b < c_n < \mu$ for all $n \geq 1$ and $p \in F(T)$. Then*

(i) *there exists $c \geq 0$ such that $\lim_{n\to\infty}\|x_n - p\| = c$.*
(ii) $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. It follows from (2.5) and Lemma 2.6 that

$$\begin{aligned}\|x_n - p\| &= \|(1-c_n)(x_{n-1}-p) + c_n(T_\alpha x_n - p)\| \\ &\leq (1-c_n)\|x_{n-1}-p\| + c_n\|T_\alpha x_n - p\| \\ &\leq (1-c_n)\|x_{n-1}-p\| + c_n\|x_n - p\|. \end{aligned} \tag{2.7}$$

It follows from (2.7) that $\|x_n - p\| \leq \|x_{n-1} - p\|$. Therefore, there exists $c \geq 0$ such that $\lim_{n\to\infty}\|x_n - p\| = c$.

Since T_α is a nonexpansive mapping, we have $\limsup_{n\to\infty}\|T_\alpha x_n - p\| \leq c$. Note that $x_n - p = (1-c_n)(x_{n-1}-p) + c_n(T_\alpha x_n - p)$, it follows from Lemma 2.2 that we have $\lim_{n\to\infty}\|x_{n-1} - T_\alpha x_n\| = 0$. From (2.5), we have $x_n - x_{n-1} = c_n(x_{n-1} - T_\alpha x_n)$, then we obtain $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$.

It follows from $\lim_{n\to\infty}\|x_{n-1} - T_\alpha x_n\| = 0$, $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$ and $\|x_n - T_\alpha x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_\alpha x_n\|$ that we get $\lim_{n\to\infty}\|x_n - T_\alpha x_n\| = 0$. Since $x_n - T_\alpha x_n = \alpha(x_n - Tx_n)$ and $0 < \alpha < 1$, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. This completes the proof. □

3. Main Results

In this section, we will first prove the strong convergence for a strictly pseudo-contractive mapping in uniformly convex Banach spaces.

Theorem 3.1. *Let K be a closed and convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable, b be a constant with $0 < b < 1$. Let $T : K \to K$ be a semi-compact $\lambda-$strictly pseudo-contractive mapping, and $F(T) \neq \emptyset$, and let $\{x_n\}$ be defined by (2.5) with $b < c_n < \mu$ for all $n \geq 1$. Then $\{x_n\}$ converges strongly to some point in $F(T)$.*

Proof. Since T is a semi-compact and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$. Since T_α is a nonexpansive mapping, we have

$$\begin{aligned} 0 \leq \|p - T_\alpha p\| &= \|(p - x_{n_k}) + (x_{n_k} - T_\alpha x_{n_k}) + (T_\alpha x_{n_k} - T_\alpha p\| \\ &\leq \|p - x_{n_k}\| + \|x_{n_k} - T_\alpha x_{n_k}\| + \|T_\alpha x_{n_k} - T_\alpha p\| \\ &\leq 2\|p - x_{n_k}\| + \|x_{n_k} - T_\alpha x_{n_k}\| \\ &\to 0, \quad as \quad k \to \infty. \end{aligned}$$

This implies that p is a fixed point of T_α. It follows from Lemma 2.6 that p is a fixed point of T. It follows from Lemma 2.8(i) that $\lim_{n\to\infty}\|x_n - p\|$ exists, then we have $\lim_{n\to\infty}\|x_n - p\| = 0$. This completes the proof. □

Example 3.2. *Let E be a real line R and $K = [-1,1]$. Define $T : K \to K$ by $Tx = -x$. Then T is a $\frac{1}{2}-$strictly pseudo-contractive mapping with $F(T) = \{0\}$. Take $\alpha_n = \frac{1}{n}$, define $x_1 = 1$ and $x_n = (1-\alpha_n)x_{n-1} + \alpha_n T x_n$ for $n \geq 2$, then $\{x_n\}$ converges strongly to fixed point of T.*

Lemma 3.3. *Let K be a closed and convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable, b be a constant with $0 < b < 1$. Let $T : K \to K$ be a $\lambda-$strictly pseudo-contractive mapping, and $F(T) \neq \emptyset$, and let $\{x_n\}$ be defined by (2.5) with $b < c_n < \mu$ for all $n \geq 1$. Then for ω_1, $\omega_2 \in F(T_\alpha)$, $\lim_{n\to\infty} \|tx_n + (1-t)\omega_1 - \omega_2\|$ exists for all $t \in [0,1]$.*

Proof. Let $d_n(t) = \|tx_n + (1-t)\omega_1 - \omega_2\|$. Obviously $\lim_{n\to\infty} \|\omega_1 - \omega_2\|$ exists and from Lemma 2.8(i) that $\lim_{n\to\infty} d_n(1) = \lim_{n\to\infty} \|x_n - \omega_2\|$ exists. It now remains to prove the lemma for $t \in (0,1)$. Set

$$Q_n^\omega(u) := (1-c_n)\omega + c_n T_\alpha u,$$

where $n \in \mathbb{N}$, $u \in K$, $\omega \in K$. Since $Q_n^\omega(u) : K \to K$ is a contraction mapping, by the Banach Contraction Principle there exists a unique $u_{n,\omega} \in K$ such that $Q_n^\omega(u_{n,\omega}) = u_{n,\omega}$. Then for $n \in \mathbb{N}$, we can define the mapping $S_n : K \to K$ by $S_n(\omega) = u_{n,\omega}$ for each $\omega \in K$. And we have $x_{n+1} = S_n(x_n)$ for all $n \in \mathbb{N}$. Next we will prove that each S_n is nonexpansive. Indeed, for any v_1, $v_2 \in K$. Note that $S_n(v_i) = u_i$ $(i = 1,2)$ if and only if

$$u_i = (1-c_n)v_i + c_n T_\alpha u_i.$$

Therefore

$$\begin{aligned} \|S_n(v_1) - S_n(v_2)\| &= \|u_1 - u_2\| \\ &\leq (1-c_n)\|v_1 - v_2\| + c_n\|T_\alpha u_1 - T_\alpha u_2\| \\ &\leq (1-c_n)\|v_1 - v_2\| + c_n\|u_1 - u_2\|. \end{aligned}$$

It follows from that

$$\|u_1 - u_2\| \leq (1-c_n)\|v_1 - v_2\| + c_n\|u_1 - u_2\|,$$

which implies that

$$\|S_n(v_1) - S_n(v_2)\| = \|u_1 - u_2\| \leq \|v_1 - v_2\|. \tag{3.1}$$

Hence, each S_n is a nonexpansive mapping.

Let $\omega \in F(T_\alpha)$, then we have $T_\alpha \omega = \omega$. Hence $(1-c_n)\omega + c_n T_\alpha \omega = \omega$. This implies that $F(T_\alpha) \subset \bigcap_{n=1}^\infty F(S_n)$. Define

$$V_{n,m} = S_{n+m-1} \circ S_{n+m-2} \circ \cdots \circ S_n, \quad for \quad m \geq 1.$$

Then $V_{n,m} : K \to K$ is a nonexpansive mapping and $V_{n,m}\omega = \omega$, $V_{n,m}x_n = x_{n+m}$ for $\omega \in F(T_\alpha)$, m, $n \in \mathbb{N}$. If $\|x_n - \omega_1\| = 0$ for some $n_0 \in \mathbb{N}$, then we have $x_n = \omega_1$ for all $n \in \mathbb{N}$. Indeed, if $n < n_0$, it follows from (2.5) that we have $x_{n_0-1} = x_{n_0-2} = \cdots = x_1 = \omega$. If $n > n_0$, since the sequence $\{\|x_n - \omega_1\|\}_{n=1}^\infty$ is nonincreasing, the we also have $x_n = \omega_1$. Therefore, we may assume that $\|x_n - \omega_1\| > 0$ for all $n \in \mathbb{N}$. Set

$$\begin{aligned} &u_n^t := tx_n + (1-t)\omega_1, \\ &\alpha_{n,m} := V_{n,m}(u_n^t) - tV_{n,m}x_n - (1-t)V_{n,m}\omega_1, \\ &\beta_n := t(1-t)\|x_n - \omega_1\|, \\ &\gamma_{n,m} := \frac{V_{n,m}\omega_1 - V_{n,m}(u_n^t)}{t\|x_n - \omega_1\|}, \\ &\delta_{n,m} := \frac{V_{n,m}(u_n^t) - V_{n,m}x_n}{(1-t)\|x_n - \omega_1\|}. \end{aligned}$$

Since $V_{n,m} : K \to K$ is a nonexpansive mapping, then we obtain that $\|\gamma_{n,m}\| \leq 1$ and $\|\delta_{n,m}\| \leq 1$. By the definition of uniformly convex space, we have

$$\begin{aligned} \|tx + (1-t)y\| &\leq 1 - 2\min\{t, 1-t\}\delta_E(\|x-y\|) \\ &\leq 1 - 2t(1-t)\delta_E(\|x-y\|) \end{aligned} \tag{3.2}$$

for all $t \in (0,1)$ and x, $y \in E$ such that $\|x\| \leq 1$ and $\|y\| \leq 1$. Hence, it follows from (3.2) that we have

$$2t(1-t)\delta_E(\|\gamma_{n,m} - \delta_{n,m}\|) \leq 1 - \|t\gamma_{n,m} + (1-t)\delta_{n,m}\|. \tag{3.3}$$

Note that

$$\gamma_{n,m} - \delta_{n,m} = \frac{\alpha_{n,m}}{\beta_n},$$
$$t\gamma_{n,m} + (1-t)\delta_{n,m} = \frac{\|V_{n,m}x_n - V_{n,m}\omega_1\|}{\|x_n - \omega_1\|} = \frac{\|x_{n+m} - \omega_1\|}{\|x_n - \omega_1\|}.$$

It follows from (3.3) that

$$2\beta_n \delta_E\left(\frac{\|\alpha_{n,m}\|}{\beta_n}\right) \le \|x_n - \omega_1\| - \|x_{n+m} - \omega_1\|. \tag{3.4}$$

It follows from Lemma 2.3 that $\frac{\delta_E(s)}{s}$ is nondecreasing, and $\lim_{n\to\infty}\|x_n - \omega_1\|$ exists, $\delta_E(0) = 0$. The continuity of δ_E gives from equation (3.4) that $\liminf_n(\limsup_m \|\alpha_{n,m}\|) = 0$ uniformly for all m. That is $\liminf_n(\limsup_m \|V_{n,m}(u_n^t) - tV_{n,m}x_n - (1-t)V_{n,m}\omega_1\|) = 0$. On the other hand, we have

$$\begin{aligned} d_{n+m}(t) &\le \|tx_{n+m} + (1-t)\omega_1 - \omega_2 + (V_{n,m}(u_n^t) - tV_{n,m}x_n - (1-t)V_{n,m}\omega_1)\| \\ &\quad + \|-(V_{n,m}(u_n^t) - tV_{n,m}x_n - (1-t)V_{n,m}\omega_1)\| \\ &= \|V_{n,m}(u_n^t) - \omega_2\| + \|V_{n,m}(u_n^t) - tV_{n,m}x_n - (1-t)V_{n,m}\omega_1\| \\ &\le d_n(t) + \|V_{n,m}(u_n^t) - tV_{n,m}x_n - (1-t)V_{n,m}\omega_1\|. \end{aligned}$$

Therefore, $\limsup_{n\to\infty} d_n(t) \le \liminf_{n\to\infty} d_n(t)$. This implies that $\lim_{n\to\infty} d_n(t)$ exists for all $t \in [0,1]$. This completes the proof. □

Now, we prove the weak convergence of the implicit iterative processes (2.5) for $\lambda-$strictly pseudo-contractive mappings.

Theorem 3.4. *Let E be a uniformly convex Banach space E whose norm is Fréchet differentiable such that its dual E^* has the Kadec-Klee property, let K be a closed and convex subset of E. Let $T : K \to K$ be a $\lambda-$strictly pseudo-contractive mapping, and $F(T) \neq \emptyset$, and let $\{x_n\}$ be defined by* (2.5) *with $b < c_n < \mu$ for all $n \ge 1$, where b be a constant with $0 < b < 1$. Then there exists a fixed point $p \in F(T)$ such that $x_n \rightharpoonup p$.*

Proof. It follows from Lemma 2.8(i) that the sequence $\{x_n\}$ is bounded. Since E is a uniformly convex Banach space, then $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_i}\}$. We assume that there exists $\omega \in E$ such that $\{x_{n_i}\}$ converges weakly to ω for $i \to \infty$. Note that $\{x_n\} \subset K$ and K is weakly closed, then $\omega \in K$. It follows from Lemma 2.8(ii) that $\lim_{n\to\infty}\|x_n - T_\alpha x_n\| = 0$. By Lemma 2.5, we obtain $\omega \in F(T_\alpha)$. Assume that $\{x_n\}$ does not converge weakly to ω. Then there exists $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $p \neq \omega$. As in the case of ω, we can obtain $p \in K$ and $p \in F(T_\alpha)$. By Lemma 3.3, $\lim_{n\to\infty}\|tx_n + (1-t)p - \omega\|$ exists for $t \in [0,1]$. It follows from Lemma 2.4 that $\omega = p$. Therefore $\{x_n\}$ converges weakly to p. This completes the proof. □

Remark 3.5. *Since the dual of reflexive Banach spaces with the Opial property or a Fréchet differentiable norm has the Kadec-Klee property, Theorem 3.4 generalizes the known ones.*

4. **Conclusions**

The problem of finding a fixed point of operators is of particular importance since it covers many monotone inclusion and optimization problems that appear in applications. In this paper, we introduced a new implicit iterative algorithm with strong convergence and weak convergence for $\lambda-$strictly pseudo-contractive operators. An interesting question of future research is to investigate to what extent the algorithm with variable step sizes can be applied the solving of monotone inclusion problems.

Acknowledgements

The authors thank the anonymous referees and the editor for their valuable comments and suggestions which have improved the quality of the paper.

This work was supported in part by the National Natural Science Foundation of China (Grand No. 61973092), the National Social Science Fund of China (Grand No. 19BGL094), the Natural Science Foundation of Guangdong Province (Grant No. 2019A1515012104 and 2015A030313485), and the Guangzhou Science and Technology Project (Grant No. 201707010494).

The authors declare that there is no conflict of interest regarding the publication of this paper.

REFERENCES

[1] *F. E. Browder*, Nonlinear operators and nonlinear equations of evolution in Banach spaces, in Nonlinear Functional Analysis (Chicago, 1968), Amer. Math. Soc. Providence, 1976, 1-308.

[2] *F. E. Browder and W. V. Petryshyn*, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl., **20** (1967), 197-228.

[3] *C. Chidume*, Geometric Properties of Banach Spaces and Nonlinear Iterations, Springer-Verlag, London, 2009.

[4] *P. Cholamjiak and S. Suantai*, Weak convergence theorems for a countable family of strict pseudo-contractions in Banach spaces, Fixed Point Theory Appl., **2010** (2010), Art. ID 632137, 16 pages.

[5] *W. Kaczor*, Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups, J. Math. Anal. Appl., **272**(2002), No. 2, 565-574.

[6] *H. Lakzian and B. E. Rhoades*, Some fixed point theorems using weaker Meir-Keeler function in metric spaces with w-distance, Appl. Math. Comput., **342** (2019), 18-25.

[7] *M. O. Osilike*, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl., *294*(2004), 73-81.

[8] *D. R. Sahu, A. Pitea and M. Verma*, A new iteration technique for nonlinear operators as concerns convex programming and feasibility problems, Numer. Algor., **83** (2020), No. 2, 421-449.

[9] *O. Scherzer*, Convergence criteria of iterative methods based on Landweber iteration for solving nonliner problems, J. Math. Anal. Appl., **194**(1991), 911-933.

[10] *J. Schu*, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, Bull. Aust. Math. Soc., **43**(1991), No. 1, 153-159.

[11] *Y. Su and S. Li*, Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl., **320**(2006), 882-891.

[12] *T. Suzuki*, On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces, Proc. Am. Math. Soc., **131**(2002), 2133-2136.

[13] *J. F. Tian, M. H. Ha and D. Z. Tian*, Tripled fuzzy metric spaces and fixed point theorem, Inform. Sciences, **518**(2020), 113-126.

[14] *G. I. Usurelu*, Split feasibility handled by a single-projection three-step iteration with comparative analysis, J. Nonlinear Convex Anal., **22**(2021), No. 3, 544-558.

[15] *G. I. Usurelu, A. Bejenaru and M. Postolache*, Newton-like methods and polynomiographic visualization of modified Thakur processes, Int. J. Comput. Math., **98**(2021), No. 5, 1049-1068.

[16] *G. I. Usurelu, A. Bejenaru and M. Postolache*, Operators with property (E) as concerns numerical analysis and visualization, Numer. Funct. Anal. Optim., **41** (2020), 1398-1411.

[17] G. I. Usurelu, T. Turcanu, Best proximity results for (EP)-operators with qualitative analysis and simulation. Math. Comput. Simulat. **187**(2021), 215-230.

[18] *J. H. Wang, F. Y. Meng, L. P. Pang and X. H. Hao*, An adaptive fixed-point proximity algorithm for solving total variation denoising models, Inform. Sciences, **402**(2017), 69-81.

[19] *Y. Wang, J. Chen and A. Pitea*, The split equality fixed point problem of demi-contractive operators with numerical example and application, Symmetry-Basel, **12** (2020), No. 6, Art. No. 902.

[20] *L. Yang*, Strong convergence theorems of an iterative scheme for strictly pseudocontractive mappings in Banach spaces, Optimization, **67**(2018), 855-863.

[21] *L. Yang and P. Chen*, Composite implicit iteration process for two finite families of generalized asymptotically quasi-nonexpansive mappings, Numer. Funct. Anal. Optim., **35**(2014), No. 2, 240-261.

[22] *Y. Yao, R. P. Agarwal, M. Postolache and Y. C. Liu*, Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem, Fixed Point Theory Appl., **2014** (2014), Art. No. 183, 1-14.

[23] *H. Zhou*, Weak convergence theorems for strict pseudo-contractions in Banach spaces, Acta Math. Sin (Engl. Series), **30**(2014), 755-766.

[24] *H. Zhou*, Convergence theorems for $\lambda-$strict pseudo-contractions in $2-$uniformly smooth Banach spaces, Nonlinear Anal., **69**(2008), 3160-3173.