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ZERO SET OF IDEALS IN BEURLING ALGEBRAS

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In this paper, for a locally compact Abelian group G with a non quasi analytic weight w of G, we investigate the zero set of ideals for Beurling algebra $L^1(G,w)$. Also, we obtain the weighted version of Plancherel theorem. In particular, we show that if the zero set of an ideal I of $L^1(G,w)$ is empty then $I = L^1(G,w)$.

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1. Introduction

Various aspects of the cohomologies of Beurling algebras have been studied by several authors, most notable are Gronbaek [6], Dales and Lau [2], Grahramani, Loy and Zhang [5] and Mewomo and Maepa [12]. For details on the cohomologies of Banach algebras, see [10, 11, 13, 14] and the references cited therein. Beurling algebras are L^1-algebras associated with locally compact groups G with an extra weight ω on the groups.
Dales and Lau in [2] studied these algebras and their second duals. Weak amenability and 2-weak amenability of Beurling algebras are studied by Samei in [16]. In the Abelian case, their harmonic analysis properties have been studied among others by Beurling [1] and Domar [3]. Domar [3] proved the following results:

Theorem 1.1. *Let G be a locally compact Abelian group and w a weight on G. For every neighborhood N of the identity e in $\widehat{G}$ (the dual group of G), there exists $f_N \in L^1(G,w)$ such that supp$\widehat{f_N} \subset N$, $\widehat{f_N}(e) \neq 0$, if and only if w satisfies*

$$\sum_{n=1}^{\infty} \frac{\log w(nt)}{n^2} < \infty \quad \forall t \in G.$$

In case this is true, the algebra $L^1(G,w)$ has the wiener property, i.e. for every proper closed ideal I of $L^1(G,w)$, there exists $\gamma \in \widehat{G}$ such that $I \subset \{f \in L^1(G,w) | \widehat{f}(\gamma) = 0\}$.

Rudin in [17] studied the zero set of ideals I of group algebra $L^1(G)$ for some locally compact Abelian group G. With this we can easily characterize the closed ideals of $L^1(G)$.
Motivated by the work of Rudin in [17] and the above result, in this paper, for a locally compact Abelian group G, we use the fact that the Beurling algebra $L^1(G,w)$ is a regular Banach algebra if the weight w is non quasi analytic and study the notion of zero set of ideals for ideals of Beurling algebra $L^1(G,w)$. Also in Theorem 3.2, we obtain the weighted version of Plancherel theorem [17]. Finally we show that if the zero set of an ideal I of $L^1(G,w)$ is empty then $I = L^1(G,w)$.

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2. **Beurling Algebras**

In this section, we establish some results on Beurling algebras needed in the next section. Let G be a locally compact group. We denote by $L^1(G)$ the group algebra of G.
Let w be a Borel measurable function, $w : G \to [1,\infty)$ such that $w(s+t) \leq w(s).w(t)$ and $w(0) = 1$, then w is called a weight function on G. The Beurling algebra $L^1(G,w)$ is defined as the set of all (equivalence classes of) measurable functions $f : G \to \mathbb{C}$ such that

$$\|f\|_w = \int_G \mid f(x) \mid w(x)dx < \infty,$$

and equipped with the convolution product $\star$ of functions; that is for $f, g \in L^1(G,w)$ and $x \in G$,

$$f \star g(x) = \int_G f(x-y)g(y)dy.$$

Also, let $L^\infty(G, \frac{1}{w})$ be the space of all measurable complex-valued function ϕ on G, such that $\dfrac{\phi}{w}$ is essentially bounded, and for $\phi \in L^\infty(G, \frac{1}{w})$, define

$$\| \phi \|_{\infty, \frac{1}{w}} = \| \frac{\phi}{w} \|_\infty = esssup\{|\frac{\phi(x)}{w(x)}|; x \in G\}.$$

The spaces $L^1(G,w)$ and $L^\infty(G, \frac{1}{w})$ are in duality by

$$\langle f, \phi \rangle = \int_G f(x)\phi(x)dx \qquad (f \in L^1(G,w), \phi \in L^\infty(G, \frac{1}{w})).$$

Recall that, f_x is the translate of f defined by $f_x(y) = f(y-x) \quad (y \in G)$.

Theorem 2.1. *Every closed translation-invariant subspace of $L^1(G,w)$ is an ideal, Conversely any closed ideal in $L^1(G,w)$ is translation invariant.*

Proof. The proof is similar to that of [17, Theorem 7.1.2]. □

Lemma 2.1. *Suppose $f \in L^1(G,w)$, then the map $x \mapsto f_x$ is a continuous map of G into $L^1(G,w)$.*

Proof. Let $\epsilon > 0$ be given and fix $x \in G$. Since $C_c(G)$ is dense in $L^1(G,w)$, there exists $g \in C_c(G)$ such that $\| g - f \|_w < \dfrac{\epsilon}{3w(x)}$. By translation we assume that $g(0) \neq 0$, where $0 \in G$ is the identity of G. Let m be the Haar measure of G, and $W = \{y \in G : g(y) \neq 0\}$, then W is an open neighborhood of $0 \in G$ and $\overline{W} = K := supp(g)$. By continuity of g at $0 \in G$, there is a neighborhood U of $0 \in G$ such that $\| g - g_z \|_\infty < \dfrac{\epsilon}{M} \quad (z \in U)$, where $M = 3m(K+K)(\sup_{t \in K+K} w(t)).w(x)$. Here $K+K = \{y+z : y, z \in K\}$. Now $V = U \cap W$ is an open neighborhood of $0 \in G$ and $K + V \subseteq K + W \subseteq K + K$. Also

$$supp(g - g_z) \subseteq K + V \subseteq K + K \qquad (z \in V).$$

Next

$$\begin{aligned} \| g - g_z \|_w &= \int_G |g(u) - g(u-z)|w(u)du = \int_{K+K} |g(u) - g(u-z)|w(u)du \\ &\leq \| g - g_z \|_\infty \int_{K+K} w(u)du \leq \| g - g_z \|_\infty (\sup_{t \in K+K} w(t))m(K+K) \\ &\leq \frac{\epsilon}{M}(\sup_{t \in K+K} w(t)) = \frac{\epsilon}{3w(x)} \qquad (z \in V). \end{aligned}$$

Also we have

$$\begin{aligned}\| g_z - f_z \|_w &= \int_G (g-f)(t-z)w(t)dt = \int_G (g-f)(\alpha)w(z+\alpha)d\alpha \\ &\le w(z)\int_G (g-f)(\alpha)w(\alpha)d\alpha \le w(z) \| g-f \|_w \le w(z).\frac{\epsilon}{3w(x)}.\end{aligned}$$

Since $w \ge 1$ we get

$$\begin{aligned}\| f - f_z \|_w &\le \| f-g \|_w + \| g - g_z \|_w + \| g_z - f_z \|_w \\ &\le \frac{\epsilon}{3w(x)} + \frac{\epsilon}{3w(x)} + w(z).\frac{\epsilon}{3w(x)} \le w(z).\frac{\epsilon}{w(x)} \quad (z \in V).\end{aligned}$$

Finally we have $f_x - f_y = (f - f_{y-x})_x$ for $y \in G$. If we assume that $y - x \in V$ and put $z = y - x$, then we get

$$\begin{aligned}\| f_x - f_y \|_w &\le w(x) \| f - f_{y-x} \|_w \le w(x) \| f - f_z \|_w \\ &\le \sup_{z\in V} w(z)\epsilon \le \sup_{z\in K+K} w(z)\epsilon.\end{aligned}$$

Since w is continuous and continuous functions are bounded on compact sets, w is bounded on V, also it is bounded on $K+K$, since ϵ is arbitrary the proof is complete. □

Lemma 2.2. *Given $f \in L^1(G,w)$ and $\epsilon > 0$, there exists a neighborhood V of 0 in G with the following property: if $u \in L^1(G,w)$ vanishes outside V, $u \ge 0$ and $\int_G u(x)dx = 1$, then $\| f - f \star u \|_w < \epsilon$.*

Proof. For $\epsilon > 0$, choose V by Lemma 2.1 with $\| f - f_y \|_w < \epsilon$ for all $y \in V$. If u satisfies the hypotheses, we have

$$(f \star u)(x) = \int_G f(x-y)u(y)dy.$$

Then we have

$$(f \star u)(x) - f(x) = \int_G [f(x-y) - f(x)]u(y)dy.$$

Hence

$$\begin{aligned}\| f \star u - f \|_w &= \int_G \int_G |[f(x-y) - f(x)]u(y)|w(x)dydx \\ &= \int_G u(y) \int_G |f(x-y) - f(x)|w(x)dxdy \\ &\le \int_G u(y) \| f - f_y \|_w \, dy = \int_V u(y) \| f - f_y \|_w \, dy \\ &\le \int_V u(y)\epsilon dy = \epsilon \int_G u(y)dy \le \epsilon.\end{aligned}$$

□

For $1 \le p < \infty$, a Beurling space on a locally compact group G is defined as follows

$$L^p(G,\omega) := \{f : f \in L^p(G); f.w \in L^p(G)\}.$$

When $p = 1$, then it is Beurling algebra. For $f \in L^p(G,w)$, we define $\| f \|_{p,\omega} := \| f\omega \|_p$.

3. Zero Set of Ideals

For $f \in L^1(G,w)$, we define $Z(f)$ to be the set of all $\gamma \in \widehat{G}$ such that $\widehat{f}(\gamma) = 0$, and if I is an ideal in $L^1(G,w)$, we define the zero set of I by $Z(I) = \bigcap_{f\in I} Z(f)$. Thus $\gamma \in Z(I)$ if and only if $\widehat{f}(\gamma) = 0$ for all $f \in I$. Since $\widehat{f}$ is continuous on $\widehat{G}$, each $Z(f)$ is closed, hence $Z(I)$ is closed for every I. It was shown by Domar in [3, Theorem 2.11] that $L^1(G,w)$ is a regular Banach algebra if and only if the weight w is non quasi analytic (n.q.a.), i.e., $\sum_{n=1}^{\infty} \frac{\log w(nt)}{n^2} < \infty$ for all $t \in G$. This generalizes a theorem of Beurling [1, Theorem V B] for $G = \mathbb{R}$. It follows that if w is a n.q.a. weight on G, then the ideal
$\{f \in L^1(G,w) : \hat{f}$ has compact support $\}$ is dense in $L^1(G,w)$ [15].
A function ϕ defined on G is said to be positive-definite if the inequality

$$\sum_{n,m=1}^{N} c_n\overline{c_m}\phi(x_n - x_m) \geq 0 \tag{1}$$

holds for every choice of $x_1, ..., x_N$ in G and for every choice of complex numbers $c_1, ..., c_N$.

Theorem 3.1. *(Bochner [17]) A continuous function ϕ on G is positive-definite if and only if there is a non-negative measure $\mu \in M(\widehat{G})$ such that*

$$\phi(x) = \int_{\widehat{G}} (x,\gamma)d\mu(\gamma) \qquad (x \in G). \tag{2}$$

Let $B(G)$ be the set of all functions f on G which are representable in the form

$$f(x) = \int_{\widehat{G}} (x,\gamma)d\mu(\gamma) \qquad (x \in G, \gamma \in M(\widehat{G})). \tag{3}$$

$B(G)$ is exactly the set of all finite linear combinations of continuous positive-definite functions on G. If $f \in L^1(G) \cap B(G)$, then $\widehat{f} \in L^1(\widehat{G})$. If the Haar measure of G is fixed, the Haar measure of $\widehat{G}$ can by normalized such that inversion formula

$$f(x) = \int_{\widehat{G}} \widehat{f}(\gamma)(x,\gamma)d\gamma \quad (x \in G) \tag{4}$$

is valid for every $f \in L^1(G) \cap B(G)$ [17].

Definition 3.1. [15] *A weight w on the locally compact Abelian group G is called semi-bounded if there exist a constant K and a subsemigroup $S \subseteq G$ with $S - S = G$, such that $w(s) \leq K$ and*

$$lim_{n\to+\infty} \frac{\log w(-ns)}{\sqrt{n}} = 0 \quad s \in S. \tag{5}$$

By definition, a semi-bounded weight is always non-quasi analytic. For the rest of the paper, we assume that w is semi-bounded.

Theorem 3.2. *The Fourier transform restricted to $L^1(G,\omega) \cap L^2(G,\omega)$ is a map onto a dense linear subspace of $L^2(\widehat{G})$, where for all $f \in L^1(G,\omega) \cap L^2(G,\omega)$, $\| f \|_{2,\omega} = \| \widehat{f\omega} \|_2$. Hence it may be extended to a map of $L^2(G,\omega)$ onto $L^2(\widehat{G})$.*

Proof. If $f \in L^1(G,\omega) \cap L^2(G,\omega)$ then $f\omega \in L^1(G) \cap L^2(G)$. Set $g := f\omega \star \overbrace{f\omega}$ (where $\overbrace{f}(x) = \overline{f(-x)}$). Then $g \in L^1(G)$, g is continuous and positive definite, $|\widehat{g}| = |\widehat{f\omega}|^2$, and the inversion theorem gives

$$\int_G |f\omega(x)|^2 dx = \int_G f\omega(x)\overbrace{f\omega}(-x)dx = g(0) = \int_{\widehat{G}} \widehat{g}(\gamma)d\gamma = \int_{\widehat{G}} |\widehat{f\omega}|^2 d\gamma$$

or $\|f\|_{2,\omega} = \| \widehat{f\omega} \|_2$.

Now we show that the image of Fourier transform is a dense set in $L^2(\widehat{G})$. That will imply that it can be extended to a map from $L^2(G,\omega)$ onto $L^2(\widehat{G})$. It is sufficient to show that the only element Φ orthogonal to all $\widehat{f}$, $f \in L^1(G,\omega) \cap L^2(G,\omega)$ is $\Phi = 0$. Using the fact that

$$\widehat{(L_x f)}(\gamma) = \overline{(x,\gamma)}\widehat{f}(\gamma).$$

For all$f \in L^1(G,\omega) \cap L^2(G,\omega)$, we obtain

$$0 = \int \Phi(\gamma)\overline{\widehat{(L_x f)}}(\gamma)d\gamma = \int (x,\gamma)\Phi\widehat{f}(\gamma)d\gamma.$$

Then it can be shown (by uniqueness of measure, Theorem1.3.6 [17]) that $\Phi f = 0$ a.e. for all $f \in (L^1 \cap L^2)(G,\omega)$, so $\Phi = 0$. □

The right translation representation $\mathcal{R}$ of G on $L^1(G)$ given by

$$\mathcal{R}_t f(x) := f(x-t), \qquad (t \in G),$$

is a strongly continuous representation satisfies $\| \mathcal{R}_t \| \leq 1$ for all $t \in G$.

Definition 3.2. [15] *Let G be a locally compact Abelian group and $\mathcal{R}$ be the right translation representation of G on $L^1(G)$. Let $\mathcal{S}(G)$ denote the class of all Banach subalgebras $\mathcal{A} \subset L^1(G)$ satisfying the following properties:*
(1) *$\mathcal{A}$ is norm dense in $L^1(G)$ and the injection $\mathcal{A} \hookrightarrow L^1(G)$ is continuous.*
(2) *$\mathcal{A}$ is translation-invariant, i.e., $\mathcal{R}_t\mathcal{A} \subseteq \mathcal{A}$ for each $t \in G$.*
(3) *For each $f \in \mathcal{A}$, the mapping $G \ni t \longmapsto \mathcal{R}_t f \in \mathcal{A}$ is continuous.*
(4) *There exists a semi-bounded weight w on G such that $\| \mathcal{R}_t \|_{\mathcal{A}} \leq w(t)$ for each $t \in G$.*
(5) *The intersection $\mathcal{A} \cap L^1(G,w)$ is norm dense in both $\mathcal{A}$ and $L^1(G,w)$.*

The set $L^2(G,w) \cap L^1(G,w)$ is dense in $L^1(G)$ because it contains continuous functions with compact support, and the injection $L^2(G,w) \cap L^1(G,w) \hookrightarrow L^1(G)$ is continuous.
Let $f \in L^2(G,w) \cap L^1(G,w)$, then $\int \mid \mathcal{R}_t f(x) \mid w(x)dx \leq w(t) \parallel f \parallel_{L^1(G,w)}$, so $\mathcal{R}_t f$ is in $L^1(G,w)$. Also

$$\begin{aligned}
(\int \mid \mathcal{R}_t f(x) \mid^2 w^2(x)dx)^{1/2} &= (\int \mid f(x-t) \mid^2 w^2(x)dx)^{1/2} \\
&= (\int \mid f(x) \mid^2 w^2(x+t)dx)^{1/2} \\
&\leq (\int \mid f(x) \mid^2 (w(x).w(t))^2 dx)^{1/2} \\
&\leq w(t) \parallel f \parallel_{L^2(G,w)} \quad (A).
\end{aligned}$$

This means that $\mathcal{R}_t(L^2(G,w) \cap L^1(G,w)) \subseteq L^2(G,w) \cap L^1(G,w)$. It is obvious by Lemma 2.1 that for each $f \in L^2(G,w) \cap L^1(G,w)$, the mapping $G \ni t \mapsto \mathcal{R}_t f$ is continuous.
For every weight w on G,

$$\begin{aligned}
\parallel \mathcal{R}_t \parallel_{L^1(G,w)} &= sup\{\parallel \mathcal{R}_t f \parallel_{L^1(G,w)} ; \quad \parallel f \parallel_{L^1(G,w)} = 1\} \\
&= sup\{\int \mid \mathcal{R}_t f(x) \mid w(x)dx ; \quad \parallel f \parallel_{L^1(G,w)} = 1\} \\
&= sup\{\int \mid f(x-t) \mid w(x)dx ; \quad \parallel f \parallel_{L^1(G,w)} = 1\} \\
&= sup\{\int \mid f(x) \mid w(x+t)dx ; \quad \parallel f \parallel_{L^1(G,w)} = 1\} \\
&\leq sup\{w(t) \parallel f \parallel_{L^1(G,w)} ; \quad \parallel f \parallel_{L^1(G,w)} = 1\} = w(t). \quad (B)
\end{aligned}$$

The norm on $L^2(G,w) \cap L^1(G,w)$ is usually considered as $\parallel . \parallel = \parallel . \parallel_{1,w} + \parallel . \parallel_{2,w}$. We can set $\parallel . \parallel = \frac{1}{2}(\parallel . \parallel_{1,w} + \parallel . \parallel_{2,w})$, which is equivalent to the usual one.

Due to (1) and (2), we have

$$\begin{aligned}\| \mathcal{R}_t \|_{L^2(G,w)\cap L^1(G,w)} &= sup\{\frac{1}{2}(\| \mathcal{R}_t f \|_{L^1(G,w)} + \| \mathcal{R}_t f \|_{L^2(G,w)});\ \ \| f \|_{L^1(G,w)}=\| f \|_{L^2(G,w)}= 1\}\\ &\leq \frac{1}{2}(sup\{\| \mathcal{R}_t f \|_{L^1(G,w)};\ \ \| f \|_{L^1(G,w)}= 1\}\\ &+ sup\{\| \mathcal{R}_t f \|_{L^2(G,w)};\ \ \| f \|_{L^2(G,w)}= 1\})\\ &\leq \frac{1}{2}(w(t) + w(t)) = w(t).\end{aligned}$$

We conclude that $L^2(G,w) \cap L^1(G,w)$ is in the class $\mathcal{S}(G)$.

Theorem 3.3. *Suppose $f \in L^1(G,w)$ and $\epsilon > 0$. There exists $\nu \in L^1(G,w)$ such that $\widehat{\nu}$ has compact support and $\| f \star \nu - f \|_w < \epsilon$.*

Proof. Let X be the set of all $g \in L^2(G,w) \cap L^1(G,w)$ such that $\widehat{g}$ has compact support. Since w is semi-bounded, by ([15] Proposition 2.4 (iii)), X is dense in $L^2(G,\omega) \cap L^1(G,w)$. If $\nu = gh$, with $g, h \in X$ then $\widehat{\nu} = \widehat{g} \star \widehat{h}$, hence $\widehat{\nu}$ has compact support, since X is dense in $L^2(G,\omega) \cap L^1(G,w)$, the set of all such ν is dense in $L^1(G,w)$. Let V be the neighborhood of identity as in Lemma 2.2, using Urysohn lemma, choose $u_0 \in C_c(G)_+$ such that $u_0 = 0$ off V. Let $\alpha = \int_G u_0 dm$, put $u = \frac{1}{\alpha} u_0$ then $u \in L^1(G,w)$ satisfies the conditions of Lemma 2.2, thus $\| f \star u - f \|_w < \epsilon/2$, also we can choose $\nu \in L^1(G,w)$ such that $\widehat{\nu}$ has compact support and $\| u - \nu \|_w < \dfrac{\epsilon}{2 \| f \|_w}$, then

$$\| f - f \star \nu \|_w \leq \| f - f \star u \|_w + \| f \star (u - \nu) \|_w < \epsilon.$$

□

Proposition 3.1. *Suppose C is a compact subset of $\widehat{G}$, $V \subset \widehat{G}$ and w is a weight function on G and $0 < m(V) < \infty$, where m is the Haar measure of $\widehat{G}$. Then there exists $k \in L^1(G,w)$ such that*

(1) $\widehat{k}(\gamma) = 1$ *on* C, $\widehat{k}(\gamma) = 0$ *outside* $C + V - V$ *and* $0 \leq \widehat{k}(\gamma) \leq 1$ *for all* $\gamma \in \widehat{G}$;
(2) $\|k\|_w \leq 2$.

Proof. Let g, h be the functions in $L^2(G,w)$ whose Fourier transforms of gw and hw are the characteristic functions of V and $C - V$ respectively and gw^2 and hw^2 be in $L^2(G)$. Define

$$k(x) = \frac{gw(x)hw(x)}{m(V)}, \quad (x \in G). \tag{6}$$

Then $\widehat{k} = m(V)^{-1}(\widehat{gw} \star \widehat{hw})$ or $\widehat{k}(\gamma) = \dfrac{1}{m(V)} \int_V \widehat{hw}(\gamma - \alpha)d\alpha$, $\gamma \in \widehat{G}$. If $\gamma \in C$ then $\widehat{hw}(\gamma - \alpha) = 1$ for all $\alpha \in V$. Hence $\widehat{k}(\gamma) = 1$. If γ is not in $C + V - V$, then $\widehat{hw}(\gamma - \alpha) = 0$, for all $\alpha \in V$, since $0 \leq \widehat{hw} \leq 1$ (1) follows.
By Theorem 3.2 $\| g \|_{2,w} = m(V)^{1/2}$, $\| h \|_{2,w} = m(C-V)^{1/2}$, since gw^2 and hw^2 are in $L^2(G)$, there exists $L, M > 0$ such that $(\int \mid hw^2(x) \mid^2 dx)^{1/2} \leq L$ and $(\int \mid gw^2(x) \mid^2 dx)^{1/2} \leqslant M$. Let $\alpha = max\{L, M\}$, then by applying Schwartz inequality to (3.6), we have

$$\begin{aligned}\| k \|_w &= m(V)^{-1} \int \mid gw(x).hw(x) \mid w(x)dx\\ &\leq m(V)^{-1}(\int \mid gw(x) \mid^2 dx)^{1/2}.(\int \mid hw^2(x) \mid^2 dx)^{1/2}\\ &\leq m(V)^{-1} \| g \|_{2,w} .L \leq m(V)^{-1/2}\alpha.\end{aligned}$$

We can choose V such that $m(V) \geq \frac{9}{16}\alpha^2$ this implies (2). □

Theorem 3.4. *If W is an open set in $\widehat{G}$ which contains a compact set C, then there exists $f \in L^1(G,w)$ such that $\widehat{f} = 1$ on C and $\widehat{f} = 0$ outside W.*

Proof. Choose a neighborhood V of 0 in $\widehat{G}$ such that $C+V-V \subset W$ and apply Proposition 3.1. □

Theorem 3.5. *Suppose $f \in L^1(G,w)$, $\gamma_0 \in \widehat{G}$, $\widehat{f}(\gamma_0) = 0$, W is a neighborhood of γ_0, and $\epsilon > 0$. Then there exists $k \in L^1(G,w)$ such that*
(1) $\| k \|_w < 2$;
(2) $\widehat{k} = 1$ *in a neighborhood of* γ_0 *and* $\widehat{k} = 0$ *outside* W;
(3) $\| f \star k \|_w < \epsilon$.

Proof. We assume $\gamma_0 = 0$. Put $\delta = \dfrac{\epsilon}{4(1+ \| f \|_1)}$, there exists a compact set E in G such that the integral of $| fw |$ over the complement $G-E$ of E is less than δ. We can find C and V as in the proof of Proposition 3.1 subject to the following conditions (i) 0 is an interior point of C; (ii) $m(C-V) < 4m(V)$; (iii) $C+V-V \subset W$; and (iv) $| 1-(x,\gamma) | < \delta$ whenever $x \in E$ and $\gamma \in C+V-V$. Define k as in the proof of Proposition 3.1 then (1) and (2) hold and since $\widehat{f}(0) = 0$, we have

$$f \star k(x) = \int_G f(y)(k(x-y)-k(x))dy \qquad (x \in G),$$

so that $\| f \star k \|_w \leq \int_G | f(y) | . \| k_y - k \|_w \, dy = \int_E + \int_{G-E}$. Since

$$\| k_y - k \|_w \leq \| k_y \|_w + \| k \|_w \leq w(y) \| k \|_w + \| k \|_w \leq 2w(y) \| k \|_w .$$

The integral over $G-E$ is less than $2 \| k \|_w \delta \leq 4\delta$ and the integral over E does not exceed $\| f \|_1 \, sup_{y \in E} \| k_y - k \|_w$. Hence the inequality $\| k_y - k \|_w < 4\delta \quad (y \in E)$ will complete the proof. Similarly, as in Proposition 3.1, we have

$$m(V)(k_y - k) = gw((hw)_y - hw) + ((gw)_y - gw)(hw)_y \quad (y \in E).$$

Schwartz inequality implies,

$$m(V) \int (k_y - k)(x)w(x)dx \leq (\int | gw^2(x) |^2 \, dx)^{1/2}.(\int | (hw)_y(x) - hw(x) |^2 \, dx)^{1/2}$$
$$+ (\int | (gw)_y(x) - gw(x) |^2 \, dx)^{1/2}.(\int | (hw)_y(x).w(x) |^2)^{1/2}$$

By Theorem 3.2 and (iv), we have that

$$(\int | ((hw)_y - hw) |^2)^{1/2} = (\int_{C-V} | 1-(y,\gamma) |^2 \, d\gamma)^{1/2} < \delta m(C-V)^{1/2}$$

and

$$(\int | ((gw)_y - gw) |^2)^{1/2} = (\int_V | 1-(y,\gamma) |^2 \, d\gamma)^{1/2} < \delta m(V)^{1/2} \tag{7}$$

. By Proposition 3.1, we have that

$$(\int | hw^2(x) |^2 \, dx)^{1/2} \leq L$$

and

$$(\int | gw^2(x) |^2 \, dx)^{1/2} \leqslant M,$$

so that

$$
\begin{aligned}
(\int \mid (hw)_y(x).w(x) \mid^2 dx)^{1/2} &= (\int \mid hw(x-y).w(x) \mid^2 dx)^{1/2} \\
&= (\int \mid h(x-y)w(x-y)w(x) \mid^2 dx)^{1/2} \\
&= (\int \mid h(x)w(x)w(x+y) \mid^2 dx)^{1/2} \\
&\leq (\int \mid h(x)w(x)w(x)w(y) \mid^2 dx)^{1/2} \\
&\leq w(y)(\int \mid hw^2(x) \mid^2)^{1/2} = w(y)L.
\end{aligned}
$$

We obtain

$$m(V) \parallel k_y - k \parallel_w < \alpha\delta(w(y)(m(V))^{1/2} + (m(C-V))^{1/2}). \tag{5}$$

Since $m(C-V) < 4m(V)$, (3.0) implies

$$\parallel k_y - k \parallel_w \leq 3\alpha\delta w(y)(m(V))^{1/2} \quad (y \in E).$$

Let $N = sup_{y \in E} w(y)$, since w is continuous and E is compact, so N is finite, we can choose V such that $m(V) \geq \dfrac{9}{16}\alpha^2 N^2$ and this complete the proof. □

Theorem 3.6. *Suppose $f \in L^1(G,w)$, $\gamma_0 \in \widehat{G}$, $\widehat{f}(\gamma_0) = 0$. There exists $\nu \in L^1(G,w)$, such that $\widehat{\nu} = 0$ in a neighborhood of γ_0, $\parallel \nu \parallel < 3$ and $\parallel f - f \star \nu \parallel_w < \epsilon$.*

Proof. As in the proof of Theorem 3.3, there exists $u \in L^1(G,w)$, such that $\parallel u \parallel_1 = 1$ and $\parallel f - f \star u \parallel_w < \epsilon/2$, since $\widehat{f}\widehat{u}(\gamma_0) = 0$, Theorem 3.5 applies to $f \star u$ and so there exists $k \in L^1(G,w)$ such that $\widehat{k} = 0$ in a neighborhood of γ_0, $\parallel k \parallel_w < 2$ and $\parallel f \star u \star k \parallel_w < \epsilon/2$. Put $\nu = u - u \star k$, then $\widehat{\nu} = 0$ when $\widehat{k} = 1$ and

$$\parallel f - f \star \nu \parallel_w \leqslant \parallel f - f \star u \parallel_w + \parallel f \star u \star k \parallel_w < \epsilon.$$

□

Theorem 3.7. *Suppose $f \in L^1(G,w)$, $\gamma_0 \in \widehat{G}$, W is a neighborhood of γ_0 and $\epsilon > 0$. There there exists $h \in L^1(G,w)$ such that $\parallel h \parallel_w < \epsilon$, $\widehat{h} = 0$ outside W and $\widehat{f}(\gamma) - \widehat{h}(\gamma) = \widehat{f}(\gamma_0)$ in some neighborhood γ_0.*

Proof. Choose $g \in L^1(G,w)$ such that $\widehat{g}(\gamma) = \widehat{f}(\gamma_0)$ in some neighborhood of γ_0 Theorem 3.5 applies to $f - g$, and so there exists $k \in L^1(G,w)$ such that $\widehat{k} = 1$ in a neighborhood of γ_0 , $\widehat{k} = 0$ outside W and $\|(f-g) \star k \parallel_w < \epsilon$. Put $h = (f-g) \star k$. Then $\widehat{h} = (\widehat{f} - \widehat{g})\widehat{k}$ and so there is a neighborhood of γ_0 in which $\widehat{h} = \widehat{f} - \widehat{g} = \widehat{f} - \widehat{f}(\gamma_0)$. □

Suppose I is an ideal of $L^1(G,w)$, if $f \in L^1(G,w)$ and $\gamma_0 \in \widehat{G}$, we say that f is locally in I at γ_0 if there exists $g \in I$ such that $\widehat{f} = \widehat{g}$ in a neighborhood of γ_0.

Theorem 3.8. *Suppose that $f \in L^1(G,w)$, I is a closed ideal of $L^1(G,w)$ and $\gamma_0 \in \widehat{G}$. Then $\widehat{f}$ belongs to $\widehat{I}$ locally at γ_0 if either of the following conditions is satisfied*

(a) γ_0 is not in $Z(I)$,

(b) γ_0 is in the interior of $Z(I)$.

Proof. If (a) holds, there exists $g \in I$ with $\widehat{g}(\gamma_0) = 1$, and Theorem 3.7 shows that there exists $h \in L^1(G,w)$ such that $\parallel h \parallel_w < 1/2$ and $\widehat{h}(\gamma) = 1 - \widehat{g}(\gamma)$ in some neighborhood V of γ_0. The series $\sum_0^\infty \widehat{f}\widehat{h}^n$ converges in the norm of $A(\widehat{G})$, to a function $\widehat{j} \in A(\widehat{G})$, and $\widehat{j}(\gamma) = (1 - \widehat{h}(\gamma))^{-1}\widehat{f}(\gamma)$ for all $\gamma \in \widehat{G}$, if $\gamma \in V$ then $\widehat{g}(\gamma)\widehat{j}(\gamma) = \widehat{f}(\gamma)$, since $\widehat{g} \in \widehat{I}$ and $\widehat{I}$ is an ideal, $\widehat{g}\widehat{j} \in \widehat{I}$, and so $\widehat{f}$ belongs to $\widehat{I}$ locally at γ_0. If (b) holds, then $\widehat{f} = 0$ in a neighborhood of γ_0 and since $\widehat{I}$ contains the constant 0, $\widehat{f}$ belongs to $\widehat{I}$ locally at γ_0. □

Theorem 3.9. *Suppose G is compact, I is a closed ideal of $L^1(G,w)$ and $Z(I) \subseteq Z(f)$, then $f \in I$.*

Proof. If γ_0 is not in $Z(I)$, there exists $g \in I$ with $\widehat{g}(\gamma_0) = 1$, and hence $g \star \gamma_0 = \gamma_0$ regarding γ_0 as a member of $L^1(G,w)$, since I is an ideal, $g \star \gamma_0 \in I$ and so $\gamma_0 \in I$, it follows that I contains every trigonometric polynomial on G of the form $\sum a_\gamma(x,\gamma)$, provided that $a_\gamma = 0$, for all $\gamma \in Z(I)$. If $Z(I) \subset Z(f)$, then $f \star \kappa$ satisfies this condition for every trigonometric polynomial κ on G. Since $\| f - f \star \kappa \|_w$ can be arbitrarily small by Theorem 3.3, and I is a closed, we conclude that $f \in I$. □

Theorem 3.10. *Suppose I is an ideal of $L^1(G,w)$ and $f \in L^1(G,w)$. If $\widehat{f}$ has compact support and f is locally in I at every $\gamma \in \widehat{G}$, then $f \in I$.*

Proof. For every $\gamma \in \widehat{G}$, choose $g_\gamma \in I$ such that $\widehat{f} = \widehat{g_\gamma}$ on an open set U_γ containing γ. By passing to a finite subcover of supp$\widehat{f}$, we obtain open sets $U_1, U_2, ..., U_n$ in $\widehat{G}$ and $g_1, g_2, ..., g_n$ in I such that $\widehat{g_j} = \widehat{f}$ on U_j and supp$\widehat{f} \subset \cup_1^n U_j$, next each $\xi \in U_j \cap$ supp$\widehat{f}$ has a compact neighborhood contained in U_j, by passing to a finite subcover supp$\widehat{f}$ again, we obtain compact sets $K_1, K_2, ..., K_n$ such that $K_j \subset U_j$ and supp$\widehat{f} \subset \cup_1^n K_j$. By Theorem 3.4, there exists $h_1, ..., h_n$ in $L^1(G,w)$ such that $\widehat{h_j} = 1$ on K_j and supp$\widehat{h_j} \subset U_j$. Then $\prod_1^n (1 - \widehat{h_j}) = 0$ on supp$\widehat{f}$, so $\widehat{f} = \widehat{f}(1 - \prod_1^n (1 - \widehat{h_j}))$. If we multiply out the product inside the brackets, each term of the resulting sum is a product of h_j's, i.e., the Fourier transform of convolution of h_j's. Collecting terms, we see that $f = \sum f \star H_j$ where $H_j \in L^1(G,w)$ and supp$\widehat{H_j} \subset U_j$. But then $\widehat{(f \star H_j)} = \widehat{f}\widehat{H_j} = \widehat{g_j}\widehat{H_j} = \widehat{(g_j \star H_j)}$, so $f \star H_j = g_j \star h_j \in I$, and hence $f \in I$. □

Theorem 3.11. *If I is a closed ideal of $L^1(G,w)$ and γ_0 is not in $Z(I)$, then every $f \in L^1(G,w)$ is locally in I at γ_0.*

Proof. Pick g in I such that $\widehat{g}(\gamma_0) = 1$. By Theorem 3.7, there exists $\nu \in L^1(G,w)$ such that $\| \nu \| < 1/2$ and $\widehat{\nu} + \widehat{g} = 1$ in a neighborhood of γ_0. Let $\nu_n = \nu \star \nu \star ... \star \nu$ (n factors) then $\| \widehat{\nu_n} \|_{\infty,1/w} \leq \| \nu_n \|_w < 2^{-n}$. Hence if $f \in L^1(G,w)$, the series $f + \sum_1^\infty f \star \nu_n$ converges in $L^1(G,w)$ to a function h such that $\widehat{h} = \sum_0^\infty \widehat{f}\widehat{\nu_n} = \sum_0^\infty \widehat{f}(\widehat{\nu})^n = \dfrac{\widehat{f}}{1 - \widehat{\nu}}$. But $1 - \widehat{\nu} = \widehat{g}$ near γ_0, so $\widehat{f} = \widehat{h}(1 - \widehat{\nu}) = \widehat{h}\widehat{g} = \widehat{(h \star g)}$ near γ_0. Since $h \star g \in I$, f is locally in I at γ_0. □

Theorem 3.12. *If I is a closed ideal of $L^1(G,w)$ and $Z(I)$ is empty, then $L^1(G,w) = I$.*

Proof. Since w is non quasi analytic, the set $\{f \in L^1(G,w) : supp\widehat{f}$ is compact$\}$ is dense in $L^1(G,w)$, by Theorem 3.3, and is contained in I by Theorem 3.10 and Theorem 3.11. □

If $\phi \in L^\infty(G, 1/w)$, the statement ($\phi(x) \longrightarrow a$ as $x \longrightarrow \infty$) will mean that to every $\epsilon > 0$ there exists a compact set K in G such that $| \phi(x) - a | < \epsilon$ in the complement of K.

Example 3.1. *Suppose $\phi \in L^\infty(G, 1/w)$ and $f \in L^1(G,w)$, $\widehat{f}(\gamma) \neq 0$ for all $\gamma \in \widehat{G}$ and $(f \star \phi)(x) \longrightarrow a\widehat{f}(0)$, as $x \longrightarrow \infty$, then $(g \star \phi)(x) \longrightarrow a\widehat{g}(0)$, as $x \to \infty$, for every $g \in L^1(G,w)$.*

Proof. Replacing ϕ by $\phi - a$, we may assume without loss of generality, that $a = 0$, the set I of all $g \in L^1(G,w)$ such that $(g \star \phi)(x) \longrightarrow a\widehat{g}(0)$ as $x \longrightarrow \infty$ is a linear subspace of $L^1(G,w)$ which is clearly translation invariant. I is closed, for if $\| g_n - g \|_w \longrightarrow 0$ then

$\| g_n \star \phi - g \star \phi \|_{\infty,1/w} \to 0$, and $f \in I$. Hence I is a closed ideal in $L^1(G,w)$ with $Z(I)$ empty so by Theorem 3.12, $I = L^1(G,w)$. □

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