BARRIER FUNCTION METHOD AND SADDLE-POINT FOR FRACTIONAL OPTIMIZATION PROBLEM

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In this paper, we discuss a novel approach to solve the fractional optimization problem. For this purpose, the fractional optimization problem is transformed into a non-fractional optimization problem through parametrization, which is further converted into a unconstrained optimization problem by using the barrier function method. Thereafter, the convergence of the barrier penalized optimization problem is discussed and it is shown that a sequence constructed by the barrier penalized optimization problem has a limit point which solves the fractional optimization problem. Moreover, the saddle-point for fractional optimization problem with the help of barrier function method is also discussed. We also framed some non-trivial examples to validate the hypotheses of the established theory.

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1. Introduction

There are many realistic problems in which we require to optimize ratio of various linear or nonlinear functions to achieve the pre-defined goals, (e.g. the problems in economics, decision theory, game theory, information theory, data envelopment analysis, tax programming, cluster analysis, signal processing, neural networks, managment science, corporate planning, production and financial planning, etc. see for instances, [7, 26, 27]). These types of the optimization problems are called the fractional optimization problems. In the literature, there are various methods to solve the fractional optimization problems. The parametric approach is one of them, in which the given fractional optimization problem is transformed into an equivalent non-fractional optimization problem via a non-negative parameter. Many researchers took their interest to solve the fractional optimization problem and used this technique in different ways (see for instances [3, 10, 17, 18, 24, 25, 27, 28, 29]). Whereas, Ebrahimnejad *et al.* [11] had solved the fractional optimization problem by converting it into the bi-objective linear programming problem involving fuzzy functions.

An interesting approach to solve the constrained optimization problem is barrier function method, which is known as interior penalty function method. The barrier function method, transforms a constrained optimization problem into an equivalent unconstrained optimization problem via a barrier function and a non-zero non-negative barrier parameter.

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A barrier function is a continuous function whose value on a point increase to infinity as the point approaches the boundary of the feasible region of an optimization problem [22]. Roughly speaking, this method generates a sequence of feasible points whose limit is an optimal solution to the considered optimization problem. Up to now it is known two kind of barrier functions: one is an inverse barrier function and another one is a logarithm barrier function. In the literature, many authors had used this method to study different types of the optimization problems. (see for instance, [4, 6, 15, 20])

Iri and Imai [16] proposed a Newton-like descent algorithm to solve the linear programming problem of its results of preliminary computational experiments on small- and medium-size problems via barrier function method. Den Hertog *et al.* [9] used classical logarithmic barrier function method for the convex programming problem. Nash and Sofer [21] applied the logarithmic barrier method to solve the nonlinear programming problem with inequality constraints, in which the primal-dual method is also discussed under convexity assumption. Goldfard *et al.* [14] presented an interior point method for quadratically constrained convex quadratic programming that is based on a logarithmic barrier function approach. Further, the two-stage stochastic linear program was solved by Zhao [30] via the log-barrier method involving Benders decomposition.

Motivated by the above works, we focus our study to find the optimal solution for the class of fractional optimization problem by the barrier function method. We discuss the equivalence between the optimal solution to the fractional optimization problem and its associated barrier penalized optimization problem for which we prove that the limit of a sequence constructed by the barrier penalized optimization problem is an optimal solution to the fractional optimization problem. Furthermore, we define the Lagrange function for fractional optimization problem and establish the relationship between a saddle-point of the Lagrange function defined for the fractional optimization problem and barrier penalized optimization problem. Suitable examples are given to justify the established results.

The paper is organized as follows. In Section 2, we recall few results which will we used in the sequel of the paper. The equivalence between an optimal solution to the fractional optimization problem and a limit point of the sequence constructed by the barrier penalized optimization problem is discussed in Section 3. Section 4 demonstrates the relationship between a saddle-point of the Lagrange function defined for the fractional optimization problem and a limit point of the sequence constructed by the barrier penalized optimization problem. Finally, in Section 5 we summarize our results obtained in the present paper.

2. Problem description and preliminaries

We consider the following fractional optimization problem

$$\text{(FOP) minimize } \frac{p(x)}{q(x)},\ (q(x) \neq 0),$$
$$\text{subject to } r_i(x) \leq 0,\ i \in I = \{1, \dots, s\},$$

where the functions $p(x), q(x)$ and $r_i(x) : X \subseteq R \to R$ are continuous on X. Let $D = \{x \in X : r_i(x) \leq 0,\ i \in I\}$ be the set of all feasible solutions to the problem (FOP) and moreover, we assume that $p(x) \geq 0$, $q(x) > 0$, for all $x \in D$. Further, let $D' = \{x \in X : r_i(x) < 0,\ i \in I\}$ be a non-empty set of all interior feasible solutions to (FOP).

The parametric form of the above problem (FOP) with a parameter $\tau \in R_+$ is given by

$$(\text{FOP})_\tau \text{ minimize } f(x) = p(x) - \tau q(x),$$
$$\text{subject to } r_i(x) \leq 0,\ i \in I,$$

where R_+ is the non-negative orthant of R.

Definition 2.1. *A feasible point x_0 is said to be an optimal solution to the problem (FOP)$_\tau$ if for all $x \in D$*

$$p(x) - \tau q(x) \geq p(x_0) - \tau q(x_0).$$

The following lemma shows the equivalence between an optimal solution to the fractional optimization problem (FOP) and its associated parametric form (FOP)$_\tau$.

Lemma 2.1. [10] $\tau = \frac{p(x_0)}{q(x_0)} = \min\{\frac{p(x)}{q(x)} : x \in D\}$ *if and only if* $f(x_0) = \min\{p(x) - \tau q(x) : x \in D\}$=0.

On the line of Davar and Mehra [8], we construct the following Karush-Kuhn-Tucker (KKT) necessary optimality conditions for the problem (FOP).

Theorem 2.1. [KKT Necessary Optimality Conditions] *Let x_0 be an optimal solution to the problem* (FOP) *and the suitable constraint qualification be satisfied at x_0. Then there exist Lagrange multipliers $\mu_0 \in R^s_+$ such that*

$$\nabla(p(x_0) - \tau q(x_0)) + \sum_{i=1}^{s} \mu_{0i} \nabla r_i(x_0) = 0, \quad (\mathbf{2}.1)$$

$$\mu_{0i} r_i(x_0) = 0, \ \forall i \in I. \quad (\mathbf{2}.2)$$

3. Barrier function method for fractional optimization problem

For fractional optimization problem (FOP) we consider its associated parametric form (FOP)$_\tau$ and for (FOP)$_\tau$ we consider its associated barrier penalized optimization problem defined as :

(FOP)$_{\tau\gamma}$ minimize $\{p(x) - \tau q(x) + \gamma b(x)\}$

where $b(x)$ is a non-negative barrier function and γ is non-zero non-negative barrier parameter. Denote by

$$\beta(\gamma) = \min\{p(x) - \tau q(x) + \gamma b(x)\}$$

the optimal value of the objective function of problem (FOP)$_{\tau\gamma}$.

Note that $b(x)$ is zero in the interior of the feasible set and infinity on its boundary and is defined by either of the two ways in general as follows:
(i) $b(x) = \sum_{i=1}^{s} \frac{-1}{r_i(x)}$ (inverse barrier function),
(ii) $b(x) = -\sum_{i=1}^{s} \ln(-r_i(x))$ (Frisch's logarithmic barrier function).

The inverse barrier function was introduced by Caroll and Fiacco [5] which was further developed by Fiacco and McCormick [12] and implemented by McCormick et al. [19] in the SUMT-2 and SUMT-3 codes.

The logarithmic barrier function was introduced by Frisch [13] and Parisot [23] for linear programming.

For the barrier penalized optimization problem (FOP)$_{\tau\gamma}$, we define $b(x)$ as follows:

$$b(x) = \phi(r_i(x)), \forall x \in D', \ \forall i \in I,$$

where $\phi : R^s \to R_+$ is a continuously differentiable function.

Clearly, if x^k solves the barrier penalized optimization problem (FOP)$_{\tau\gamma}$, then $\nabla\beta(\gamma) = 0$. Therefore,

$$\nabla(p(x^k) - \tau q(x^k)) + \gamma_k \sum_{i=1}^{s} \nabla\phi(r_i(x^k)) \nabla r_i(x^k) = 0,$$

equivalently,

$$\nabla(p(x^k) - \tau q(x^k)) + \sum_{i=1}^{s} \mu_{0i}^k \nabla r_i(x^k) = 0,$$

where $\mu_{0i}^k = \gamma_k \nabla\phi(r_i(x^k))$ is known as Lagrange multipliers and $\mu_{0i}^k \to \mu_{0i}$ as $k \to \infty$, $\forall i \in I$.

Now, we show that an optimal solution to the problem (FOP) is a limit point of the convergent sequence of the barrier penalized optimization problem $(\text{FOP})_{\tau\gamma}$.

Theorem 3.1. *Let x_0 be an optimal solution to the problem* (FOP) *and assume that for any neighborhood N of x_0, there exist a point $\bar{x} \in N \cap D'$. Further, if the functions $p(x), q(x), r_i(x), \forall i \in I$ and $b(x)$ are continuous at x_0 on X, then x_0 is a limit point of any convergent sequence of the barrier penalized optimization problem* $(\text{FOP})_{\tau\gamma}$.

Proof. Let x_0 be an optimal solution to the problem (FOP). Since $p(x)$ and $q(x)$ are continuous functions at x_0 on X, so, for any $\epsilon > 0$ there exists a point $\bar{x} \in N \cap D'$ such that

$$p(x_0) - \tau q(x_0) + \epsilon > p(\bar{x}) - \tau q(\bar{x}).$$

From the positivity of γ we have

$$p(x_0) - \tau q(x_0) + \epsilon + \gamma b(\bar{x}) > p(\bar{x}) - \tau q(\bar{x}) + \gamma b(\bar{x}) \geq \beta(\gamma).$$

Since $\epsilon > 0$ is a very small quantity, neglecting it and taking limit $\gamma \to 0^+$ on both side of the above inequality, it follows that

$$p(x_0) - \tau q(x_0) \geq \lim_{\gamma \to 0^+} \beta(\gamma). \quad (\mathbf{3}.1)$$

On the other hand, by the barrier penalized optimization problem $(\text{FOP})_{\tau\gamma}$, we have

$$\beta(\gamma) = \min \{p(x) - \tau q(x) + \gamma b(x)\}.$$

As $\gamma > 0$ and for all $x \in D'$, $b(x) \geq 0$, the above equality yields

$$\beta(\gamma) \geq \min \{p(x) - \tau q(x)\}.$$

Since x_0 is an optimal solution to the problem (FOP), we conclude from the above inequality that

$$\beta(\gamma) \geq p(x_0) - \tau q(x_0).$$

Again, taking limit $\gamma \to 0^+$ on the both side of the above inequality, we get

$$\lim_{\gamma \to 0^+} \beta(\gamma) \geq p(x_0) - \tau q(x_0). \quad (\mathbf{3}.2)$$

On combining the inequalities ($\mathbf{3}$.1) and ($\mathbf{3}$.2), we have

$$\lim_{\gamma \to 0^+} \beta(\gamma) = p(x_0) - \tau q(x_0).$$

This completes the proof. □

Now, by using the logarithmic barrier function, we present the following example in order to validate the established result in Theorem **3**.*1*.

Example 3.1. *Let $X = R$ and consider the following fractional optimization problem*

$$(\text{FOP1}) \ \textit{minimize} \ \frac{x+1}{\frac{1}{2}x + 2}$$

$$\textit{subject to} \ (-x-1, \ \frac{1}{2} - x) \leq \mathbf{0},$$

where $\mathbf{0} = (0, 0)$. Note that $D = \{x \in X : x \geq \frac{1}{2}\}$ is the set of feasible solution to (FOP1) and $p(x) \geq 0, q(x) > 0, \forall x \in D$.

The parametric form of the above fractional optimization problem (FOP1) is

$$(\text{FOP1})_\tau \ \textit{minimize}\ f(x) = (x+1) - \tau(\frac{1}{2}x+2)$$
$$\textit{subject to}\ (-x-1,\ \frac{1}{2}-x) \leq \mathbf{0},\ \tau \in R_+.$$

Let $\tau = \frac{2}{3}$. *One can verify that* $x_0 = \frac{1}{2}$ *is an optimal solution to (FOP1)*$_\tau$ *and so by* Lemma **2**.*1*, $x_0 = \frac{1}{2}$ *is also an optimal solution to (FOP1).*

Further, the barrier penalized optimization problem, using Frisch's logarithmic barrier function, with the problem (FOP1)$_\tau$ *is*

$$(\text{FOP1})_{\tau\gamma} \ \textit{minimize}\ \{(x+1) - \tau(\frac{1}{2}x+2) + \gamma(-\ln(x+1) - \ln(x-\frac{1}{2}))\},$$
$$\forall x \in D' \ \textit{and}\ \gamma > 0.$$

On solving the above barrier penalized optimization problem (FOP1)$_{\tau\gamma}$, *we get* $x_{k_1} = \frac{-1-16\gamma+\sqrt{9+256\gamma^2}}{4}$ *and* $x_{k_2} = \frac{-1-16\gamma-\sqrt{9+256\gamma^2}}{4}$. *Clearly,* $x_{k_2} \notin D'$ *as* $\gamma \to 0$. *Further, it can be observed that as* $\gamma \to 0$, $\{x_k\} \to x_0 = \frac{1}{2}$ *is an optimal solution to (FOP1). Hence, the result.*

Now, we derive the contrary of the Theorem **3**.*1*.

Theorem 3.2. *Let* x^k *be a solution to the barrier penalized optimization problem* $(\text{FOP})_{\tau\gamma}$ *for any* $\gamma > 0$ *and sequence* γ_k *satisfy the condition* $0 < \gamma_{k+1} < \gamma_k$, *in which* $\gamma_k \to 0$ *as* $k \to \infty$. *Further, if the functions* $p(x)$, $q(x)$, $r_i(x), i \in I$ *and* $b(x)$ *are continuous functions at* x_0 *on* X, *then any limit point* x_0 *of* $\{x^k\}$ *is also an optimal solution to the problem* (FOP).

Proof. Let x_0 be a limit point of the sequence $\{x^k\}$. Firstly, we prove that x_0 is a feasible solution to (FOP). Since $p(x)$, $q(x)$ and $r_i(x), \forall i \in I$ are continuous functions at x_0 on X, we have

$$\lim_{k\to\infty} (p(x^k) - \tau q(x^k)) = p(x_0) - \tau q(x_0), \tag{3.3}$$
$$\text{and} \ \lim_{k\to\infty} r_i(x^k) = r_i(x_0) \leq 0, \forall i \in I.$$

Thus, x_0 is a feasible solution to (FOP). Now, we remain to show that x_0 is an optimal solution to (FOP). Assume to contrary that x_0 is not an optimal solution to the problem (FOP). Then, by Lemma **2**.*1*, there exists a point $\bar{x} \in D$ such that

$$p(\bar{x}) - \tau q(\bar{x}) < p(x_0) - \tau q(x_0). \tag{3.4}$$

By assumption, x^k is a solution to $(\text{FOP})_{\tau\gamma}$. Therefore, we have

$$p(x^k) - \tau q(x^k) + \gamma_k b(x^k) \leq p(\bar{x}) - \tau q(\bar{x}) + \gamma_k b(\bar{x}).$$

Taking limit $k \to \infty$ on the both side of the above inequality and using (**3**.3), we get

$$p(x_0) - \tau q(x_0) \leq p(\bar{x}) - \tau q(\bar{x}),$$

which contradicts the inequality (**3**.4). Hence, x_0 is an optimal solution to the problem (FOP). This completes the proof. □

Now, by using the logarithmic barrier function, we present an example to validate the above Theorem **3**.*2*.

TABLE 1

Iter. k	γ_k	$x_{\gamma_k} = x_{k+1}$	$f(x_{\gamma_k})$	$b(x_{\gamma_k})$	$\beta(\gamma_k)$	$\gamma_k b(x_{\gamma_k})$
1	*1.0*	*1.3819*	*0.3819*	*1.4436*	*1.8256*	*1.4437*
2	*0.1*	*1.0901*	*0.0901*	*2.5012*	*0.3402*	*0.2501*
3	*0.01*	*1.0099*	*0.0099*	*4.6251*	*0.0552*	*0.0462*
4	*0.001*	*1.0009*	*0.0009*	*7.0140*	*0.0079*	*0.0070*
5	*0.0001*	*1.0001*	*0.0001*	*9.2104*	*0.0001*	*0.0009*

Example 3.2. *Let* $X = R$ *and consider the following fractional optimization problem*

$$\text{(FOP2)}\ \textit{minimize}\ \frac{x^2+x+1}{x^2+2}$$
$$\textit{subject to}\ (1-x,\ x-2) \leq \mathbf{0}.$$

Note that $D = \{x \in X : 1 \leq x \leq 2\}$ *is the set of feasible solution to the problem (FOP2) and* $p(x) \geq 0, q(x) > 0, \forall x \in D$.

The parametric form of the above fractional optimization problem (FOP2) is

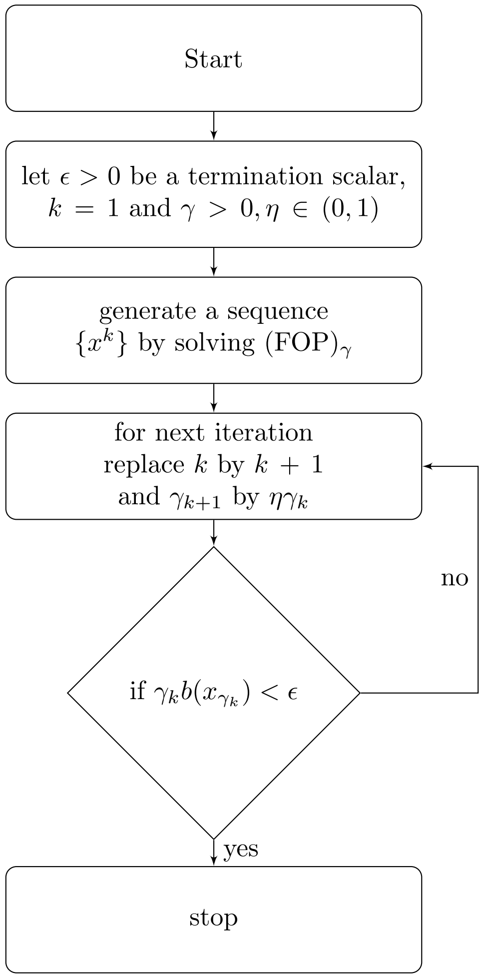
$$\text{(FOP2)}_\tau\ \textit{minimize}\ f(x) = (x^2+x+1) - \tau(x^2+2)$$
$$\textit{subject to}\ (1-x,\ x-2) \leq \mathbf{0},\ \tau \in R_+.$$

Further, the barrier penalized optimization problem, using Frisch's logarithmic barrier function, with the parametric form of the problem (FOP2) is

$$\text{(FOP2)}_{\tau\gamma}\ \textit{minimize}\ \{(x^2+x+1) - \tau(x^2+2) + \gamma(-\ln(x-1) - \ln(2-x))\},$$
$$\forall x \in D'\ \textit{and}\ \gamma > 0.$$

Let $\tau = 1$, $\gamma_1 = 1$, *the parameter* $\eta = 0.1$ *and* $\epsilon = 0.0010$. *By Table 1, it is easily seen that* $\gamma_k \to 0$ *as* $k \to \infty$. *Further, we get the sequence* $\{x_k\}$ *which converges to* $x_0 = 1$. *By Definition 2.1, we conclude that* $x_0 = 1$ *is an optimal solution to (FOP2)*$_\tau$ *and so by* Lemma **2**.*1,* $x_0 = 1$ *is also an optimal solution to (FOP2). Thus, we conclude that all hypotheses of Theorem* **3**.*2 are hold and the limit point of sequence* $\{x^k\}$ *is an optimal solution to (FOP2).*

The following Flowchart describe the algorithm to solve the barrier penalized optimization problem.



4. Saddle-point for fractional optimization problem with barrier function method

The theory of a saddle-point has an important place in Operational Research to find an optimal solution for the optimization problems.

Motivated by Antczak [1, 2], in this section we present the saddle-point of the Lagrange function defined for the problem (FOP) with the help of barrier function method. We show the relationship between a saddle-point defined for the problem (FOP) and a limit of a convergent sequence constructed by the barrier penalized optimization problem $(\text{FOP})_{\tau\gamma}$. First, we recall basic definitions of Lagrange function $L(x, \mu)$ and a saddle-point for (FOP) on the line of Antczak [2], which will be usefully to prove the main results .

Definition 4.1. *The Lagrange function $L(x, \mu) : D \times R^s_+ \to R$ for the problem (FOP) is defined by*

$$L(x, \mu) = p(x) - \tau q(x) + \sum_{i=1}^{s} \mu_i r_i(x), \tau \in R_+.$$

Definition 4.2. *A point $(x_0, \mu_0) \in D \times R^s_+$ is said to be a saddle-point of the Lagrange function defined for the problem (FOP), if the following inequalities*

(*i*) $L(x_0, \mu) \leq L(x_0, \mu_0), \forall \mu \in R^s_+,$

(ii) $L(x_0, \mu_0) \le L(x, \mu_0), \forall x \in D$
hold.

Now, we discuss the equivalence between a saddle-point of the Lagrange function defined for (FOP) and a limit point of the convergent sequence constructed by the barrier penalized optimization problem $(\text{FOP})_{\tau\gamma}$.

Theorem 4.1. *Let (x_0, μ_0) be a saddle-point of the Lagrange function defined for the problem (FOP) and let $p(x)$, $q(x)$ and $r_i(x), \forall i \in I$ be continuous functions at x_0 on X. If γ is sufficient large barrier parameter (it is sufficient to set $\gamma \ge \max\{\mu_{0i}, i \in I\}$), then x_0 is a limit of the convergent sequence to the barrier penalized optimization problem $(\text{FOP})_{\tau\gamma}$.*

Proof. Since (x_0, μ_0) is a saddle-point of the Lagrange function defined for the problem (FOP), therefore, by Definition **4**.2 (i) we have

$$L(x_0, \mu) \le L(x_0, \mu_0), \ \forall \mu \in R^s_+,$$

$$\text{or, } p(x_0) - \tau q(x_0) + \sum_{i=1}^{s} \mu_i r_i(x_0) \le p(x_0) - \tau q(x_0) + \sum_{i=1}^{s} \mu_{0i} r_i(x_0), \ \forall \mu \in R^s_+.$$

We set $\mu_i = 0, \forall i \in I$, thus, we get

$$\sum_{i=1}^{s} \mu_{0i} r_i(x_0) \ge 0. \tag{4.1}$$

Since x_0 is a feasible point to the problem (FOP), we have

$$\sum_{i=1}^{s} \mu_{0i} r_i(x_0) \le 0. \tag{4.2}$$

On combining the inequalities (**4**.1) and (**4**.2), we obtain

$$\sum_{i=1}^{s} \mu_{0i} r_i(x_0) = 0. \tag{4.3}$$

Again, by Definition **4**.2 (ii) we have

$$L(x_0, \mu_0) \le L(x, \mu_0), \ \forall x \in D,$$

$$\text{or, } p(x_0) - \tau q(x_0) + \sum_{i=1}^{s} \mu_{0i} r_i(x_0) \le p(x) - \tau q(x) + \sum_{i=1}^{s} \mu_{0i} r_i(x), \ \forall x \in D,$$

which together with equality (**4**.3), yields

$$p(x_0) - \tau q(x_0) \le p(x) - \tau q(x) + \sum_{i=1}^{s} \mu_{0i} r_i(x), \ \forall x \in D. \tag{4.4}$$

Here, we consider two cases.
Case I: Let $r_i(x) \le \frac{-1}{r_i(x)}$, for all $r_i(x) \le 0$, $\forall i \in I$. Since $\mu_{0i} \in \mathbb{R}_+, \forall i \in I$, then, the following inequality

$$\mu_{0i} r_i(x) \le \mu_{0i} \frac{-1}{r_i(x)}, \forall i \in I$$

holds. Equivalently, we have

$$\sum_{i=1}^{s} \mu_{0i} r_i(x) \le \sum_{i=1}^{s} \mu_{0i} \frac{-1}{r_i(x)}.$$

On combining the above inequality along with inequality (**4**.4), we have

$$p(x_0) - \tau q(x_0) \leq p(x) - \tau q(x) + \sum_{i=1}^{s} \mu_{0i} \frac{-1}{r_i(x)}, \ \forall x \in D.$$

By the definition of inverse barrier function $b(x) = \sum_{i=1}^{s} \frac{-1}{r_i(x)}$. The above inequality reduces to

$$p(x_0) - \tau q(x_0) \leq p(x) - \tau q(x) + \mu_{0i} b(x), \ \forall x \in D. \quad (\mathbf{4}.5)$$

Case II: Let $r_i(x) \leq -\ln(-r_i(x))$, for all $r_i(x) \leq 0, \forall i \in I$. Since $\mu_{0i} \in \mathbb{R}_+$, then, the following inequality

$$\mu_{0i} r_i(x) \leq -\mu_{0i} \ln(-r_i(x)), \forall i \in I$$

holds. Equivalently, we have

$$\sum_{i=1}^{s} \mu_{0i} r_i(x) \leq -\sum_{i=1}^{s} \mu_{0i} \ln(-r_i(x)).$$

On combining the above inequality along with inequality (**4**.4), we have

$$p(x_0) - \tau q(x_0) \leq p(x) - \tau q(x) - \sum_{i=1}^{s} \mu_{0i} \ln(-r_i(x)), \ \forall x \in D.$$

By the definition of logarithmic barrier function $b(x) = -\sum_{i=1}^{s} \ln(-r_i(x))$. The above inequality reduces to

$$p(x_0) - \tau q(x_0) \leq p(x) - \tau q(x) + \mu_{0i} b(x), \ \forall x \in D. \quad (\mathbf{4}.6)$$

The inequalities (**4**.5) and (**4**.6) conclude the similar inequality in both the cases. Since $\gamma \geq \max\{\mu_{0i}, i \in I\}$, then the inequalities (**4**.5) and (**4**.6) can be written as

$$p(x_0) - \tau q(x_0) \leq p(x) - \tau q(x) + \gamma b(x), \ \forall x \in D.$$

Thus, by the definition of $\beta(\gamma)$, we get

$$p(x_0) - \tau q(x_0) \leq \beta(\gamma), \ \forall x \in D.$$

Taking limit $\gamma \to 0^+$ on the both side of the above inequality, it follows that

$$p(x_0) - \tau q(x_0) \leq \lim_{\gamma \to 0^+} \beta(\gamma). \quad (\mathbf{4}.7)$$

On the other hand, the functions $p(x)$, $q(x)$ and $r_i(x), \forall i \in I$, are continuous, therefore, by Theorem **3**.*1* it follows that

$$p(x_0) - \tau q(x_0) \geq \lim_{\gamma \to 0^+} \beta(\gamma). \quad (\mathbf{4}.8)$$

On combining the inequalities (**4**.7) and (**4**.8) we get

$$p(x_0) - \tau q(x_0) = \lim_{\gamma \to 0^+} \beta(\gamma).$$

Hence, this completes the proof. □

Now, we shall prove the converse of the above Theorem **4**.*1*.

Theorem 4.2. *Let x_0 be the limit point to the convergent sequence $\{x^k\}$ of the barrier penalized optimization problem* (FOP)$_{\tau\gamma}$. *Assume that*

(i) *x^k is a solution to* (FOP)$_{\tau\gamma}$ *and γ_k satisfied the condition $0 < \gamma_{k+1} < \gamma_k$, in which $\gamma_k \to 0$ as $k \to \infty$,*

(ii) *the functions $p(x), q(x)$ and $r_i(x), \forall i \in I$ are continuous at x_0 on X.*

Furthermore, if γ is assumed to be a sufficient large barrier parameter, then there exist Lagrange multipliers $\mu_0 \in R_+^s$ such that (x_0, μ_0) is a saddle-point of the problem (FOP).

Proof. Firstly, we show that x_0 is an optimal solution to (FOP). By using assumptions $(i), (ii)$ and following the footsteps of Theorem **3**.*2*, we see that x_0 is feasible solution in (FOP) and hence it is an optimal solution to (FOP). Henceforth, by Lemma **2**.*1*, x_0 is also an optimal solution to $(\text{FOP})_\tau$.

Since x_0 is an optimal solution to the problem (FOP), therefore, there exist Lagrange multipliers $\mu_0 \in R^s_+$ such that the KKT necessary optimality conditions (**2**.1)-(**2**.2) are satisfied at x_0. In what follows, we prove that (x_0, μ_0) is a saddle-point of (FOP). From the feasibility of x_0 to (FOP) and equality (**2**.2), we have

$$\sum_{i=1}^{s} \mu_i r_i(x_0) \leq \sum_{i=1}^{s} \mu_{0i} r_i(x_0), \ \forall \mu \in R^s_+,$$

or,

$$p(x_0) - \tau q(x_0) + \sum_{i=1}^{s} \mu_i r_i(x_0) \leq p(x_0) - \tau q(x_0) + \sum_{i=1}^{s} \mu_{0i} r_i(x_0), \ \forall \mu \in R^s_+.$$

That is

$$L(x_0, \mu) \leq L(x_0, \mu_0), \ \forall \mu \in R^s_+.$$

Definition **4**.2 implies that the above inequality satisfied the first condition of saddle-point for (FOP). Now, we left to prove the remaining second condition of saddle-point for (FOP). Let us suppose that the second condition is holds, then we have

$$L(x_0, \mu_0) \leq L(x, \mu_0), \ \forall x \in D,$$

or,

$$p(x_0) - \tau q(x_0) + \sum_{i=1}^{s} \mu_{0i} r_i(x_0) \leq p(x) - \tau q(x) + \sum_{i=1}^{s} \mu_{0i} r_i(x), \ \forall x \in D.$$

Using the KKT necessary condition (**2**.2), the above inequality reduces to

$$p(x_0) - \tau q(x_0) \leq p(x) - \tau q(x) + \sum_{i=1}^{s} \mu_{0i} r_i(x), \ \forall x \in D,$$

equivalently, for all $x \in D$

$$p(x_0) - \tau q(x_0) - (p(x) - \tau q(x)) \leq \sum_{i=1}^{s} \mu_{0i} r_i(x) \leq 0.$$

It follows that

$$p(x_0) - \tau q(x_0) - (p(x) - \tau q(x)) \leq 0,$$

which shows that x_0 is an optimal solution to $(\text{FOP})_\tau$. Therefore, by Lemma **2**.*1*, x_0 is an optimal solution to (FOP). Hence, we conclude that (x_0, μ_0) is a saddle-point of the problem (FOP). □

Now, to authenticate the validity of the above Theorem **4**.*2*, we furnish the following example in which we use the logarithmic barrier function.

Example 4.1. *Let $X = R$ and consider the following fractional optimization problem*

$$\text{(FOP3)} \ \textit{minimize} \ \frac{2x+3}{x+3}$$
$$\textit{subject to} \ (x-1, \ x^2-1) \leq \mathbf{0}.$$

Note that $D = \{x \in X : -1 \leq x \leq 1\}$ is the set of feasible solution to (FOP3) and $p(x) \geq 0, q(x) > 0, \forall x \in D$.

TABLE 2

Iter. k	γ_k	$x_{\gamma_k} = x_{k+1}$	$f(x_{\gamma_k})$	$b(x_{\gamma_k})$	$\beta(\gamma_k)$	$\gamma_k b(x_{\gamma_k})$	$\mu_0^k = (\mu_{0_1}^k, \mu_{0_2}^k)$
1	*0.5*	*-0.7583*	*0.3625*	*0.1265*	*0.4258*	*0.0632*	$\begin{bmatrix} 0.2844, \\ 1.7843 \end{bmatrix}$
2	*0.05*	*-0.9677*	*0.0484*	*0.9029*	*0.0935*	*0.0451*	$\begin{bmatrix} 0.0254, \\ 1.5225 \end{bmatrix}$
3	*0.005*	*-0.9966*	*0.0051*	*1.8679*	*0.0144*	*0.0093*	$\begin{bmatrix} 0.0002, \\ 1.4680 \end{bmatrix}$
4	*0.0005*	*-0.9996*	*0.0006*	*2.7960*	*0.0020*	*0.0014*	$\begin{bmatrix} 0.0002, \\ 1.2496 \end{bmatrix}$
5	*0.00005*	*-0.9999*	*0.0001*	*3.3980*	*0.0003*	*0.0001*	$\begin{bmatrix} 0.0000, \\ 0.4999 \end{bmatrix}$

The parametric form of the above fractional optimization problem (FOP3) is

$$(\text{FOP3})_\tau \ \textit{minimize} \ f(x) = (2x+3) - \tau(x+3)$$
$$\textit{subject to} \ (x-1, \ x^2-1) \leq \mathbf{0}, \ \tau \in R_+.$$

Further, the barrier penalized optimization problem, using Frisch's logarithmic barrier function, with the parametric form of the problem (FOP3) is

$$(\text{FOP3})_{\tau\gamma} \ \textit{minimize} \ \{(2x+3) - \tau(x+3) + \gamma(-\ln(1-x) - \ln(1-x^2))\},$$
$$\forall x \in D' \ \textit{and} \ \gamma > 0.$$

We take $\tau = \frac{1}{2}$ *and let* $\gamma_1 = 0.5$*, the parameter* $\eta = 0.1$ *and* $\epsilon = 0.0002$*, therefore, by Table 2, it is easily seen that* $\gamma_k \to 0$ *as* $k \to \infty$*. Further, we get the sequence of* $\{x_k\}$ *which converge to* $x_0 = -1$*. Hence, by Definition 2.1, we conclude that* $x_0 = -1$ *is an optimal solution to (FOP3)τ, t herefore, by* Lemma **2**.*1,* $x_0 = -1$ *is also an optimal solution to (FOP3). The Lagrange function* $L(x, \mu)$ *is given by* $L(x, \mu) = p(x) - \tau q(x) + \sum_{i=1}^{s} \mu_i r_i(x)$*.. It can be easily verified that* (x_0, μ_0) *is satisfied all the conditions of Definition 4.2, where* $\mu_0 = (0, \frac{1}{2})$*. Thus, we conclude that all hypotheses of Theorem 4.2 are hold and* (x_0, μ_0) *is the saddle-point of the Lagrange function defined for the considered fractional problem (FOP3).*

5. Conclusions

In this paper, we employed the barrier function method to solve the fractional optimization problem after transforming it into a parametric form. We have established the equivalence between an optimal solution to (FOP) and a limit point of sequence constructed by its associated problem $(\text{FOP})_{\tau\gamma}$ and proved that an optimal solution to (FOP) is a limit point of the sequence, which is constructed by its associated problem $(\text{FOP})_{\tau\gamma}$. Thereafter, we have also shown the relationship between a saddle-point of Lagrange function defined for (FOP) and a limit point of sequence constructed by its associated problem $(\text{FOP})_{\tau\gamma}$. Moreover, we checked the established results with the help of non-trivial examples.
Future research will orient to discuss the same method for vector fractional optimization problem.

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