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EXISTENCE OF THREE SOLUTIONS FOR ELLIPTIC DIRICHLET PROBLEMS INVOLVING THE p-LAPLACIAN

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The aim of this note is to establish the existence of at least three weak solutions for a Dirichlet boundary value problem involving the p-Laplacian. The approach is based on variational methods.

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1. Introduction

In this paper, we are interested in ensuring the existence of at least three weak solutions for the following Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a non-empty bounded open set with a smooth boundary $\partial\Omega$, $2 \leq p < N$, λ is a positive parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function.

Motivated by the fact that such problems are used to describe a large class of physical phenomena, many authors have studied the existence and multiplicity of solutions for (1).

The aim of this paper is to establish a precise interval of parameters λ for which problem (1) admits at least three non-zero solutions. Our analysis is based mainly on a three critical points theorem due to Bonanno and Marano [5] to transfer the existence of critical points of the Euler functional to the existence of three solutions to problem (1).

In the literature many papers (see, for instance, the papers [1, 2, 3, 4, 6, 9, 10] and references therein) deal with nonlinear elliptic problems with Dirichlet boundary conditions. For example, Li and Tang in [10], using a three critical points theorem of Ricceri [11], established the existence of an interval $\Lambda \subseteq [0, +\infty[$ and a positive real number ρ such that for each $\lambda \in \Lambda$ the quasilinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda F_u(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = \lambda F_v(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

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admits at least three weak solutions whose norms in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ are less than ρ, and in [1] some similar results for the quasilinear elliptic system

$$\begin{cases} \Delta_{p_i} u_i + \lambda F_{u_i}(x, u_1, \dots, u_n) = 0 & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

for $1 \leq i \leq n$, were obtained. In [2], the authors using a three critical points theorem of Averna and Bonanno [3], proved the existence of a definite interval, in which λ lies, system (2) admits at least three weak solutions. Also in [4], the authors based on a very recent three critical points theorem due to Bonanno and Maranno [5], established the existence of an open interval Λ_1 for each λ of which, the quasilinear elliptic system

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda F_u(x, u, v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a non-empty bounded open set with a smooth boundary $\partial\Omega$, $p, q > N$, λ is a positive parameter, $a, b \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a \geq 0$ and $\operatorname{ess\,inf}_\Omega b \geq 0$, and $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(\cdot, t_1, t_2)$ is continuous in $\overline{\Omega}$ for all $(t_1, t_2) \in \mathbb{R}^2$ and $F(x, \cdot, \cdot)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, admits at least three distinct weak solutions in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

We note that the ideas used here are motivated by the corresponding ones in [6].

This paper is arranged as follows. In Section 2, we recall some basic notations and definitions and our main tool (Theorem 2.1), while Section 3 is devoted to our main result (Theorem 3.1) and a consequence in the autonomous case.

2. **Preliminaries**

Our main tool is the following three critical points theorem due to Bonanno and Marano [5]. Here, X^* denotes the dual space of X.

Theorem 2.1 (Theorem 3.6 of [5]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$, such that:

(a_1) $\frac{\sup_{\Phi(x) \leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a_2) *for each* $\lambda \in \Lambda_r = \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$, the functional $J_\lambda := \Phi - \lambda\Psi$ has at least three distinct critical points in X.

In the sequel, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) *there exist two non-negative constants* a_1, a_2 *and* $q \in]1, pN/(N-p)[$ *such that*

$$|f(x,t)| \le a_1 + a_2|t|^{q-1},$$

for every $(x,t) \in \Omega \times \mathbb{R}$.

Throughout this paper, X will denote the Sobolev space $W_0^{1,p}(\Omega)$ with the norm

$$\|u\|_p = \Big(\int_\Omega |\nabla u(x)|^p dx\Big)^{1/p}.$$

We say that $u : \Omega \to \mathbb{R}$ is a (weak) solution to problem (1) if $u \in X$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \lambda \int_\Omega f(x,u(x))v(x)dx = 0$$

for all $v \in X$.

Let $p^* = pN/(N-p)$ be the critical Sobolev exponent and denote, as usual, with Γ the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz, \qquad \forall\, t > 0.$$

From the Sobolev embedding theorem (see, for instance [13, Theorem A.5]) there exists $c \in \mathbb{R}^+$ such that

$$\|u\|_{L^{p^*}(\Omega)} \le c\|u\|_p\,, \qquad u \in X. \tag{3}$$

The best constant that appears in (3) is

$$c = \pi^{-1/2}N^{-1/p}\Big(\frac{p-1}{N-p}\Big)^{1-1/p}\left\{\frac{\Gamma(1+N/2)\Gamma(N)}{\Gamma(N/p)\Gamma(1+N-N/p)}\right\}^{1/N}, \tag{4}$$

see, for instance, [14].

Fixing $q \in [1, p^*[$, again from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \le c_q\|u\|_p\,, \qquad u \in X, \tag{5}$$

and, in particular, the embedding $X \hookrightarrow L^q(\Omega)$ is compact.

Since $q < p^*$, by using Hölder's inequality we have

$$\|u\|_{L^q(\Omega)} \le m(\Omega)^{\frac{p^*-q}{p^*q}}\|u\|_{L^{p^*}(\Omega)},$$

and due to (4), $c_q \le c\, m(\Omega)^{\frac{p^*-q}{p^*q}}$, where $m(\Omega)$ denotes the Lebesgue measure of the set Ω.

Now, fix $x^0 \in \Omega$ and pick r_1, r_2 with $0 < r_1 < r_2$ such that $S(x^0, r_1) \subset S(x^0, r_2) \subseteq \Omega$, where $S(x^0, r_i)$ denotes the ball with center at x^0 and radius r_i for $i = 1, 2$.

Finally, put

$$\kappa := \frac{r_2 - r_1}{\pi^{N/2p}}\Big(\frac{p\Gamma(1+N/2)}{r_2^N - r_1^N}\Big)^{1/p}, \tag{6}$$

and

$$K_1 := \frac{p^{\frac{1-p}{p}} c_1\big((r_2/r_1)^N - 1\big)}{(r_2 - r_1)^p}, \qquad K_2 := \frac{p^{\frac{q-p}{p}} c_q^q\big((r_2/r_1)^N - 1\big)}{q(r_2 - r_1)^p}. \tag{7}$$

3. Main results

We need the following proposition in the proof of our main result.

Proposition 3.1. *Let $T : X \to X^*$ be the operator defined by $T(u)(v) = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x)\nabla v(x)dx$ for every $u, v \in X$. Then T admits a continuous inverse on X^*.*

Proof. If $x, y \in \mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$, owing to (2.2) of [12], there exists a positive constant c_p such that $\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \geq c_p|x-y|^p$. Thus, it is easy to verify that $(T(u) - T(v))(u - v) \geq c_p\|u - v\|_p^p$, for every $u, v \in X$. This actually means that T is a uniformly monotone operator in X. For every $u \in X \setminus \{0\}$ we have

$$\frac{\langle T(u), u\rangle}{\|u\|_p} = \frac{\int_\Omega |\nabla u(x)|^p dx}{\|u\|_p} = \|u\|_p^{p-1},$$

hence T is coercive. Also, T is hemicontinuous, since it is continuous. Therefore, the conclusion follows immediately by applying Theorem 26.A of [15]. □

By standard arguments, we can prove the following result.

Proposition 3.2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that assumption* (i) *holds and F defined by*

$$F(x, \xi) := \int_0^\xi f(x, t)dt, \qquad \forall (x, \xi) \in \Omega \times \mathbb{R}. \tag{8}$$

Then $\Psi : X \to \mathbb{R}$ defined by $\Psi(u) := \int_\Omega F(x, u(x))dx$ is a Gâteaux differentiable functional on X with compact derivative $\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)dx$ for every $v \in X$.

Proof. Suppose $u, v \in X$ and $t \neq 0$. Then, by the Mean Value Theorem

$$\begin{aligned}
&\left|\frac{\Psi(u + tv) - \Psi(u)}{t} - \int_\Omega f(x, u(x))v(x)dx\right| \\
&\leq \int_\Omega \big|f(x, u(x) + t\zeta(x)v(x)) - f(x, u(x))\big||v(x)|dx \\
&\leq \|v\|_\infty \int_\Omega \big|f(x, u(x) + t\zeta(x)v(x)) - f(x, u(x))\big|dx,
\end{aligned}$$

where $0 < \zeta(x) < 1$ for every $x \in \Omega$ for which $F(x, \xi)$ is differentiable with respect to ξ. Since f is a Carathéodory function, we have $\lim_{t\to 0} f(x, u(x) + t\zeta(x)v(x)) = f(x, u(x))$ for a.e. $x \in \Omega$. Then, by the assumption (i) on $f(x, t)$ we have

$$\begin{aligned}
&\big|f(x, u(x) + t\zeta(x)v(x)) - f(x, u(x))\big| \\
&\leq 2a_1 + a_2\big((k + 1)|u(x)|^{q-1} + k|v(x)|^{q-1}\big)
\end{aligned}$$

for any $|t| < 1$ and some $k > 0$. Therefore, the Lebesgue Convergence Theorem implies $\lim_{t\to 0} \frac{\Psi(u+tv)-\Psi(u)}{t} = \int_\Omega f(x,u(x))v(x)dx$. So, $\Psi : X \to \mathbb{R}$ is a Gâteaux differentiable functional at every $u \in X$ with derivative $\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$ for every $v \in X$. To show that $\Psi' : X \to X^*$ is a compact operator, it suffices to show that Ψ' is strongly continuous on X. Let $u_n \rightharpoonup u$ in X. Then $u_n \to u$ uniformly on Ω. Now since $F(x,\xi)$ is differentiable with respect to ξ for a.e. $x \in \Omega$, the derivative of F is continuous in $\mathbb{R}$ for a.e. $x \in \Omega$, so $f(x,u_n) \to f(x,u)$. Thus $\Psi'(u_n) \to \Psi'(u)$, i.e., Ψ' is strongly continuous on X and this implies that Ψ' is a compact operator by [15, Proposition 26.2]. □

Our main result is the following theorem.

Theorem 3.1. *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that assumption* (i) *holds. Furthermore, assume that*

(ii) *$F(x,\xi) \geq 0$ for every $(x,\xi) \in \Omega \times \mathbb{R}^+$, where F is given by (8);*

(iii) *there exist two positive constants b and $s < p$ such that $F(x,\xi) \leq b(1 + |\xi|^s)$, for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$;*

(iv) *there exist two positive constants γ and δ, with $\delta > \gamma\kappa$ such that*

$$\frac{\inf_{x\in\Omega} F(x,\delta)}{\delta^p} > a_1\frac{K_1}{\gamma^{p-1}} + a_2K_2\gamma^{q-p}$$

where a_1, a_2 are given in (i) *and κ, K_1, K_2 are given by* (6) *and* (7).

Then, for each λ belonging to

$$\Lambda_1 := \left]\frac{(r_2/r_1)^N - 1}{p(r_2-r_1)^p}\frac{\delta^p}{\inf_{x\in\Omega} F(x,\delta)}, \frac{(r_2/r_1)^N - 1}{p(r_2-r_1)^p}\frac{1}{a_1\frac{K_1}{\gamma^{p-1}} + a_2K_2\gamma^{q-p}}\right[,$$

problem (1) admits at least three weak solutions in X.

Proof. In order to apply Theorem 2.1, we define $\Phi, \Psi : X \to \mathbb{R}$ by

$$\Phi(u) := \frac{\|u\|_p^p}{p}, \qquad \text{and} \qquad \Psi(u) := \int_\Omega F(x,u(x))dx, \qquad \forall u \in X.$$

The functional Φ is continuously Gâteaux differentiable with $\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x)\nabla v(x)dx$ for every $u, v \in X$ (see, e.g., [7, page 133]). By Proposition 3.1, Φ' admits a continuous inverse on X^*, and since Φ' is monotone, Φ is sequentially weakly lower semicontinuous (see [15, Proposition 25.20]). Also, Ψ is continuously Gâteaux differentiable functional with $\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)dx$ for every $u, v \in X$, and Ψ' is a compact operator (see Proposition 3.2).

A critical point of the functional $\Phi - \lambda\Psi$ is a function $u \in X$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0,$$

for every $v \in X$. Hence the critical points of the functional $\Phi-\lambda\Psi$ are weak solutions of problem (1).

Clearly Φ is coercive and $\Phi(0) = \Psi(0) = 0$. By (i), we have

$$F(x,\xi) \leq a_1|\xi| + a_2\frac{|\xi|^q}{q}, \tag{9}$$

for every $(x,\xi)\in\Omega\times\mathbb{R}$. Taking into account (5) and (9) it follows that

$$\Psi(u)=\int_\Omega F(x,u(x))dx\le a_1\|u\|_{L^1(\Omega)}+\frac{a_2}{q}\|u\|^q_{L^q(\Omega)}\le a_1c_1\|u\|_p+\frac{a_2}{q}c_q^q\|u\|_p^q.$$

Then, for every $u\in X$ such that $\Phi(u)\le r$, we get

$$\Psi(u)\le a_1c_1\sqrt[p]{rp}+\frac{a_2}{q}c_q^q(rp)^{q/p}.$$

Hence

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)\le a_1c_1\sqrt[p]{rp}+\frac{a_2}{q}c_q^q(rp)^{q/p}. \tag{10}$$

So, according to (10), we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r}\le a_1c_1\sqrt[p]{\frac{p}{r^{p-1}}}+\frac{a_2}{q}c_q^qp^{q/p}r^{q/p-1}, \tag{11}$$

for every $r>0$.

Next, put

$$w(x):=\begin{cases}0 & \text{if } x\in\Omega\setminus B(x^0,r_2),\\ \frac{\delta}{r_2-r_1}\left(r_2-\sqrt{\sum_{j=1}^N(x_j-x_j^0)^2}\right) & \text{if } x\in B(x^0,r_2)\setminus B(x_0,r_1),\\ \delta & \text{if } x\in B(x^0,r_1).\end{cases}$$

It is easy to verify that $w\in X$ and we have

$$\begin{aligned}\Phi(w) &= \frac{1}{p}\int_{B(x^0,r_2)\setminus B(x^0,r_1)}\frac{\delta^p}{(r_2-r_1)^p}dx\\ &= \frac{\delta^p}{p(r_2-r_1)^p}\Big(m\big(B(x^0,r_2)\big)-m\big(B(x^0,r_1)\big)\Big)\\ &= \frac{\delta^p}{p(r_2-r_1)^p}\frac{\pi^{N/2}}{\Gamma(1+N/2)}(r_2^N-r_1^N). \end{aligned} \tag{12}$$

Bearing in mind that $\delta>\gamma\kappa$, it follows that $\Phi(w)>\gamma^p=:r$.

At this point, thanks to (ii), we obtain

$$\Psi(w)\ge\int_{B(x^0,r_1)}F(x,\delta)dx\ge\inf_{x\in\Omega}F(x,\delta)\frac{r_1^N\pi^{N/2}}{\Gamma(1+N/2)}. \tag{13}$$

Hence, in view of (12) and (13), one has $\frac{\Psi(w)}{\Phi(w)}\ge\frac{p(r_2-r_1)^p}{(r_2/r_1)^N-1}\frac{\inf_{x\in\Omega}F(x,\delta)}{\delta^p}$. By virtue of (11) and taking into account (iv), we get

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,\gamma^p])}\Psi(u)}{\gamma^p} &\le a_1c_1\sqrt[p]{p}\frac{1}{\gamma^{p-1}}+\frac{a_2}{q}c_q^qp^{q/p}\gamma^{q-p}\\ &= \frac{p(r_2-r_1)^p}{(r_2/r_1)^N-1}\left(a_1\frac{K_1}{\gamma^{p-1}}+a_2K_2\gamma^{q-p}\right)\\ &< \frac{p(r_2-r_1)^p}{(r_2/r_1)^N-1}\frac{\inf_{x\in\Omega}F(x,\delta)}{\delta^p}\\ &\le \frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Therefore, assumption (a_1) of Theorem 2.1 is satisfied.

Moreover, if $s < p$, for every $u \in X$, $|u|^s \in L^{p/s}(\Omega)$ and the Hölder's inequality gives

$$\int_\Omega |u(x)|^s dx \leq \|u\|^s_{L^p(\Omega)} m(\Omega)^{\frac{p-s}{p}}, \qquad \forall\, u \in X.$$

Then, by (5), one has

$$\int_\Omega |u(x)|^s dx \leq c_p^s \|u\|_p^s m(\Omega)^{\frac{p-s}{p}}, \qquad \forall\, u \in X. \tag{14}$$

Finally, from (14) and due to assumption (iii), we obtain

$$J_\lambda(u) \geq \frac{\|u\|_p^p}{p} - \lambda b c_p^s\, m(\Omega)^{\frac{p-s}{p}} \|u\|_p^s - \lambda b\, m(\Omega), \qquad \forall\, u \in X.$$

Therefore, $\Phi - \lambda\Psi$ is a coercive functional for every positive parameter λ, in particular, for every $\lambda \in \Lambda_1 \subseteq \left]\frac{\Phi(w)}{\Psi(w)}, \frac{\gamma^p}{\sup_{\Phi(u)\leq\gamma^p}\Psi(u)}\right[$.

Then, also assumption (a_2) holds. Hence all assumptions of Theorem 2.1 are satisfied, so that, for each $\lambda \in \Lambda_1$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points that are weak solutions of problem (1). □

Remark 3.1. *We note that if $f(x, 0) \not\equiv 0$ in Ω, then Theorem 3.1 ensures the existence of at least three non-zero weak solutions for problem (1). Moreover if, in addition, f is a non-negative function, the Strong Maximum Principle (see* [8, Theorem 8.19]*) and Theorem 3.1 guarantee the existence of at least three positive weak solutions.*

Finally, we give a particular consequence of Theorem 3.1 as follows.

Corollary 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative function such that $f(0) \neq 0$. Assume that*

(j) *there exist two non-negative constants a_1, a_2 and $q \in]1, p^*[$ such that*

$$f(t) \leq a_1 + a_2|t|^{q-1},$$

for every $t \in \mathbb{R}$;

(jj) *there exists a positive constant δ with $\delta > \kappa$ such that*

$$\frac{\int_0^\delta f(t)dt}{\delta^p} > a_1 K_1 + a_2 K_2;$$

(jjj)

$$\lim_{t\to+\infty} \frac{f(t)}{t^\alpha} = 0,$$

for some $0 \leq \alpha < 1$.

Then, for each λ belonging to

$$\Lambda_2 := \left] \frac{(r_2/r_1)^N - 1}{p(r_2 - r_1)^p} \frac{\delta^p}{F(\delta)}, \frac{(r_2/r_1)^N - 1}{p(r_2 - r_1)^p} \frac{1}{a_1 K_1 + a_2 K_2} \right[,$$

the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

possesses at least three positive weak solutions in X.

Proof. The conclusion readily follows from Theorem 3.1 and Remark 3.1 by choosing $\gamma = 1$ and taking into account that assumption (jjj) implies (iii). □

4. **Conclusion**

Based on a recent three critical points theorem obtained by Bonanno and Marano [5], we established the existence of an open interval $]\lambda', \lambda''[$ for each λ in the interval a class of Dirichlet boundary value problems involving the p-Laplacian and depending on λ admits at least three weak solutions.

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