CODAZZI-CARTAN CONNECTIONS ON FINSLER SPACES

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The aim of this paper is to introduce Codazzi-Cartan connections on Finsler spaces. These connections are generalizations of Cartan connections in the sense of substituting the Codazzi symmetry instead of parallelism in the vertical sub-bundle. The β-change is considered on Finslerian spaces and the induced structures on the indicatrix spaces will be investigated by using a natural diffeomorphism.

Keywords: Codazzi-Cartan connection, Finsler space, statistical manifold, tangent bundle.

1. Introduction

Finsler geometry is one of the generalizations of Riemannian geometry. The history of Finsler geometry is contained in consistent monographs, such as [3, 4, 5]. In this framework, a characteristic trait is the considerable amount of tensors and notations. A defining mapping $E : \mathrm{T}M \to \mathbb{R}^+$ endowed with certain properties provides the most noteable geometric objects of the theory. The occurrence of E may be motivated by introducing a distance function $\mathrm{d} : \mathrm{T}M \to \mathbb{R}^+$ which yields $E := \lim_{t\to 0} \frac{\mathrm{d}(x,\gamma(t))}{t}$, where γ is a curve on M, with initial rate $y \in \mathrm{T}_xM$ (a more detailed motivation can be read in [7]). As well, a defining alternative might be a PDE having E as a solution having the properties of a Finsler metric (e.g., many of dynamical systems are examples of such differential equations). Several example-based developments of this kind are provided in [7].

A statistical manifold is a triple (M, g, ∇), where g is a Riemannian metric on a manifold M and ∇ is a symmetric linear connection, such that the cubic tensor field $\mathfrak{C} = \nabla g$ is totally symmetric, namely the following Codazzi equations hold:

$$\mathfrak{C}_{ijk} = \partial_k(g_{ij}) - \Gamma^h{}_{ik}g_{jh} - \Gamma^h{}_{jk}g_{ih}, \qquad \mathfrak{C}_{ijk} = \mathfrak{C}_{jki} = \mathfrak{C}_{kij},$$

where $\Gamma^i{}_{jk}$ are the Christoffel symbols of ∇.

Several works naturally extend statistical Riemannian structures to the Finsler tangent bundle (e.g., [11]). However, our approach in this paper is a different one. Indeed, we define the Codazzi-Cartan connections for a Finslerian metric, which include in particular the special case of the Cartan connection.

By using a β-change on a Finsler metric, one can get another Cartan connection. So, starting from a Cartan connection we have two distinct connections; one of them is a Codazzi-Cartan connection, and the other is defined by a β-change. Using the relations between these connections, we derive some necessary conditions, under which the Finslerian metric reduces to a Riemannian one. We know that the indicatrix space modifies under a β-change. We would like to define a natural diffeomorphism between these two indicatrix

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spaces, and then pullback the Codazzi structures in order to see how they mutually connect. This procedure will be ensured by a pseudo-Cartan connection whose nonlinear component is derived by this natural diffeomorphism.

This paper is organized as follows: the second Section includes preliminaries on nonlinear Finslerian connections and the related structures. The third Section is devoted to the study of Codazzi structures and some of their features. The final Section explains the β-change. This Section introduces a natural diffeomorfism between the indicatrix spaces, which are related to a β-change. Moreover, we study the induced structures and obtain relations under which they coincide.

2. Preliminaries

This Section contains the basic concepts needed to proceed. Details of these preliminaries can be found in e.g. [10]. In the following, we assume that the dimension of the base manifold M is n.

2.1. Split tangent bundle and (non)linear connections

Let N be a nonlinear connection with respect to a Finsler metric. This is provided by n^2 functions $N^i_j(x,y) \in C^\infty(\mathrm{T}M)$; for a vector field $\partial_i := \frac{\partial}{\partial x^i}$, its horizontal lift $(\partial_i)^H$ is

$$\delta_i := \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x,y)\frac{\partial}{\partial y^j}.$$

The local fields δ_i and $\partial_{\bar{i}} := \frac{\partial}{\partial y^i} = (\partial_i)^V$ form a local basis in $H\mathrm{T}M$ and $V\mathrm{T}M$, respectively; thus

$$\{\delta_i, \partial_{\bar{i}}\}, \quad i = 1, \cdots, n, \quad \bar{i} = \bar{1}, \cdots, \bar{n}, \tag{1}$$

is a local basis in $\mathrm{TT}M$ adapted to the splitting $\mathrm{TT}M = H\mathrm{T}M + V\mathrm{T}M$. We denote its dual basis by $(\mathrm{d}x^i, \delta y^i)$, where $\delta y^i := \mathrm{d}y^i + N^i_j(x,y)\mathrm{d}x^j$. The torsion of the nonlinear connection N is defined by $t^i{}_{jk} := \partial_{\bar{k}} N^i_j - \partial_{\bar{j}} N^i_k$. By setting $R^i{}_{jk} := \delta_k N^i_j - \delta_j N^i_k$, the Lie brackets of the vector fields from the adapted basis (1) are given by

$$[\delta_j, \delta_k] = R^i{}_{jk}\partial_{\bar{i}}, \quad [\delta_j, \partial_{\bar{k}}] = \partial_{\bar{k}} N^i_j \partial_{\bar{i}}, \quad [\partial_{\bar{j}}, \partial_{\bar{k}}] = 0.$$

The vanishing of $R^i{}_{jk}$ is an iff condition for $[\delta_j, \delta_k] \in H\mathrm{T}M$. Hence, we call $R^i{}_{jk}$ as the *curvature tensor* field components of the nonlinear connection N.

2.2. Finsler spaces

Let (M, E) be a Finsler space; we consider the bilinear form

$$g_{ij}(x,y) := \frac{1}{2}\partial_{\bar{i}}\partial_{\bar{j}} E^2, \tag{2}$$

called the metric (or the fundamental) tensor, with the inverse g^{ij} defined by $g_{ij}g^{jk} = \delta^k_i$. The transvection by this metric makes it possible to move up and down the indices. For instance, if $A^i{}_{jk}$ is a tensor, then we can produce the transvected tensor $A_{jkm} := g_{im}A^i{}_{jk}$. In a similar manner, by using g^{ij}, we can move indices up. Our agreement is that in any transvection, the new index stands on the rightmost position. An essential symmetric $(0,3)$-tensor is $C_{ijk} := \frac{1}{2}\partial_{\bar{i}} g_{jk}$, called *the Cartan tensor field.* Further, by the transvection $C^k{}_{ij} := g^{km}C_{ijm}$, define the (0,2)-tensor $C(\partial_i, \partial_j) := C^k{}_{ij}\partial_k$, whose vanishing is equivalent with reducing the Finsler space to a Riemannian one. We denote the Christoffel symbols of the metric tensor (2) by

$$\gamma^i{}_{jk}(x,y) := \frac{1}{2} g^{ir}(x,y)(\partial_j g_{rk}(x,y) + \partial_k g_{rj}(x,y) - \partial_r g_{jk}(x,y)).$$

As well. there exist two important objects determined only by E: *the canonical spray* $S := y^i\partial_i - 2G^i(x,y)\partial_{\bar{i}}$ associated with the Finsler space (M, E), where $G^i(x,y) := \frac{1}{2}\gamma^i{}_{jk}(x,y)y^jy^k$, and *the Cartan nonlinear connection* N defined by

$$N^i_j(x,y) := \frac{1}{2}\partial_{\bar{j}}G^i(x,y), \tag{3}$$

which is globally defined on $\mathrm{T}M\backslash\{0\}$ (which we shall denote this by the same symbol N, while the context specifies which nonlinear connection we consider on the space). By using (2), we can prove that the local equality $g(\partial_{\bar{i}}, \partial_{\bar{j}}) = g_{ij}$. There is a natural lift (named *Sasaki lift*), that extends g to $H\mathrm{T}M$ and makes $V\mathrm{T}M$ and $H\mathrm{T}M$ to be orthogonal by putting $g(\delta_i, \delta_j) = g_{ij}$. We will use this lift hereafter. The formula $g(\partial_i, \partial_j) = g_{ij} + N^m_i N^n_j g_{mn}$ holds, as well. Now, we define the generalized Christioffel symbols

$$\begin{cases} F^i{}_{jh} := \dfrac{1}{2}g^{im}(\delta_j g_{mh} + \delta_h g_{jm} - \delta_m g_{jh}), & (4) \\ C^i{}_{jh} = \dfrac{1}{2}g^{im}(\partial_{\bar{j}} g_{mh} + \partial_{\bar{h}} g_{jm} - \partial_{\bar{m}} g_{jh}). & (5) \end{cases}$$

It is notable that the right hand side of (5) can be simplified to its left hand side. For a Finsler metric E, it is customary to denote a Finsler connection as $\mathcal{FC}$ [1] by a triple $(N^i_j, \mathcal{F}^i{}_{jk}, \mathcal{C}^i{}_{jk})$, where $\mathcal{F}^i{}_{jk}, \mathcal{C}^i{}_{jk} \in C^\infty(\mathrm{T}M)$ and N^i_j is just (3). Moreover, we denote the local expressions of the horizontal and vertical covariant derivatives of g with respect to a Finsler connection $(N^i_j, \mathcal{F}^i{}_{jk}, \mathcal{C}^i{}_{jk})$ by

$$\begin{cases} g_{ij,k} := \delta_k g_{ij} - g_{mj}\mathcal{F}^m{}_{ik} - g_{mi}\mathcal{F}^m{}_{jk} \\ g_{ij;k} := \partial_{\bar{k}} g_{ij} - g_{mj}\mathcal{C}^m{}_{ik} - g_{mi}\mathcal{C}^m{}_{jk}. \end{cases}$$

Due to the properties of the Sasaki metric, we can work on vertical sections only. Indeed, in a Finsler space (M, F) the local expression of a covariant derivative ∇ as a $\mathcal{FC}$ with respect to (1), and the Sasaki extension of the fundamental metric (2), is

$$\nabla_{\delta_i}\delta_j = \Gamma^k{}_{ij}\delta_k + \Gamma^{\bar{k}}{}_{ij}\partial_{\bar{k}}, \qquad \nabla_{\delta_i}\partial_{\bar{j}} = \Gamma^k{}_{i\bar{j}}\delta_k + \Gamma^{\bar{k}}{}_{i\bar{j}}\partial_{\bar{k}},$$
$$\nabla_{\partial_{\bar{i}}}\delta_j = \Gamma^k{}_{\bar{i}j}\delta_k + \Gamma^{\bar{k}}{}_{\bar{i}j}\partial_{\bar{k}}, \qquad \nabla_{\partial_{\bar{i}}}\partial_{\bar{j}} = \Gamma^k{}_{\bar{i}\bar{j}}\delta_k + \Gamma^{\bar{k}}{}_{\bar{i}\bar{j}}\partial_{\bar{k}},$$

where $\alpha, \beta, \gamma \in \{1, \cdots, n, \bar{1}, \cdots, \bar{n}\}$, $\Gamma^\gamma{}_{\alpha\beta} \in C^\infty(\mathrm{T}M)$ where $\bar{k} = n + k$. But ∇ is a Finsler connection, namely

$$\rho(\nabla_X Y) = \nabla_X(\rho Y), \quad \text{and} \quad \nabla_X V\mathrm{T}M \subseteq V\mathrm{T}M,$$

where $\rho(\delta_i) := \partial_{\bar{i}}$ and $\rho(\partial_{\bar{i}}) := -\delta_i$. So

$$\begin{cases} \Gamma^{\bar{k}}{}_{ij} = \Gamma^k{}_{i\bar{j}} = \Gamma^{\bar{k}}{}_{\bar{i}j} = \Gamma^k{}_{\bar{i}\bar{j}} = 0, \\ \Gamma^k{}_{ij} = \Gamma^{\bar{k}}{}_{i\bar{j}}, \quad \Gamma^k{}_{\bar{i}j} = \Gamma^{\bar{k}}{}_{\bar{i}\bar{j}}. \end{cases}$$

Thus we can focus only on the situations

$$\begin{cases} \nabla_{\delta_i}\partial_{\bar{j}} = \Gamma^{\bar{k}}{}_{i\bar{j}}\partial_{\bar{k}}, \\ \nabla_{\partial_{\bar{i}}}\partial_{\bar{j}} = \Gamma^{\bar{k}}{}_{\bar{i}\bar{j}}\partial_{\bar{k}}. \end{cases}$$

Also, the torsion of the Riemannian subcase shall naturally generalize to the h-torsion and to the v-torsion

$$\begin{cases} T^i{}_{jk} := \mathcal{F}^i{}_{jk} - \mathcal{F}^i{}_{kj}, \\ S^i{}_{jk} := \mathcal{C}^i{}_{jk} - \mathcal{C}^i{}_{kj}. \end{cases}$$

[1] We prefer using the notation $\mathcal{FC}$ for its components instead of FC, which denotes the abbreviation for "Finsler connection".

According to [9], a Finsler connection $(N^i_j, \mathcal{F}^i_{jk}, \mathcal{C}^i_{jk})$ has only three local components of curvature $R^i{}_{hjk}, P^i{}_{hjk}$ and $S^i{}_{hjk}$ given by

$$\begin{cases} R^i{}_{hjk} = \dfrac{\delta \mathcal{F}^i{}_{hj}}{\delta x^k} - \dfrac{\delta \mathcal{F}^i{}_{hk}}{\delta x^j} + \mathcal{F}^m{}_{hj}\mathcal{F}^i{}_{mk} - \mathcal{F}^m{}_{hk}\mathcal{F}^i{}_{mj} + \mathcal{C}^i{}_{hm}R^m{}_{jk}, \\ P^i{}_{hjk} = \dfrac{\partial \mathcal{F}^i{}_{hj}}{\partial y^k} - \mathcal{C}^i{}_{hk,j} + \mathcal{C}^i{}_{hm}P^m{}_{jk}, \\ S^i{}_{hjk} = \dfrac{\partial \mathcal{C}^i{}_{hj}}{\partial y^k} - \dfrac{\partial \mathcal{C}^i{}_{hk}}{\partial y^j} + \mathcal{C}^m{}_{hj}\mathcal{C}^i{}_{mk} - \mathcal{C}^m{}_{hk}\mathcal{C}^i{}_{mj}, \end{cases}$$

where, $P^k{}_{ji} := \partial_{\bar{i}} N^k_j - F^k_{ij}$.

3. The Codazzi-Cartan connection

This section is devoted to defining the Codazzi-Cartan connection and finding its local expression.

Definition 3.1. *A Codazzi-Cartan connection $\overset{C}{\nabla}$ on a Finsler manifold (M, E) is defined by the following axioms*

(A1) $\overset{C}{\nabla}$ *is h-metric; namely $g_{ij,k} = 0$,*

(A2) *There exist $\mathfrak{A}_{ijk} \in C^\infty(\mathrm{T}M)$ invariant under any permutation of its indices, such that*

$$g_{ij;k} = \mathfrak{A}_{ijk},$$

(A3) $\overset{C}{\nabla}$ *is h-torsion free, namely $T^i{}_{jk} = 0$,*

(A4) $\overset{C}{\nabla}$ *is v-torsion free, namely $S^i{}_{jk} = 0$,*

(A5) $N^i_j = F^i{}_{kj} y^k$.

It is obvious that Definition 3.1 is a generalization of the Cartan Finsler connection; namely $(N^i_j, F^i{}_{jk}, C^i{}_{jk})$, and as in the situation of the Cartan connection, the cases like $(\nabla_{X^V} g)(Y^V, Z^H)$ will be derived from the axioms. So, we deal with a family of connections that are generalizations Cartan-like connections in a natural way. It is notable that these axioms are mutually independent (just similar to the Cartan connection). Axiom $A5$ has another equivalent in literature too. The reference [2] contains a detailed discussion about these axioms on the Cartan connection. The axioms $A1$ and $A2$ are called as h-parallel and v-parallel properties of g with respect to the Cartan connection, and it is important to note that one can not gather them into a single axiom. More precisely, the global expression of $A1$ and $A2$ is exactly

$$\begin{cases} (\overset{C}{\nabla}_{X^V} g)(Y^V, Z^V) = (\overset{C}{\nabla}_{X^V} g)(Y^H, Z^H) = 0 \\ (\overset{C}{\nabla}_{X^H} g)(Y^V, Z^V) = (\overset{C}{\nabla}_{X^H} g)(Y^H, Z^H) = 0, \end{cases}$$

that is a special case, which is not equivalent to $(\overset{C}{\nabla}_X g)(Y, Z) = 0$, where $X^H, Y^H \in H\mathrm{T}M$, $X^V, Y^V \in V\mathrm{T}M$ and $X, Y, Z \in \mathrm{TT}M$ are arbitrarily chosen. Obviously, other cases like $(\overset{C}{\nabla}_{X^V} g)(Y^V, Z^H)$ can be determined by the axioms.

Hence, the Codazzi-Cartan and the Cartan connections will coincide when the metric g is parallel with respect to the Cartan connection, namely $\mathfrak{A} = 0$. Moreover, one can see that the approach from Definition 3.1 is different from the Codazzi equation (6.6) from [8].

Lemma 3.1. *$\mathfrak{A}$ has the following local expression*

$$\mathfrak{A}(\partial_{\bar{i}}, \partial_{\bar{j}}, \partial_{\bar{k}}) = 2C_{ijk} - 2g_{mk} \overset{C}{\Gamma}{}^{\bar{m}}{}_{\bar{i}\bar{j}} .$$

Proof. We have

$$\mathfrak{A}(\partial_{\bar{i}}, \partial_{\bar{j}}, \partial_{\bar{k}}) = g_{ij;k} = (\overset{C}{\nabla}_{\partial_{\bar{k}}} g)(\partial_{\bar{i}}, \partial_{\bar{j}}) = \partial_{\bar{k}} g_{ij} - g_{mj} \overset{C}{\Gamma}{}^{\bar{m}}{}_{\bar{k}\bar{i}} - g_{mi} \overset{C}{\Gamma}{}^{\bar{m}}{}_{\bar{k}\bar{j}},$$

which yields

$$\begin{aligned}\mathfrak{A}(\partial_{\bar{i}}, \partial_{\bar{j}}, \partial_{\bar{k}}) =& g_{ik;j} + g_{jk;i} - g_{ij;k} \\ =& \partial_{\bar{i}} g_{jk} + \partial_{\bar{j}} g_{ik} - \partial_{\bar{k}} g_{ij} - 2g_{mk} \overset{C}{\Gamma}{}^{\bar{m}}{}_{\bar{i}\bar{j}} \\ =& 2C_{ijk} - 2g_{mk} \overset{C}{\Gamma}{}^{\bar{m}}{}_{\bar{i}\bar{j}} .\end{aligned}$$

□

The axioms $A_1, \cdots, A_5$ give us an opportunity to calculate the local expression of $\overset{C}{\nabla}$ in the upcoming theorem.

Theorem 3.1. $\overset{C}{\nabla}$ *has the local expression*

$$\begin{cases} \overset{C}{\nabla}_{\delta_i} \partial_{\bar{j}} = F^k{}_{ij} \partial_{\bar{k}}, \\ \overset{C}{\nabla}_{\partial_{\bar{i}}} \partial_{\bar{j}} = V^k{}_{ij} \partial_{\bar{k}}, \end{cases} \tag{6}$$

where

$$V^k{}_{ij} = \frac{1}{2} g^{mk} (2C_{ijm} - \mathfrak{A}(\partial_{\bar{i}}, \partial_{\bar{j}}, \partial_{\bar{m}})). \tag{7}$$

Proof. To derive the equation (6), we use the reference [12]. Since this part of the proof is independent from the tensor $\mathfrak{A}$, and all conditions are the same as for the Cartan connection, according to [12] we have

$$2g(\overset{C}{\nabla}_{\delta_i} \delta_j, \partial_{\bar{k}}) = \delta_i g(\partial_{\bar{j}}, \partial_{\bar{k}}) + \delta_j g(\partial_{\bar{k}}, \partial_{\bar{i}}) - \delta_k g(\partial_{\bar{i}}, \partial_{\bar{j}}),$$

and hence we get the result (6). To prove (7), the using of Lemma 3.1 suffices. □

So, by using the transvection $\mathfrak{A}^k{}_{ij} := g^{mk} \mathfrak{A}_{ijm}$, $\overset{C}{\nabla}$ can be determined by the triplet $(N^i_j, F^k{}_{ij}, C^k{}_{ij} - \frac{1}{2}\mathfrak{A}^k{}_{ij})$.

In the following, we will investigate how two Cartan and Codazzi-Cartan connections are mutually related when their curvatures coincide.

Proposition 3.1. *The triplets* $(R^i{}_{hjk}, P^i{}_{hjk}, S^i{}_{hjk})$ *and* $(\overset{C}{R}{}^i{}_{hjk}, \overset{C}{P}{}^i{}_{hjk}, \overset{C}{S}{}^i{}_{hjk})$ *as curvature components of the Cartan and the Codazzi-Cartan connections, coincide if and only if*

$$\begin{cases} \mathfrak{A}_{hmt} R^m{}_{jk} = 0 \\ \delta_j(\mathfrak{A}^i{}_{hk}) + F^i{}_{pj} \mathfrak{A}^p{}_{hk} - F^p{}_{hj} \mathfrak{A}^i{}_{pk} - F^p{}_{kj} \mathfrak{A}^i{}_{hp} = \mathfrak{A}^i{}_{hm} P^m{}_{jk} \\ 2(\partial_{\bar{j}} \mathfrak{A}^i{}_{hk} - \partial_{\bar{k}} \mathfrak{A}^i{}_{hj}) = (C^m{}_{hj} \mathfrak{A}^i{}_{mk} - C^i{}_{mk} \mathfrak{A}^m{}_{hj} + \mathfrak{A}^m{}_{hj} \mathfrak{A}^i{}_{mk}) \\ \qquad + (C^m{}_{hk} \mathfrak{A}^i{}_{mj} - C^i{}_{mj} \mathfrak{A}^m{}_{hk} + \mathfrak{A}^m{}_{hk} \mathfrak{A}^i{}_{mj}). \end{cases}$$

Corollary 3.1. *If* $\overset{C}{S}{}^i{}_{hjk} = S^i{}_{hjk}$ *and* $\mathfrak{A}_{ijk} = C_{ijk}$, *then* E *reduces to a Riemannian metric.*

Proof. It suffices to multiply the two sides of $\overset{C}{S}{}^i{}_{hjk} = S^i{}_{hjk}$ with y^k. Since $C^i{}_{jh}$ is a homogeneous function of degree zero, we infer $C^i{}_{hj} = 0$, and consequently $C_{ijk} = 0$. □

Corollary 3.2. *If* $\overset{C}{S}{}^i{}_{hjk} = S^i{}_{hjk}$ *and* $y^k \mathfrak{A}_{ijk} = 0$, *then* E *reduces to a Riemannian metric.*

Proof. If we multiply the two sides of the third equation of Proposition 3.1 with y^k, we get $-\mathfrak{A}^i{}_{hj} = y^k \partial_{\bar{k}} \mathfrak{A}^i{}_{hj}$, which shows that $\mathfrak{A}^i{}_{hj}$ should identically vanish. □

4. The β-change on Finsler manifolds

Let E_1 be a Finslerian metric. Then its β-change (following the definition from [1]) is

$$E_2(x,y) = e^{\sigma(x)}E_1(x,y) + \beta(x,y),$$

where σ and $\beta := b_i y^i$ are a function and a $1-$form on M, respectively. Under this change, the Cartan tensor changes as follows

$$\bar{C}_{ijk} = \tau(C_{ijk} + \frac{1}{2E_2}h_{ijk}),$$

where $h_{ijk} := h_{ij}\mathfrak{m}_k + h_{jk}\mathfrak{m}_i + h_{ki}\mathfrak{m}_j$ and $\mathfrak{m}_i := b_i - \frac{\beta}{E_1}\partial_{\bar{i}}E_1$ (see [1] for details). Now we are ready to investigate the change of the Codazzi-Cartan connection under this proccess. We can derive the following theorem by using [1] and some direct substitutions.

Theorem 4.1. *Let $(\overset{0}{N}{}^i_j, \overset{0}{F}{}^i{}_{jk}, \overset{0}{V}{}^i{}_{jk})$ and $(\overset{1}{N}{}^i_j, \overset{1}{F}{}^i{}_{jk}, \overset{1}{V}{}^i{}_{jk})$ be the Codazzi-Cartan connections of the Finsler manifolds (M, E_1) and (M, E_2), respectively. Then the following relation holds.*

$$\begin{aligned}\overset{1}{V}{}^i{}_{jk} - \overset{0}{V}{}^i{}_{jk} = \frac{1}{2}[\tau^{-1}{}^1g^{mk} + \mu E_1{}^m E_1{}^k - \tau^{-2}(E_1{}^m b^k + E_1{}^k b^m)]\\ [2\tau(C_{ijm} + \frac{1}{2E_2}h_{ijm}) - \overset{1}{\mathfrak{A}}_{ijm}] - \frac{1}{2}{}^1g^{mk}(2C_{ijm} - \overset{0}{\mathfrak{A}}_{ijm}),\end{aligned} \tag{8}$$

where $\tau = e^{\sigma}\frac{E_2}{E_1}$, $\mu = \frac{e^{\sigma}E_1 b^2 + \beta}{E_2\tau^2}$, $b^2 = b_i b^i$, $b^i = {}^1g^{ij}b_j$ and $E_1^i = {}^1g^{ij}\partial_{\bar{j}}E_1$.

Corollary 4.1. *Let E_1 change to E_2 conformally. Then $\overset{1}{V}{}^i{}_{jk} - \overset{0}{V}{}^i{}_{jk} = 0$ if and only if the tensor $\overset{0}{\mathfrak{A}}$ changes conformally to $\overset{1}{\mathfrak{A}}$ by $\overset{1}{\mathfrak{A}} = e^{2\sigma}\overset{0}{\mathfrak{A}}$.*

Proof. If we put $\beta = 0$, then the following equations hold

$$b_i = 0, \quad \mathfrak{m} = 0, \quad h_{ijk} = 0, \quad \mu = 0.$$

By substituting these equations in (8), the assertion follows. □

We investigate the relation between two Codazzi-Cartan and Cartan connections, when their Finslerian metrics vary under a conformal change. Accordingly, by putting $\overset{0}{\mathfrak{A}} = 0$ in (8), we infer the following result

Corollary 4.2. *Let E_1 change to E_2 under a β-change. If we equip the Finslerian manifold (M, E_1) with the Cartan connection and the Finslerian manifold (M, E_2) with the Codazzi-Cartan connection, then we have the following relation*

$$\begin{aligned}\overset{1}{V}{}^i{}_{jk} = \frac{1}{2}[\tau^{-1}{}^1g^{mk} + \mu E_1{}^m E_1{}^k - \tau^{-2}(E_1{}^m b^k + E_1{}^k b^m)]\\ [2\tau(C_{ijm} + \frac{1}{2E_2}h_{ijm}) - \overset{1}{\mathfrak{A}}_{ijm}] - {}^1g^{mk}C_{ijm} + C^i{}_{jk}.\end{aligned}$$

4.1. Codazzi-Cartan connections on the indicatrix space

Let $E : \mathrm{T}M \to \mathbb{R}^{\geq 0}$ be a Finslerian metric. The indicatrix space I_E related to the E is the collection of unit vectors with respect to E. The unit vector field normal to the indicatrix space is the vector field $L : \mathrm{T}M \to \mathrm{TT}M$ defined by $L(x,y) = \frac{y^i}{E(x,y)}\partial_{\bar{i}}$. So, by considering the tangent sub-bundle $\{A - g(A,L)L | A \in \mathrm{TT}M\}$, we can easily infer

$$\mathrm{T}I_E = \underset{1\leq i\leq n}{\mathrm{Span}}\{A_i := \partial_{\bar{i}} - g(\partial_{\bar{i}}, L)L.\delta_i\}.$$

Note that $\{A_1, \cdots, A_n\}$ are not independent, but they span an $(n-1)$-dimensional space. If the Finslerian metric E_1 changes to E_2 by a β-change, then one can define a smooth mapping $f : I_1 \to I_2$ by

$$f(x, y) = \left(x, \frac{y}{e^{\sigma} + \beta(x, y)}\right). \tag{9}$$

Note that E_2 is a Finslerian metric and hence

$$E_2(x, y) = e^{\sigma(x)} E_1(x, y) + \beta(x, y) = e^{\sigma(x)} + \beta(x, y) \geq 0,$$

for $(x, y) \in I_1$, and if the base manifold is compact, then we can claim that there exists a function σ and an $1-$form β on M such that $e^{\sigma} + \beta(x, y)$ is non-zero and positive on non-zero vectors. So, we make a β-change with such a function and an 1-form.

In [4], the authors proved that there exists a coordinate system

$$(x^1, \cdots, x^n, u^1, \cdots, u^{n-1}, u^n),$$

on $\mathrm{T}M$, such that $(x^1, \cdots, x^n, u^1, \cdots, u^{n-1})$ is the coordinate system on the indicatrix space, where $(x^1, \cdots, x^n)$ denotes the position. From now on, we will work with such a coordinate system, which we employ in order to provide a coordinate system on I_2.

The following proposition shows that f is a diffeomorphism.

Proposition 4.1. *The function $f : I_1 \to I_2$ given by* (9)*, is a diffeomorphism.*

Proof. It suffices to show that f is a topological homeomorphism, since it is a differentiable mapping. Let $(x_1, y_1), (x_2, y_2) \in I_1$ be such that $f(x_1, y_1) = f(x_2, y_2)$. By using (9), we get $x_1 = x_2$ and we will have $y_1 = \frac{E_2(x_1,y_1)}{E_2(x_2,y_2)} y_2$. But, $E_1(x_1, y_1) = E_1(x_2, y_2) = 1$ and so ${}^1g_{ij} y_1^i y_1^j = 1 = {}^1g_{ij} y_2^i y_2^j$. This shows that $(\frac{E_2(x_1,y_1)}{E_2(x_2,y_2)})^2 = 1$. Since $E_2 \geq 0$, we infer $\frac{E_2(x_1,y_1)}{E_2(x_2,y_2)} = 1$, and then $y_1 = y_2$. □

Since f is a diffeomorphism, by using the coordinate system

$$(x^1, \cdots, x^n, u^1, \cdots, u^{n-1}),$$

on I_1, we can define a coordinate system $(\tilde{x}^1, \cdots, \tilde{x}^n, \tilde{u}^1, \cdots, \tilde{u}^{n-1})$ on I_2, such that $f_*(\frac{\partial}{\partial u^i}) = \frac{\partial}{\partial \tilde{u}^i} = \tilde{\partial}_{\bar{i}}$. It is obvious that $\tilde{y}^i(x, y) = E_2(x_0, y_0) y^i$, where $(x, y) \in I_2$ and $(x_0, y_0) \in I_1$. We extend this coordinate system to $\mathrm{T}M$, and we denote it as $(\tilde{x}^1, \cdots, \tilde{x}^n, \tilde{u}^1, \cdots, \tilde{u}^{n-1}, \tilde{u}^n)$.

Now, we would like to compute the differential mapping of f. Let $\gamma : (-\varepsilon, \varepsilon) \to I_1$ pass through $(x_0, y_0) \in I_1$ with

$$\gamma(t) = \left(x_0, \frac{y_0 + K(\partial_{\bar{i}})t}{E_2(x_0, y_0 + K(\partial_{\bar{i}})t)}\right),$$

where $K : \mathrm{TT}M \to \mathrm{T}M$ is the connection map. Then we have

$$\begin{aligned} f_*(A_i = \partial_{\bar{i}}) &= \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(x_0, \frac{y_0 + K(\partial_{\bar{i}})t}{E_2(x_0, y_0 + K(\partial_{\bar{i}})t) e^{\sigma(x_0)} + \beta(x_0, y_0 + K(\partial_{\bar{i}})t)}) \\ &= \frac{(e^{\sigma(x_0)} + \beta(x_0, y_0))\tilde{\partial}_{\bar{i}} - [\partial_{\bar{i}} E_2(x_0, y_0) e^{\sigma(x_0)} + \beta(x_0, K(\partial_{\bar{i}}))] y_0}{(e^{\sigma(x_0)} + \beta(x_0, y_0))^2}. \end{aligned} \tag{10}$$

Now, let $\gamma(t) = (x^1(t), \cdots, x^n(t), y^1(t), \cdots, y^n(t))$ be a curve in I_1 which passes through $(x_0^1, \cdots, x_0^n, y_0^1, \cdots, y_0^n) \in I_1$, such that $\gamma'(0) = \delta_i(x_0, y_0)$. Then we have

$$f_*(\delta_i) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(x(t), \frac{y(t)}{E_2(x(t), y(t))}\right).$$

By differentiation, from the above equation we get

$$\begin{aligned} f_*(\delta_i) &= \frac{\partial}{\partial \tilde{x}^i} - \frac{1}{E_2(x_0,y_0)} N_i^j(x_0,y_0)\tilde{\partial}_{\bar{j}} \\ &- \frac{1}{E_2^2(x_0,y_0)} \left[\frac{\partial \sigma}{\partial x^i} e^{\sigma(x_0)} + \frac{\partial \beta_k}{\partial x^i} y_0^k - N_i^k(x_0,y_0)\beta_k(x_0)\right] \tilde{y}^j \tilde{\partial}_{\bar{j}}. \end{aligned} \tag{11}$$

Now, we focus on the case $\beta = 0$. We first use several abbreviations on (10) and (11). We perform a new nonlinear splitting by $\{\tilde{\partial}_{\bar{i}}, \tilde{\delta}_i, \frac{\tilde{y}^s}{E_2}\partial_{\bar{s}} = {}^2L\}$, where

$$\tilde{\delta}_i := \frac{\partial}{\partial \tilde{x}^i} - \frac{1}{E_2(x_0,y_0)} N_i^j(x_0,y_0)\tilde{\partial}_{\bar{j}}. \tag{12}$$

So, under the assumption $\beta = 0$ and using (12), we have

$$f_*(\delta_i) = \tilde{\delta}_i - \left(\frac{\partial \sigma}{\partial x^i} e^{\sigma(x_0)}\right) \tilde{y}^j \tilde{\partial}_{\bar{j}}.$$

If we use the new horizontal space $\tilde{H}$ from above, in order to define a new metric $\bar{G}_{ij} = {}^2g_{ij}$, as well as the induced metric from the Finsler metric E_1, then we can pullback it to I_1. Its components are provided by

$$\begin{aligned} f^*\bar{G}(\partial_{\bar{i}}, \partial_{\bar{j}}) &= e^{2\sigma}\ {}^1g_{ij}, \\ f^*\bar{G}(\partial_{\bar{i}}, \delta_j) &= -e^{2\sigma} \frac{\partial \sigma}{\partial x^j} \partial_{\bar{i}} E_1, \\ f^*\bar{G}(\delta_i, \delta_j) &= e^{2\sigma}\ {}^1g_{ij} + \frac{\partial \sigma}{\partial x^i}\frac{\partial \sigma}{\partial x^j} e^{2\sigma}. \end{aligned}$$

We shall further work in the above introduced coordinate systems on $(\mathrm{T}M \setminus \{0\}, {}^2E)$, and use the horizontal space $\tilde{H}$ spanned by

$$\left\{\tilde{\delta}_i = \frac{\partial}{\partial \tilde{x}^i} - \frac{1}{E_2(x_0,y_0)} N_i^j(x_0,y_0)\tilde{\partial}_{\bar{j}}\right\}_{i=1}^n.$$

By using these notations, we define a pseudo-Cartan connection $(\tilde{N}_j^i, \tilde{\mathfrak{F}}^i{}_{jk}, \tilde{\mathfrak{C}}^i{}_{jk})$ on $\mathrm{T}M\setminus\{0\}$, by

$$\begin{cases} \tilde{\mathfrak{F}}^i{}_{jh} := \dfrac{1}{2}\ {}^2g^{im}(\delta_j\ {}^2g_{mh} + \delta_h\ {}^2g_{jm} - \delta_m\ {}^2g_{jh}), \\ \tilde{\mathfrak{C}}^i{}_{jh} = \dfrac{1}{2}\ {}^2g^{im}(\partial_{\bar{j}}\ {}^2g_{mh} + \partial_{\bar{h}}\ {}^2g_{jm} - \partial_{\bar{m}}\ {}^2g_{jh}). \end{cases}$$

We can transfer the pseudo-Cartan connection on I_1, as follows

$$(\dot{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}})^k := (\mathrm{proj}\tilde{\nabla}_{\tilde{\partial}_{\bar{i}}}\tilde{\partial}_{\bar{j}})^k, \qquad (\dot{\nabla}_{\delta_i}\partial_{\bar{j}})^k := (\mathrm{proj}\tilde{\nabla}_{f_*(\delta_i)}\tilde{\partial}_{\bar{j}})^k,$$

where $\mathrm{proj}\tilde{\nabla}$ is the induced pseudo-Cartan connection on I_2. We shall investigate when does this connection coincide with the Codazzi-Cartan connection on I_1.

By straight computations, we infer the following theorem.

Theorem 4.2. *The connection* $\dot{\nabla}$ *is provided by*

$$\begin{aligned} (\dot{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}})^k &= \tilde{C}_{ij}^k - \tilde{C}_{ij}^a(\tilde{\partial}_{\bar{a}}E_2)\tilde{y}^k, \\ (\dot{\nabla}_{\delta_i}\partial_{\bar{j}})^k &= \tilde{F}_{ij}^k - \tilde{F}_{ij}^a(\tilde{\partial}_{\bar{a}}E_2)\tilde{y}^k - \frac{\partial \sigma}{\partial \tilde{x}^i} e^{\sigma}\tilde{y}^a\tilde{C}_{aj}^k \\ &+ \frac{\partial \sigma}{\partial \tilde{x}^i} e^{\sigma}\tilde{y}^a\tilde{C}_{aj}^s(\tilde{\partial}_{\bar{s}}E_2)\tilde{y}^k. \end{aligned}$$

Now, we are ready to get the essential sufficient condition on the tensor $\mathfrak{A}$, for $\dot{\nabla}$ to coincide with the Codazzi-Cartan connection on I_1.

Theorem 4.3. *Let* $\text{proj}\dot{\nabla}$ *be the defined connection and let* $\text{proj}\overset{C}{\nabla}$ *be the induced Codazzi-Cartan connection on* I_1. *Then these two connections coincide if and only if*

$$\tilde{C}^s_{ij} - \tilde{C}^a_{ij}(\tilde{\partial}_{\bar{a}}E_2)\tilde{y}^s - [\tilde{C}^k_{ij} - \tilde{C}^a_{ij}(\tilde{\partial}_{\bar{a}}E_2)\tilde{y}^k]\partial_{\bar{k}}E_1 y^s = V^s_{ij} - V^k_{ij}(\partial_{\bar{k}}E_1)y^s,$$

and

$$\tilde{F}^s_{ij} - \tilde{F}^a_{ij}(\tilde{\partial}_{\bar{a}}E_2)\tilde{y}^s - \frac{\partial\sigma}{\partial\tilde{x}^i}e^\sigma\tilde{y}^a\tilde{C}^s_{aj} + \frac{\partial\sigma}{\partial\tilde{x}^i}e^\sigma\tilde{y}^a\tilde{C}^s_{aj}(\tilde{\partial}_{\bar{i}s}E_2)\tilde{y}^s\frac{\partial\sigma}{\partial\tilde{x}^i}e^\sigma$$
$$- [\tilde{F}^k_{ij} - \tilde{F}^a_{ij}(\tilde{\partial}_{\bar{a}}E_2)\tilde{y}^k - \frac{\partial\sigma}{\partial\tilde{x}^i}e^\sigma\tilde{y}^a\tilde{C}^k_{aj} + \frac{\partial\sigma}{\partial\tilde{x}^i}e^\sigma\tilde{y}^a\tilde{C}^s_{aj}(\tilde{\partial}_{\bar{i}s}E_2)\tilde{y}^k\frac{\partial\sigma}{\partial\tilde{x}^i}e^\sigma](\partial_{\bar{k}}E_1)y^s$$
$$= F^s_{ij} - F^k_{ij}(\partial_{\bar{k}}E_1)y^s,$$

for $s = 1, \cdots, n$, *where* $V^k{}_{ij} = \frac{1}{2}g^{mk}(2C_{ijm} - \mathfrak{A}(\partial_{\bar{i}}, \partial_{\bar{j}}, \partial_{\bar{m}}))$.

Proof. First, we should project the local vector fields $\dot{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}}$ for $i, j = 1, \cdots, n-1$ and $\dot{\nabla}_{\delta_i}\partial_{\bar{j}}$ for $i = 1, \cdots, n$, $j = 1 \cdots, n-1$ on I_1 along the normal vector field to I_1. Moreover, we should project $\overset{C}{\nabla}$ on I_1, too. The remaining part of the proof achieved by setting these projections equal. □

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Appendix

For a fast and easy view, we mention all the used notations.
Suppose (M, E) is an n-dimensional Finsler manifold.

∂_i: partial derivative with respect to x^i;

$\partial_{\bar{i}}$: partial derivative with respect to y^i;

δ_i: partial derivative with respect to horizontal base local fields $\frac{\delta}{\delta x^i}$;

N^i_j: non-linear connection in section 2 and Cartan nonlinear connection induced by E in other sections;

${R^k}_{ij}$: curvature tensor fields of the non-linear connection N;

g_{ij}: used for both the metric tensor of E and its Sasaki lift;

${\gamma^k}_{ij}$: the Christoffel symbols of the metric tensor g;

G^i: the components of canonical spray S of E;

$_{,}$, $_{;}$: the horizontal and the vertical derivatives, respectively;

$(N^i_j, {\mathcal{F}^k}_{ij}, {\mathcal{C}^k}_{ij})$: a Finsler connection $\mathcal{FC}$;

${C^k}_{ij}$: the Cartan tensor field;

$\overset{C}{\nabla}$: the Codazzi-Cartan connection;

$\mathfrak{A}_{ijk}$: Codazzi components of the Codazzi-Cartan connection;

∇: the Cartan connection;

$\dot{\nabla}$: the defined connection on the tangent bundle, by using the pseudo-Cartan connection;

I_i: the indicatrix induced by the Finsler metric E_i;

${}^{I_i}\overset{C}{\nabla}$: the restriction of $\overset{C}{\nabla}$ to I_i;

${}^{I_i}\nabla$: the restriction of ∇ to I_i;

${}^i g$: the metric tensor of E_i and its Sasaki lift;

${}^{I_i} g$: the restriction of the metric tensor of E_i and its Sasaki lift to I_i;

$\bar{G}$: the defined Riemannian metric using the horizontal space $\tilde{H}$, the vertical space and the metric ${}^2 g$.