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SOME SHARP INEQUALITIES ON REAL HYPERSURFACES OF SOME QUADRICS

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The objective of this paper is two-fold: Firstly, we establish some optimal inequalities involving the normalized scalar curvature and generalized normalized $\delta-$Casorati curvatures for real hypersurfaces in the complex hyperbolic quadrics Q^{q*} (and the complex quadrics Q^q). Secondly, we study a notion of the Ricci tensor derived from a curvature of real hypersurfaces in Q^{q*} (and Q^q). Then we classify real hypersurfaces with isometric Reeb flow in Q^{q*} (and Q^q) by using an integral formula related to the Ricci curvature. Also, we give a complete proof of non-existence of real hypersurfaces in Q^{q*} (and Q^q).

Keywords: complex hyperbolic quadrics, complex quadrics, Ricci curvature, Casorati curvatures, quasi-umbilical, integral formula.

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1. Introduction

In a class of Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $G_2(\mathbb{C}^{q+2}) = SU_{q+2}/S(U_2U_q)$ and $G_2^*(\mathbb{C}^{q+2}) = SU_{2,q}/S(U_2U_q)$. The $q-$dimensional complex quadric $Q^q = SO_{q+2}/SO_qSO_2$ and complex hyperbolic quadric $Q^{q*} = SO^{\circ}_{2,q}/SO_qSO_2$ can be regarded as another kind of Hermitian symmetric spaces with rank 2 of compact type and of noncompact type, respectively. Q^q and Q^{q*} are the complex hypersurfaces in complex projective space $\mathbb{CP}^{q+1}$ and complex hyperbolic space $\mathbb{CH}^{q+1}$, respectively. Many results have been obtained on real hypersurfaces in $G_2(\mathbb{C}^{q+2})$, $G_2^*(\mathbb{C}^{q+2})$, Q^q and Q^{q*} with different conditions (for example, see [9]). In 2005, Y. J. Suh and Y. Watanabe [13] proved the non-existence properties related to the Ricci curvature along the direction of structure vector ξ of any compact real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{q+2}), q \geq 3$. In the present paper, motivated by this result, we will show a non-existence property for compact real hypersurfaces in Q^{q*} and Q^q related to the Ricci curvature.

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On the other hand, B.-Y. Chen [4] introduced the notion of Chen invariants and obtained some optimal inequalities consisting of intrinsic invariants and some extrinsic invariants for any Riemannian submanifolds. Instead of concentrating on the sectional curvature with the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant. The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. Several geometers in [8, 15, 16, 5, 6] found geometrical meaning and the importance of the Casorati curvature and hence obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces (for example [14]). Brubaker et al. [2] obtained a nice geometric interpretation of Cauchy-Schwarz inequality in terms of Casorati curvature. Recently, Park [10] proved inequalities for the Casorati curvatures of real hypersurfaces in some Grassmannians. As a natural prolongation of our research, in the present paper, we will study these inequalities for real hypersurfaces in Q^{q*} and Q^q.

2. The complex (hyperbolic) quadrics and hypersurfaces

The homogeneous quadratic equation $z_1^2+\cdots+z_{q+2}^2=0$ on $\mathbb{C}^{q+2}$ defines a complex hypersurface Q^q in the $(q+1)-$dimensional complex projective space $\mathbb{CP}^{q+1}=SU_{q+2}/S(U_{q+1}U_1)$. The complex hypersurface Q^q is known as the $q-$dimensional complex quadric. The $1-$dimensional quadric Q^1 is isometric to the round $2-$sphere S^2. For $q\geq 2$ the triple (Q^q,J,g) is a Hermitian symmetric space of rank two and its maximal sectional curvature is equal to 4. The $2-$dimensional quadric Q^2 is isometric to the Riemannian product $S^2\times S^2$. On the other hand, the complex hypersurface Q^{q*} in $\mathbb{CH}^{q+1}$ is known as the $q-$dimensional complex hyperbolic quadric. The complex structure J on $\mathbb{CH}^{q+1}$ naturally induces a complex structure on Q^{q*} which we will denote by J as well. We equip Q^{q*} with the Riemannian metric g which is induced from the Begerman metric on $\mathbb{CH}^{q+1}$ with constant holomorphic sectional curvature -4. For $q\geq 2$ the triple (Q^{q*},J,g) is a Hermitian symmetric space of rank 2 and its minimal sectional curvature is equal to 4. The $1-$dimensional quadric Q^{1*} is isometric to the $2-$dimensional real hyperbolic space $\mathbb{RH}^2=SO^o_{1,2}/SO_1SO_2$. The $2-$dimensional complex quadric Q^{2*} is isometric to the Riemannian product of complex hyperbolic spaces $\mathbb{CH}^1\times\mathbb{CH}^1$. We will assume $q\geq 3$ for the sections 3 and 4 of this paper.

In addition to the complex structure J there is another distinguished geometric structure on Q^{q*}, namely, a parallel rank two vector bundle $\mathcal{U}$ which contains an S^1-bundle of real structures, that is, complex conjugation $\mathcal{A}$ on the tangent spaces of Q^{q*}. Both anti-commute with each other, that is, $\mathcal{A}J=-J\mathcal{A}$. We denote by $\mathcal{A}_{\overline{z}}$ the shape operator of Q^{q*} in $\mathbb{CH}^{q+1}$ with respect to $\overline{z}$. Note that $\overline{z}\in\nu_{[z]}Q^{q*}$ is a unit normal vector of Q^{q*} in $\mathbb{CH}^{q+1}$ at the point $[z]$ (for details see [12]). Then we have $\mathcal{A}_{\overline{z}}w=\overline{w}$ for all $w\in T_{[z]}Q^{q*}$, that is, $\mathcal{A}_{\overline{z}}$ is just complex conjugation restricted to $T_{[z]}Q^{q*}$. The shape operator $\mathcal{A}_{\overline{z}}$

is an anti-linear involution on the complex vector space $T_{[z]}Q^{q*}$ and $T_{[z]}Q^{q*} = V(\mathcal{A}_{\overline{z}}) \oplus JV(\mathcal{A}_{\overline{z}})$, where $V(\mathcal{A}_{\overline{z}}) = \mathbb{R}_1^{q+2} \bigcap T_{[z]}Q^{q*}$ is the $(+1)-$eigenspace and $JV(\mathcal{A}_{\overline{z}}) = i\mathbb{R}_1^{q+2} \bigcap T_{[z]}Q^{q*}$ is the $(-1)-$eigenspace of $\mathcal{A}_{\overline{z}}$.

The Riemannian curvature tensor $\overline{R}^*$ of Q^{q*} in $\mathbb{CH}^{q+1}$ concerning the Riemannian metric g, the complex structure J and a generic real structure $\mathcal{A}$ in $\mathcal{U}$ is given by [12]

$$\begin{aligned}\overline{R}^*(X,Y)Z &= -g(Y,Z)X + g(X,Z)Y - g(JY,Z)JX \\ &\quad +g(JX,Z)JY + 2g(JX,Y)JZ - g(\mathcal{A}Y,Z)\mathcal{A}X \\ &\quad +g(\mathcal{A}X,Z)\mathcal{A}Y - g(J\mathcal{A}Y,Z)J\mathcal{A}X + g(J\mathcal{A}X,Z)J\mathcal{A}Y.\end{aligned}$$

The Riemannian curvature tensor $\overline{R}$ of Q^q in $\mathbb{CP}^{q+1}$ is defined as follows [3]

$$\begin{aligned}\overline{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &\quad -g(JX,Z)JY - 2g(JX,Y)JZ + g(\mathcal{A}Y,Z)\mathcal{A}X \\ &\quad -g(\mathcal{A}X,Z)\mathcal{A}Y + g(J\mathcal{A}Y,Z)J\mathcal{A}X - g(J\mathcal{A}X,Z)J\mathcal{A}Y.\end{aligned}$$

A nonzero tangent vector $W \in T_{[z]}Q^{q*}$ is called singular if it is tangent to more than one maximal flat in Q^{q*}. There are two types of singular tangent vectors for the complex hyperbolic quadric Q^{q*} [12]

1. If there exists a real structure $\mathcal{A} \in \mathcal{U}_{[z]}$ such that $W \in V(\mathcal{A})$, then W is singular. Such a singular tangent vector is called $\mathcal{U}-$principal.

2. If there exist a real structure $\mathcal{A} \in \mathcal{U}_{[z]}$ and orthonormal vectors $X, Y \in V(\mathcal{A})$ such that $\frac{W}{||W||} = \frac{(X+JY)}{\sqrt{2}}$, then W is singular. Such a singular tangent vector is called $\mathcal{U}-$isotropic.

Let M be a real hypersurface in Q^{q*} and denoted by the data (P, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M. Note that $\xi = -J\mathcal{N}$, where $\mathcal{N}$ is a unit normal vector field of M. The vector field ξ is known as the Reeb vector field of M and η is $1-$form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field P restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $P\xi = 0$.

If the integral curves of ξ are geodesics in M, the hypersurface M is called a Hopf hypersurface. The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. If we assume that M is a Hopf hypersurface, then we have $S\xi = \beta\xi$, where S denotes the shape operator of the real hypersurfaces M with the smooth function $\beta = g(S\xi, \xi)$ on M.

By the Kaehler structure J of the complex hyperbolic quadric Q^{q*}, one can transfer any tangent vector field X on M in Q^{q*} as follows: $JX = PX + \eta(X)\mathcal{N}$, where $PX = tan(JX)$ and $\mathcal{N}$ is a unit normal vector field on M in

Q^{q*}. Then it naturally satisfies the following relations: $P^2X = -X + \eta(X)\xi$, $\eta(\xi) = 1$, and $P\xi = 0$.

Basic complex linear algebra shows that for every unit tangent vector $W \in T_{[z]}Q^{q*}$ there exist a real structure $\mathcal{A} \in \mathcal{U}_{[z]}$ and orthonormal vectors $X, Y \in V(\mathcal{A})$ such that $W = \cos(t)X + \sin(t)JY$, for some $t \in [0, \frac{\pi}{4}]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \frac{\pi}{4}$.

Now, we assume that the normal vector $\mathcal{N}_{[z]}$ of M is not $\mathcal{U}-$principal. Then there exists a real structure $\mathcal{A} \in \mathcal{U}_{[z]}$ such that $\mathcal{N}_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2$, for some orthonormal vectors $Z_1, Z_2 \in V(\mathcal{A})$, $t \in [0, \frac{\pi}{4}]$. Note that t is a function on M. First of all $\xi = -J\mathcal{N}$, we have

$$\left.\begin{array}{ll} \mathcal{N}_{[z]} = \cos(t)Z_1 + \sin(t)JZ_2, & \mathcal{A}\mathcal{N}_{[z]} = \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi_{[z]} = \sin(t)Z_2 - \cos(t)JZ_1, & \mathcal{A}\xi_{[z]} = \sin(t)Z_2 + \cos(t)JZ_1, \end{array}\right\} \tag{1}$$

and therefore $Q_{[z]} = T_{[z]}Q^{q*} \ominus ([Z_1] \oplus [Z_2])$ is strictly contained in $\mathcal{C}_{[z]}$.

Following Gauss equation, the curvature tensor R^* for $(2q-1)-$dimensional submanifold M in Q^{q*} induced from the curvature tensor $\overline{R}^*$ is expressed as follows (see [12])

$$\begin{aligned} R^*(X,Y)Z &= -g(Y,Z)X + g(X,Z)Y - g(JY,Z)JX + g(JX,Z)JY \\ &\quad +2g(JX,Y)JZ - g(\mathcal{A}Y,Z)\mathcal{A}X + g(\mathcal{A}X,Z)\mathcal{A}Y \\ &\quad -g(J\mathcal{A}Y,Z)J\mathcal{A}X + g(J\mathcal{A}X,Z)J\mathcal{A}Y + g(SY,Z)SX \\ &\quad -g(SX,Z)SY. \end{aligned} \tag{2}$$

Using (2) and contracting Y and Z, we have the Ricci tensor of M in Q^{q*}

$$\begin{aligned} Ric^*(X) &= \sum_{i=1}^{2q-1} \Big[-g(E_i,E_i)X + g(X,E_i)E_i - g(JE_i,E_i)JX \\ &\quad +g(JX,E_i)JE_i + 2g(JX,E_i)JE_i - g(\mathcal{A}E_i,E_i)\mathcal{A}X \\ &\quad +g(\mathcal{A}X,E_i)\mathcal{A}E_i - g(J\mathcal{A}E_i,E_i)J\mathcal{A}X + g(J\mathcal{A}X,E_i) \\ &\quad J\mathcal{A}E_i + g(SE_i,E_i)SX - g(SX,E_i)SE_i \Big] \\ &= -(2q-1)X + X + PX + 2P^2X + g(AN,N)AX + X \\ &\quad -g(AX,N)AN + g(JAN,N)JAX + X - g(JAX,N) \\ &\quad JAN + hSX - S^2X. \end{aligned}$$

Further, we have

$$\begin{aligned} Ric^*(X) &= -(2q-1)X + 3\eta(X)\xi + g(\mathcal{A}\mathcal{N},\mathcal{N})\mathcal{A}X \\ &\quad -g(\mathcal{A}X,\mathcal{N})\mathcal{A}\mathcal{N} + g(J\mathcal{A}\mathcal{N},\mathcal{N})J\mathcal{A}X \\ &\quad -g(J\mathcal{A}X,N)J\mathcal{A}N + hSX - S^2X, \end{aligned} \tag{3}$$

where $h = trace(S)$ denotes the mean curvature, defined by the shape operator S of M in Q^{q*}. Then the Ricci curvature $K^*(\xi,\xi)$ along the direction of ξ is given by

$$\begin{aligned} K^*(\xi,\xi) &= -(2q-4) + 2g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\xi,\xi) \\ &\quad +hg(S\xi,\xi) - g(S^2\xi,\xi). \end{aligned} \tag{4}$$

We suppose that a real hypersurface M in Q^{q*} has $\mathcal{U}-$principal unit normal vector field $\mathcal{N}$, that is, $\mathcal{A}\mathcal{N} = \mathcal{N}$ for a complex conjugation $\mathcal{A} \in \mathcal{U}$. On the other hand, for real hypersurfaces with $\mathcal{U}-$isotropic unit normal vector field $\mathcal{N}$ in Q^{q*} (respectively in Q^q), we put $\mathcal{N} = \frac{Z_1+JZ_2}{\sqrt{2}}$, for any $Z_1, Z_2 \in V(\mathcal{A})$.

3. **Optimal inequalities for the Casorati curvatures**

Let us consider a local orthonormal tangent frame $\{E_1,\ldots,E_p\}$ of TM and a local orthonormal normal frame $\{\mathcal{N}\}$ of $T^\perp M$ in Q^{q*}. At any $\wp \in M$, the scalar curvature τ^* of M is given by

$$\tau^* = \sum_{1\le i<j\le p} R^*(E_i,E_j,E_j,E_i) = \sum_{1\le i<j\le p} g(R^*(E_i,E_j)E_j,E_i). \tag{5}$$

The mean curvature vector $\mathcal{H}$ of M in Q^{q*} is $\mathcal{H} = \frac{1}{p}\sum_{i=1}^{p}\sigma(E_i,E_i)$. Conveniently, let us put $\sigma_{ij} = g(\sigma(E_i,E_j),\mathcal{N}) = g(SE_i,E_j)$, for $i,j = \{1,\ldots,n\}$. Then the squared norm of mean curvature vector of M is defined as $||\mathcal{H}||^2 = \frac{1}{p^2}\left(\sum_{i=1}^{p}\sigma_{ii}\right)^2$ and the squared norm of second fundamental form σ is denoted by $\mathfrak{C} = \frac{1}{p}||\sigma||^2$, where $||\sigma||^2 = \sum_{i,j=1}^{p}\left(\sigma_{ij}\right)^2$. It is known as the Casorati curvature $\mathfrak{C}$ of M in Q^{q*}.

If we suppose that L is an r-dimensional subspace of TM, $r \ge 2$, and $\{E_1,\ldots,E_r\}$ is an orthonormal basis of L. Then the Casorati curvature of the subspace L is $\mathfrak{C}(L) = \frac{1}{r}\sum_{i,j=1}^{r}\left(\sigma_{ij}\right)^2$.

The normalized δ-Casorati curvatures $\delta_{\mathfrak{C}}(p-1)$ and $\widehat{\delta}_{\mathfrak{C}}(p-1)$ are respectively defined as

$$[\delta_{\mathfrak{C}}(p-1)]_\wp = \frac{1}{2}\mathfrak{C}_\wp + \frac{p+1}{2p}\inf\{\mathfrak{C}(L)|L : \text{a hyperplane of } T_\wp M\},$$

and

$$[\widehat{\delta}_{\mathfrak{C}}(p-1)]_\wp = 2\mathfrak{C}_\wp - \frac{2p-1}{2p}\sup\{\mathfrak{C}(L)|L : \text{a hyperplane of } T_\wp M\}.$$

Now, we define the generalized normalized δ-Casorati curvatures $\delta_{\mathfrak{C}}(t;p-1)$ and $\widehat{\delta}_{\mathfrak{C}}(t;p-1)$ as follows:

1. For $0 < t < p^2 - p$

$$[\delta_{\mathfrak{C}}(t;p-1)]_\wp = t\mathfrak{C}_\wp + b(t)\inf\{\mathfrak{C}(L)|L : \text{a hyperplane of } T_\wp M\},$$

2. For $t > p^2 - p$

$$[\widehat{\delta}_{\mathfrak{C}}(t; p-1)]_{\wp} = t\mathfrak{C}_{\wp} + b(t)\sup\{\mathfrak{C}(L)|L : \text{a hyperplane of } T_{\wp}M\},$$

where $b(t) = \frac{1}{tp}(p-1)(p+t)(p^2-p-t)$, $t \neq p(p-1)$.

Definition 3.1. *A point $r \in M$ is said to be an invariantly quasi-umbilical point if there exist unit normal vector $\{\mathcal{N}\}$ such that the shape operator S has an eigenvalue of multiplicity $\geq (p-1)$. The real hypersurface M is said to be an invariantly quasi-umbilical hypersurface, if each of its points is an invariantly quasi-umbilical point* [1].

Theorem 3.1. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{q*}. Then*

1. *The generalized normalized δ-Casorati curvature $\delta_{\mathfrak{C}}(t; p-1)$ satisfies*

$$\rho^* \leq \frac{\delta_{\mathfrak{C}}(t; p-1)}{p(p-1)} - \left(\frac{p+1}{p}\right), \tag{6}$$

for any $t \in \mathbb{R}$ with $0 < t < p(p-1)$.

2. *The generalized normalized δ-Casorati curvature $\widehat{\delta}_{\mathfrak{C}}(t; p-1)$ satisfies*

$$\rho^* \leq \frac{\widehat{\delta}_{\mathfrak{C}}(t; p-1)}{p(p-1)} - \left(\frac{p+1}{p}\right), \tag{7}$$

for any $t \in \mathbb{R}$ with $t > p(p-1)$.

Proof. Let $\{E_1, \ldots, E_p\}$ and $\{\mathcal{N}\}$ be the orthonormal basis of TM and $T^{\perp}M$, respectively at any point $\wp \in M$. Putting $X = W = E_i$, $Y = Z = E_j$, $i \neq j$ into (2) and taking summation $1 \leq i, j \leq p$, we have

$$\begin{aligned}\sum_{1\leq i,j\leq p} g(R^*(E_i, E_j)E_j, E_i) &= \sum_{1\leq i,j\leq p}\Big[- g(E_j, E_j)g(E_i, E_i) + g(E_i, E_j)g(E_j, E_i) \\ &\quad -g(PE_j, E_j)g(PE_i, E_i) + g(PE_i, E_j)g(PE_j, E_i) \\ &\quad +2g(PE_i, E_j)g(PE_j, E_i) - g(\mathcal{A}E_j, E_j)g(\mathcal{A}E_i, E_i) \\ &\quad +g(\mathcal{A}E_i, E_j)g(\mathcal{A}E_j, E_i) - g(J\mathcal{A}E_j, E_j)g(J\mathcal{A}E_i, E_i) \\ &\quad +g(J\mathcal{A}E_i, E_j)g(J\mathcal{A}E_j, E_i) + g(SE_j, E_j)g(SE_i, E_i) \\ &\quad -g(SE_i, E_j)g(SE_j, E_i)\Big].\end{aligned}$$

By using (5), we get

$$\begin{aligned}2\tau^* &= -p^2 + 3 + g(\mathcal{A}\mathcal{N}, \mathcal{N})\sum_{i=1}^{p} g(\mathcal{A}E_i, E_i) - \sum_{i=1}^{p} g(\mathcal{A}E_i, \mathcal{N})g(\mathcal{A}\mathcal{N}, E_i) \\ &\quad +\sum_{i=1}^{p} g(J\mathcal{A}\mathcal{N}, \mathcal{N})g(J\mathcal{A}E_i, E_i) - \sum_{i=1}^{p} g(J\mathcal{A}E_i, \mathcal{N})g(J\mathcal{A}\mathcal{N}, E_i) \\ &\quad +p^2||\mathcal{H}||^2 - ||\sigma||^2\end{aligned}$$

$$
\begin{aligned}
&= -p^2+3+g(\mathcal{A}\mathcal{N},\mathcal{N})\Big[\sum_{i=1}^{p+1} g(\mathcal{A}E_i,E_i)-g(\mathcal{A}\mathcal{N},\mathcal{N})\Big] \\
&\quad -\sum_{i=1}^{p+1} g(\mathcal{A}E_i,\mathcal{N})g(\mathcal{A}\mathcal{N},E_i)+g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\mathcal{N},\mathcal{N}) \\
&\quad -\sum_{i=1}^{p+1} g(J\mathcal{A}E_i,\mathcal{N})g(J\mathcal{A}\mathcal{N},E_i)+p^2||\mathcal{H}||^2-||\sigma||^2,
\end{aligned}
$$

where $g(J\mathcal{A}\mathcal{N},\mathcal{N})=g(\mathcal{A}\xi,\mathcal{N})=0$. Further, we derive

$$
\begin{aligned}
2\tau^* &= -p^2+3-g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\mathcal{N},\mathcal{N})-\sum_{i=1}^{p+1} g(\mathcal{A}E_i,\mathcal{N})g(\mathcal{A}\mathcal{N},E_i) \\
&\quad +g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\mathcal{N},\mathcal{N})-\sum_{i=1}^{p+1} g(J\mathcal{A}E_i,\mathcal{N})g(J\mathcal{A}\mathcal{N},E_i) \\
&\quad +p^2||\mathcal{H}||^2-||\sigma||^2 \\
&= -p^2+3-\sum_{i=1}^{p+1} g(\mathcal{A}E_i,\mathcal{N})g(\mathcal{A}\mathcal{N},E_i)-\sum_{i=1}^{p+1} g(J\mathcal{A}E_i,\mathcal{N})g(J\mathcal{A}\mathcal{N},E_i) \\
&\quad +p^2||\mathcal{H}||^2-||\sigma||^2.
\end{aligned}
$$

Thus, we rewrite the above relation as

$$2\tau^* = -p^2+1+p^2||\mathcal{H}||^2-p\mathfrak{C}. \tag{8}$$

Now, we define the following function, denoted by $\mathcal{P}$, a quadratic polynomial in the components of the second fundamental form $\mathcal{P}=t\mathfrak{C}+b(t)\mathfrak{C}(L)-2\tau^*-p^2+1$. Assuming, without loss of generality, that L is spanned by $E_1,\ldots,E_{p-1}$, it follows that

$$\mathcal{P} = \frac{t}{p}\sum_{i,j=1}^{p}\sigma_{ij}^2+\frac{b(t)}{p-1}\sum_{i,j=1}^{p-1}\sigma_{ij}^2-2\tau^*-p^2+1. \tag{9}$$

On combining (8) and (9), we arrive at

$$\mathcal{P}=(\frac{p+t}{p})\sum_{i,j=1}^{p}\sigma_{ij}^2+(\frac{b(t)}{p-1})\sum_{i,j=1}^{p-1}\sigma_{ij}^2-(\sum_{i=1}^{p}\sigma_{ii})^2,$$

which can be easily rewritten as

$$
\begin{aligned}
\mathcal{P} &= \sum_{i=1}^{p-1}\Big[(a-1)(\sigma_{ii})^2+\frac{2(p+t)}{p}(\sigma_{ip})^2\Big] \\
&\quad +\Big[2a\sum_{i\le j=1}^{p-1}(\sigma_{ij})^2-2\sum_{i\le j=1}^{p}(\sigma_{ii}\sigma_{jj})+\frac{t}{p}(\sigma_{pp})^2\Big],
\end{aligned} \tag{10}
$$

where $a = \left(\frac{p+t}{p} + \frac{b(t)}{p-1}\right)$. From (10), we observe that the critical points $\sigma^c = (\sigma_{11}, \sigma_{12}, \dots, \sigma_{pp})$ of $\mathcal{P}$ are the solutions of the following system of linear homogenous equations

$$\left.\begin{array}{ll} \frac{\partial \mathcal{P}}{\partial \sigma_{ii}} = 2a(\sigma_{ii}) - 2\sum_{s=1}^{p} \sigma_{ss} = 0, & \frac{\partial \mathcal{P}}{\partial \sigma_{pp}} = \frac{2t}{p}\sigma_{pp} - 2\sum_{s=1}^{p-1} \sigma_{ss} = 0, \\ \frac{\partial \mathcal{P}}{\partial \sigma_{ij}} = 4a\sigma_{ij} = 0, & \frac{\partial \mathcal{P}}{\partial \sigma_{ip}} = 4(\frac{p+t}{p})\sigma_{ip} = 0, \end{array}\right\} \tag{11}$$

where $i, j = \{1, 2, \dots, p-1\}, i \neq j$.
Hence, every solution σ^c has $\sigma_{ij} = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, it is to see that the Hessian matrix of $\mathcal{P}$ takes the following form

$$H(\mathcal{P}) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2a-2 & -2 & \dots & -2 & -2 \\ -2 & 2a-2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & 2a-2 & -2 \\ -2 & -2 & \dots & -2 & \frac{2t}{p} \end{pmatrix},$$

O are the null matrices and H_2 and H_3 are the diagonal matrices of the respective dimensions. H_2 and H_3 are, respectively, given as

$$H_2 = diag\big(4a, 4a, \dots, 4a\big),$$

and

$$H_3 = diag\bigg(\frac{4(p+t)}{p}, \frac{4(p+t)}{p}, \dots, \frac{4(p+t)}{p}\bigg).$$

Hence, we find that $H(\mathcal{P})$ has the following eigenvalues

$$\lambda_{11} = 0, \lambda_{22} = 2a, \lambda_{33} = \dots = \lambda_{pp} = 2a,$$

$$\lambda_{ij} = 4a, \lambda_{ip} = \frac{4(p+t)}{p}, \quad \forall i, j \in \{1, 2, \dots, p-1\}, \quad i \neq j.$$

It follows that the Hessian matrix is positive semi-definite for all points and admits precisely one eigenvalue equal to zero. Therefore, we deduce that $\mathcal{P}$ is parabolic and reaches a minimum $\mathcal{P}(\sigma^c)$ for the solution σ^c of the system (11). In fact, because of the convexity, the critical point σ^c is a global minimum. But inserting (11) into (10), we obtain $\mathcal{P}(\sigma^c) = 0$. Thus, we deduce $\mathcal{P} \geq 0$ and this implies

$$2\tau^* \leq t\mathfrak{C} + b(t)\mathfrak{C}(L) - p^2 + 1.$$

The normalized scalar curvature ρ^* of M is defined as $\rho^* = \frac{2\tau^*}{p(p-1)}$. Thus, we have

$$\rho^* \leq \frac{t}{p(p-1)}\mathfrak{C} + \frac{b(t)}{p(p-1)}\mathfrak{C}(L) - \left(\frac{p+1}{p}\right)$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, our assertion (6) follows. In the same manner, we can establish an inequality (7) in the second part of the theorem. □

The characterisation of the equality cases is as follows.

Theorem 3.2. *Let M be a real hypersurface in the complex hyperbolic quadric Q^{q*}. Equalities hold in the relations (6) and (7) if and only if M is an invariantly quasi-umbilical hypersurface with trivial normal connection in Q^{q*} and, with respect to suitable tangent and normal orthonormal frames, the shape operator S takes the following form*

$$S = \begin{pmatrix} b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{p(p-1)}{t}b \end{pmatrix} \tag{12}$$

Proof. The equality sign holds in the equalities (6) and (7) if and only if

$$\sigma_{ij} = 0, \quad \forall \quad i,j \in \{1,\dots,n\}, \quad i \neq j, \tag{13}$$

and

$$\sigma_{pp} = \frac{p(p-1)}{t}\sigma_{11} = \dots = \frac{p(p-1)}{t}\sigma_{p-1\ p-1} \tag{14}$$

Equation (13) means that normal connection is flat. Furthermore, (14) means that there exist unit normal vector field $\mathcal{N}$ such that the shape operator S has an eigenvalue of multiplicity $(p-1)$, that is, M is invariantly quasi-umbilical hypersurface. This proves our assertion. □

Remark 3.1. *(a) Since the generalized normalized δ-Casorati curvatures $\delta_{\mathfrak{C}}(t;p-1)$ and $\widehat{\delta}_{\mathfrak{C}}(t;p-1)$ are the generalized versions of the normalized δ-Casorati curvatures $\delta_{\mathfrak{C}}(p-1)$ and $\widehat{\delta}_{\mathfrak{C}}(p-1)$, respectively, we can also get the results for the normalized δ-Casorati curvatures $\delta_{\mathfrak{C}}(p-1)$ and $\widehat{\delta}_{\mathfrak{C}}(p-1)$.*

(b) Note that we have derived two optimal inequalities for real hypersurface M in Q^{q*}, now by similar approach ones can easily obtain these two optimal inequalities in the case of complex quadrics Q^q.

4. Classification of real hypersurfaces in some quadrics with isometric Reeb flow

Related to integral formula, there were several pinching results being obtained using a so-called "Bochner integral formula" (see [19]). Let M be a compact Riemannian manifold. Then for any tangent vector field X on M, K. Yano [18] established the following integral formula

$$\int_M \big\{K(X,X) + \frac{1}{2}||\mathcal{L}_X g||^2 - ||\nabla X||^2 - ||div(X)||^2\big\} * 1 = 0, \tag{15}$$

where where $K(X,X)$ denotes the Ricci curvature along the direction of the vector X and $\mathcal{L}_X$ is the operator of Lie derivative with respect to X, defined by $(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)$.

Proposition 4.1. *Let M be a minimal compact real hypersurface in $Q^{q*}, q \geq 3$. If $2\tau^* \geq -(2q-1)(2q-2)-2$, then*
1. *$PS = SP$,*
2. *the unit normal vector field $\mathcal{N}$ is $\mathfrak{U}-$isotropic,*
3. $2\tau^* = -(2q-1)(2q-2)-2$.

Proof. The proof is based on the so called Bochner technique. Since M is minimal, it follows that $h=0$. Thus, the the Ricci curvature $K^*(\xi,\xi)$ in (4) along the direction ξ becomes

$$K^*(\xi,\xi) = -(2q-4) + 2g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\xi,\xi) - g(S^2\xi,\xi), \tag{16}$$

and the scalar curvature is given by

$$2\tau^* = -(2q-1)^2 + 1 - trace(S^2). \tag{17}$$

By substituting $X=\xi$ into (15) and by making use of $div(\xi) = trace(PS) = 0$, $||\nabla\xi||^2 = trace(S^2) - g(S^2\xi,\xi)$, and (17), we derive

$$\begin{aligned} 0 &= \int_M \big\{ -(2q-4) + 2g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\xi,\xi) - g(S^2\xi,\xi) \qquad (18)\\ &\quad + \frac{1}{2}||\mathcal{L}_\xi g||^2 - trace(S^2) + g(S^2\xi,\xi)\big\} * 1 \\ &= \int_M \big\{ -(2q-4) + 2g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\xi,\xi) + \frac{1}{2}||\mathcal{L}_\xi g||^2 \\ &\quad + 2\tau^* + (2q-1)^2 - 1\big\} * 1 \\ &= \int_M \big\{2\tau^* + (2q-1)(2q-2) + 2 + \frac{1}{2}||\mathcal{L}_\xi g||^2 \\ &\quad + 2g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\xi,\xi)\big\} * 1. \qquad (19) \end{aligned}$$

If $2\tau^* \geq -(2q-1)(2q-2)-2$, then it follows that integrand (18) is non-negative and we arrive at $2\tau^* = -(2q-1)(2q-2)-2$, $g(\mathcal{A}\mathcal{N},\mathcal{N})g(\mathcal{A}\xi,\xi) = 0$

and $\mathcal{L}_\xi g = 0$. Further,

$$\begin{aligned} 0 &= (\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\ &= g(PSX,Y) + g(PSY,X) \\ &= g(PSX,Y) - g(SPX,Y), \end{aligned}$$

where we have used $\nabla_X \xi = PSX$. Moreover, we have either $g(\mathcal{A}\mathcal{N},\mathcal{N}) = 0$ or $g(\mathcal{A}\xi,\xi) = 0$. Then by (1), we get $t = \frac{\pi}{4}$ and the unit normal vector field $\mathcal{N}$ becomes $\mathcal{U}-$isotropic, that is, $\mathcal{N} = \frac{Z_1 + JZ_2}{\sqrt{2}}$, for any $Z_1, Z_2 \in V(\mathcal{A})$. Hence, we get our assertions. □

Following [11] and Proposition 4.1, we state that

Theorem 4.1. *Let M be a minimal compact real hypersurface in $Q^{q*}, q \geq 3$. If $2\tau^* \geq -(2q-1)(2q-2) - 2$, then M is a tube of radius $\frac{\pi}{4}$ over $\mathbb{CH}^{\frac{q}{2}}$ in Q^{q*}.*

Similarly, we have

Theorem 4.2. *Let M be a minimal compact real hypersurface in $Q^q, q \geq 3$ with $\mathcal{U}-$isotropic unit normal vector field $\mathcal{N}$. If $2\tau^* \geq (2q-1)(2q-2) + 2$, then M is a tube of radius $\frac{\pi}{4}$ over $\mathbb{CP}^{\frac{q}{2}}$ in Q^q.*

5. Non-existence of real hypersurfaces in some quadrics

On compact Riemannian manifold M we have the following integral formula [18, 17] (in terms of Ricci curvature K).

Lemma 5.1. *Let M be a compact Riemannian manifold. Then for any vector field X defined on M we have $\int_M \left\{K(X,X) + ||\nabla X||^2\right\} * 1 \geq 0$. Then the equality holds if and only if X is a harmonic vector field.*

Before going to prove our main theorem we wish to give the following proposition with the help of above quoted Lemma 5.1.

Proposition 5.1. *Let M be a compact real hypersurface in $Q^{q*}, q \geq 3$ with $\mathcal{U}-$principal unit normal vector field $\mathcal{N}$ and trace$(S^2) \leq (2q-2) - hg(S\xi,\xi) + 2||S\xi||^2$.*
1. *If Ricci curvature $K^*(\xi,\xi)$ is positive semi-definite, then ξ is a harmonic vector field and has vanishing covariant derivative.*
2. *If Ricci curvature $K^*(\xi,\xi)$ is positive definite, then a harmonic vector field ξ (other than zero) does not exist in M.*

Proof. By applying Lemma 5.1 to the structure vector filed ξ of a compact real hypersurface M in Q^{q*} and using (16), we arrive at

$$\begin{aligned} 0 &\leq \int_M \left\{K^*(\xi,\xi) + ||\nabla\xi||^2\right\} * 1 \\ &= \int_M \left\{ -(2q-2) + hg(S\xi,\xi) - g(S^2\xi,\xi) + g(\nabla\xi,\nabla\xi)\right\} * 1. \end{aligned}$$

Since $\nabla_X \xi = PSX$ and $g(PX, PY) = g(X, Y) - \eta(X)\eta(Y)$, then we have the following

$$\begin{aligned} 0 &\leq \int_M \big\{ -(2q-2) + hg(S\xi, \xi) - g(S^2\xi, \xi) + g(\nabla\xi, \nabla\xi) \big\} * 1 \\ &= \int_M \big\{ -(2q-2) + hg(S\xi, \xi) - g(S^2\xi, \xi) + trace(S^2) - g(S^2\xi, \xi) \big\} * 1 \\ &= \int_M \big\{ -(2q-2) + hg(S\xi, \xi) - 2||S\xi||^2 + trace(S^2) \big\} * 1. \end{aligned}$$

If the trace of the shape operator S^2 satisfies $trace(S^2) \leq (2q-2) - hg(S\xi, \xi) + 2||S\xi||^2$, then the equality holds and we get both the assertions of the theorem. □

Theorem 5.1. *There do not exist any compact real hypersurfaces in $Q^{q*}, q \geq 3$, with $\mathfrak{U}-$principal unit normal vector field $\mathfrak{N}$, satisfying $K^*(\xi, \xi) \geq 0$, and $trace(S^2) \leq (2q-2) - hg(S\xi, \xi) + 2||S\xi||^2$.*

Proof. By the assumption of Proposition 5.1, that is, $K^*(\xi, \xi) \geq 0$, we conclude that $K^*(\xi, \xi) = 0$ and $\nabla\xi = 0$. Taking into account the latter case, that is, $\nabla\xi = 0$ and this implies that $SX = \eta(SX)\xi$ for any tangent vector field X on M, that is, M is a totally $\eta-$umbilical real hypersurface in Q^{q*}. From this we know that the structure vector ξ is principal, that is, $S\xi = \beta\xi$, where $\beta = \eta(S\xi)$. The trace of shape operator is

$$\begin{aligned} h &= trace(S) = \sum_{i=1}^{2q-1} g(SE_i, E_i) = \sum_{i=1}^{2q-1} \eta(SE_i)\eta(E_i) \\ &= g(S\xi, \xi) = \eta(S\xi) = \beta. \end{aligned}$$

From this and together with $K^*(\xi, \xi) = 0$, it follows that $K^*(\xi, \xi) = (2-2q) + \beta^2 - \beta^2 = 2 - 2q$. But $K^*(\xi, \xi) = 2 - 2q = 0$, which further gives $q = 1$. This contradicts our assumption $q \geq 3$. □

Proposition 5.2. *Let M be a compact real hypersurface in $Q^{q*}, q \geq 3$ with $\mathfrak{U}-$isotropic unit normal vector field $\mathfrak{N}$ and $trace(S^2) \leq (2q-4) - hg(S\xi, \xi) + 2||S\xi||^2$.*

1. *If Ricci curvature $K^*(\xi, \xi)$ is positive semi-definite, then ξ is a harmonic vector field and has vanishing covariant derivative.*

2. *If Ricci curvature $K^*(\xi, \xi)$ is positive definite, then a harmonic vector field ξ (other than zero) does not exist in M.*

Proof. The proof of this proposition is similar to Proposition 5.1. □

By using the assumption of Proposition 5.2, we can easily prove the following theorem.

Theorem 5.2. *There do not exist any compact real hypersurfaces in $Q^{q*}, q \geq 3$, with $\mathfrak{U}-$isotropic unit normal vector field $\mathfrak{N}$, satisfying $K^*(\xi, \xi) \geq 0$, and $trace(S^2) \leq (2q-4) - hg(S\xi, \xi) + 2||S\xi||^2$.*

Remark 5.1. *Note that ones can study such results for a compact real hypersurface in $Q^q, q \geq 3$ with $\mathcal{U}-$principal unit normal vector field $\mathcal{N}$ and $\mathcal{U}-$principal unit normal vector field $\mathcal{N}$.*

6. **Some open problems**

1. We note that the techniques used in this paper to obtain the sharp optimal inequalities involving generalized normalized δ-Casorati curvatures for submanifolds in some quadrics with codimension 1 are based on an optimization procedure by showing that a quadratic polynomial in the components of the second fundamental form is parabolic. Now, one can study the similar inequalities for general submanifolds (such as CR-submanifolds) as well. On the other hand, the concept of semi-symmetric metric connection on Riemannian manifold was introduced by H.A. Hayden. Chen-like inequalities for submanifolds of real, complex and Sasakian space forms endowed with semi-symmetric metric connections are derived. Moreover, some optimal inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection are obtained by using a different algebra approaches. The problem is to obtain optimal inequalities for the generalized normalized δ-Casorati curvatures of different classes of submanifolds in some quadrics admitting semi-symmetric metric connections or other connections.

2. Related to integral formula (sections 4 and 5):
(a) there were several pinching results being obtained using a so-called "Bochner integral formula" (for example, see [7]). It is expected that the inequality leads to minimal compact real hypersurfaces satisfying $PS = SP$. When the ambient space is Q^q, it provides a characterization of tubes of radius $\frac{\pi}{4}$ over $\mathbb{CP}^{q/2}$ in Q^q (by using results in [3]). One could study this result.
(b) Would it be possible to redesign the condition(s) so that it becomes a classification result?

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