SPLITTING RELAXED S-SUBGRADIENT PROJECTION ALGORITHM FOR NON-CONVEX SPLIT FEASIBILITY PROBLEMS

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In this paper, we suggest an algorithm based on subgradient projection method for solving non-convex split feasibility problems in finite dimensional spaces. The step-size of the proposed sequence is chosen according to Armijo-type line rule. Convergence result is proved under some additional conditions.

Keywords: nonconvex split feasibility problems, S-subdifferentiable, S-subgradient projection, Armijo-type line rule.

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1. **Introduction**

Find an element $x \in \mathbb{R}^n$ satisfying

$$x \in C \text{ with } Ax \in Q, \tag{1}$$

where $C \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are closed convex non-empty sets and A is a matrix from $\mathbb{R}^n$ into $\mathbb{R}^m$. This problem is called split feasibility problem (abbr. SFP). Since the problem (1) was raised by Censor and Elfving [6] in 1994, it has been successfully applied to signal processing, image restoration, especially in the field of intensity modulated radiation therapy (IMRT) [3–5, 7]. Many algorithms have been proposed for solving the SFP and the related problems, please see [1, 6, 8–13, 15–37] and their references therein. Byrne [4, 5] suggested the CQ algorithm:

$$x_{k+1} = P_C\left(x_k - \varrho_k A^T (I - P_Q) Ax_k\right),\ k \geq 1, \tag{2}$$

where P_C and P_Q are the metric projections onto C and Q, respectively and the step ϱ_k is in $(0, 2/\delta)$ with δ being the spectral radius of matrix A^TA (or $\varrho_k \in \left(0, 2/\|A\|^2\right)$ equivalently). Compared with the algorithm in [6] where the inverse A^{-1} (suppose it exists) is needed, the so-called CQ algorithm (2) is easy to implement due to it only deals with metric projections. To perform the CQ algorithm (2), the form of given closed convex subsets C and Q should be very simple so that the metric projections P_C and P_Q can be calculated easily. Now we consider the level sets as follows:

$$C_0 = \{x \in \mathbb{R}^n : c(x) \leq 0\} \text{ and } Q_0 = \{y \in \mathbb{R}^m : q(y) \leq 0\},$$

where $c : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^m \to \mathbb{R}$ are convex functions. As far as we know, the efficiency of the CQ algorithm (2) would be affected extremely whenever the closed convex sets are constructed by level sets because the metric projection onto level set do not have closed form, in other word, the computation of metric projection onto such set is not an easy task.

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To overcome this difficulty, Yang [26] presented relaxed CQ algorithm that computes the metric projection onto half-space containing the level set instead of the latter one itself. The relaxed CQ algorithm [26] is as follows:

$$x_{k+1} = P_{C_{k,0}}(x_k - \varrho_k A^T(I - P_{Q_{k,0}})Ax_k),\ k \geq 1, \tag{3}$$

where $\varrho_k \in \left(0, 2/\|A\|^2\right)$ and $C_{k,0}$, $Q_{k,0}$ are given by

$$C_{k,0} = \{x \in \mathbb{R}^n : \langle \phi_k, x - x_k\rangle + c(x_k) \leq 0\},$$

and

$$Q_{k,0} = \{y \in \mathbb{R}^m : \langle \varphi_k, y - Ax_k\rangle + q(Ax_k) \leq 0\},$$

in which $\phi_k \in \partial c(x_k)$ and $\varphi_k \in \partial q(Ax_k)$. Obviously, $C_{k,0}$ and $Q_{k,0}$ are half-spaces and hence the metric projections onto $C_{k,0}$ and $Q_{k,0}$ have the closed form, this makes algorithm (3) easy to implemented in practice. However, the step-size ϱ_k in (3) depends on matrix norm $\|A\|$, this greatly affects the applicability of the algorithm, see [14]. Thus, López [16] defined a function as follows

$$f_k(x) = \frac{1}{2}\left\|Ax - P_{Q_{k,0}}(Ax)\right\|^2,\ k \geq 1.$$

We can rewrite algorithm (3) as

$$x_{k+1} = P_{C_{k,0}}\left(x_k - \varrho_k \nabla f_k(x_k)\right), \tag{4}$$

where step-size

$$\varrho_k = \lambda_k f_k(x_k)/\|\nabla f_k(x_k)\|^2, \quad 0 < \lambda_k < 2, \tag{5}$$

and gradient

$$\nabla f_k(x) = A^T\left(I - P_{Q_{k,0}}\right)Ax.$$

The convergence of algorithm (4) with step-size (5) is guaranteed under the computation of metric projection onto half-space. In this case, they did not need to calculate the value of the matrix norm $\|A\|$. Moreover, the method of avoiding calculating matrix norm can also use the Armijo-type line rule. Inspired by the relaxed projection method and the Tseng's modified forward-backward splitting method, Wang [24] suggested the following algorithm:

$$\begin{cases} y_k = P_{C_n}\left(x_k - \tau_k f_k(x_k)\right), \\ x_{k+1} = P_{C_n}\left(y_k - \tau_k(f_k(y_k) - f_k(x_k))\right), \end{cases}$$

the step-size τ_k here is selected by the Armijo-type line rule. In this case, the matrix norm $\|A\|$ does not need to be estimated. See also [15, 17, 21] for more details about Armijo-type line rule.

On the other hand, the relaxed CQ algorithm (4) can be rewritten as subgradient projection algorithm when the concept of subgradient projection is introduced. Guo [13] denoted by G_c the subgradient projector associated with (c, 0) and by G_{f_k} the subgradient projector associated with (f_k, 0). Guo [13] proposed the following iterative step:

$$x_{k+1} = G_c\left(R_{\lambda_k f_k}(x_k)\right), \tag{6}$$

where $R_{\lambda_k f_k} = I + \lambda_k\left(G_{f_k} - I\right)$ is a relaxation of G_{f_k}, λ_k is chosen in the interval $(0, 2)$, then algorithm (6) converges to the solution of problem (1) in which the original closed convex subsets are replaced by level sets.

There is a natural question: Can the algorithm (6) and its variants be constructed in which the step-size is chosen by the Armijo-type line rule?

Motivated by the works of Wang [24] and Guo [13], we suggest in the paper a new form of subgradient projection algorithm to solve the SFP in which the step-size is chosen according to the Armijo-type line rule. Moreover, the functions c and q in (1) we consider are both continuous, S-subdifferential, locally Lipschitzian, not necessarily convex instead of the original convex.

2. Preliminaries

Let $S \subseteq \mathbb{R}^n$ be nonempty closed set, denote by P_S the orthogonal (metric) projection from $\mathbb{R}^n$ onto S; that is,

$$P_S(x) := \operatorname{argmin}_{y\in S}\|x-y\|, \quad x \in \mathbb{R}^n.$$

Definition 2.1 ([2])**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real function. We use $Levf = \{x \in \mathbb{R}^n : f(x) \le \xi\}$ to denote the level set of f.*

Definition 2.2 ([13])**.** *Given $S \subseteq \mathbb{R}^n$ and $r_f > 0$, a vector $u \in \mathbb{R}^n$ is said to be an S-subgradient of function $f : \mathbb{R}^n \to \mathbb{R}$ at x if*

$$\langle y-x,u\rangle + f(x) + \frac{r_f}{2}d_S^2(x) \le f(y) + \frac{r_f}{2}d_S^2(y), \quad y \in \mathbb{R}^n.$$

The set of all S-subgradients of function f at x is called S-subdifferential of f at x and is denoted by

$$\partial_{Sr_f}f(x) = \left\{u \in \mathbb{R}^n : \langle y-x,u\rangle + f(x) + \frac{r_f}{2}d_S^2(x) \le f(y) + \frac{r_f}{2}d_S^2(y), y \in \mathbb{R}^n\right\} \tag{7}$$

where $d_S(x) = \inf_{y\in S}\|x-y\|$ is the usual distance from the point x to the set S.

If $r_f = 0$ in (7), the S-subdifferential turns out to be the Fenchel subdifferential. If $S = \mathbb{R}^n$, the above result is still valid.

Definition 2.3 ([2])**.** *Given a (not necessarily convex) function $f : \mathbb{R}^n \to \mathbb{R}$, define its Fenchel subdifferential at x by*

$$\partial f(x) := \{u \in \mathbb{R}^n : \langle y-x,u\rangle + f(x) \le f(y), \forall y \in \mathbb{R}^n\}.$$

When f is convex, $\partial f(x)$ is the usual subdifferential.

Lemma 2.1 ([13])**.** *Let S be closed and convex and $C_\xi = Levf$ be a non-empty set such that $C_\xi \subseteq S \subseteq \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be S-subdifferential on $\mathbb{R}^n$ with respect to S. Then there exists a constant $r_f > 0$ such that for any $x \notin C_\xi$,*

$$s_f(x) \in \partial_{Sr_f}f(x) \Rightarrow s_f(x) \ne 0.$$

Definition 2.4 ([13])**.** *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and S-subdifferential on $\mathbb{R}^n$ with respect to S. Let S be closed and convex and $C_\xi = Levf$ be a non-empty set such that $C_\xi \subseteq S \subseteq \mathbb{R}^n$. Assume that $\partial_{Sr_f}f(x)$ is the S-subdifferential of f with respect to S and $s_f(x) \in \partial_{Sr_f}f(x)$. The S-subgradient projector onto C_ξ related to (f,ξ) is*

$$G_{S,f} : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \begin{cases} x + \frac{\xi - f(x)}{\|s_f(x)\|^2}s_f(x), & x \notin C_\xi \\ x, & x \in C_\xi. \end{cases}$$

Lemma 2.2 ([13])**.** *Let $S \subseteq \mathbb{R}^n$ be closed and convex and $f : \mathbb{R}^n \to \mathbb{R}$ be S-subdifferential on $\mathbb{R}^n$ with respect to S. Then, there exists a constant $r_f > 0$ such that*

$$u \in \partial_{Sr_f}f(x) \Leftrightarrow u \in \partial f(x) + r_f(I - P_S)(x).$$

3. Split Feasibility Problem in Non-convex Case

Let's now consider the split feasibility problem defined on non-convex level sets which is formulated as finding an element x satisfying the form:

$$x \in C_0 \text{ and } Ax \in Q_0,$$

where $A : \mathbb{R}^n \to \mathbb{R}^m$, C_0 and Q_0 are stated in (1) in which $c : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^m \to \mathbb{R}$ are continuous, S-subdifferential and locally Lipschitzian. In the sequel, we assume that the solution set $\Gamma := \{x \in C_0 : Ax \in Q_0\}$ is non-empty.

Assume that $S_n \subseteq \mathbb{R}^n$ and $S_m \subseteq \mathbb{R}^m$ are closed convex sets such that $C_0 \subseteq S_n$ and $Q_0 \subseteq S_m$. Since $c : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^m \to \mathbb{R}$ are continuous and S-subdifferential, we use $\partial_{S_n r_c} c(x)$ and $\partial_{S_m r_q} q(y)$ to denote S-subdifferential of c and q with respect to S_n and S_m, respectively. Let $s_c(x) \in \partial_{S_n r_c} c(x)$ be S-subgradient of c at $x \in \mathbb{R}^n$ and let $s_q(y) \in \partial_{S_m r_q} q(y)$ be S-subgradient of q at $y \in \mathbb{R}^m$. By definition 2.4, the S-subgradient projector onto C_0 associated with $(c, 0)$ can be defined by

$$G_{S_n,c} : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \begin{cases} x + \frac{-c(x)}{\|s_c(x)\|^2} s_c(x), & x \notin C_0 \\ x, & x \in C_0. \end{cases}$$

We can also define the S-subgradient projector $G_{S_m,q} : \mathbb{R}^m \to \mathbb{R}^m$ by the same way.

On the other hand, given $s_c(x_k) \in \partial_{S_n r_c} c(x_k)$ and $s_q(Ax_k) \in \partial_{S_m r_q} q(Ax_k)$, set

$$C_{k,0} = \{u \in \mathbb{R}^n : \langle s_c(x_k), u - x_k \rangle + c(x_k) \leq 0\},$$

and

$$Q_{k,0} = \{v \in \mathbb{R}^m : \langle s_q(Ax_k), v - Ax_k \rangle + q(Ax_k) \leq 0\}, k \geq 1.$$

Write $f_k(x) = \frac{1}{2} \left\| x - P_{C_{k,0}}(x) \right\|^2$ and $g_k(x) = \frac{1}{2} \left\| Ax - P_{Q_{k,0}}(Ax) \right\|^2$ for all $x \in \mathbb{R}^n$ and their Lipschitz gradients are $\nabla f_k(x) = x - P_{C_{k,0}}(x)$ and $\nabla g_k(x) = A^T \left(Ax - P_{Q_{k,0}}(Ax) \right)$ for all $x \in \mathbb{R}^n$. Denote the subgradient projector related to $(f_k, 0)$ by G_{f_k} and denote the subgradient projector associated with $(g_k, 0)$ by G_{g_k}.

Set $R_{\mu_k f_k} = I + \mu_k (G_{f_k} - I)$ and $R_{\lambda_k g_k} = I + \lambda_k (G_{g_k} - I)$.. Now we construct the following recursive procedure: for any $x_1 \in \mathbb{R}^n$,

$$\begin{cases} z_k = R_{\mu_k f_k} \left(R_{\lambda_k g_k}(x_k) \right), \\ y_k = G_{S_n,c} \left(z_k - \tau_k \nabla g_k(z_k) \right), \\ x_{k+1} = G_{S_n,c} \left(y_k - \tau_k (\nabla g_k(y_k) - \nabla g_k(z_k)) \right), \end{cases} \tag{8}$$

for all $k \geq 1$, where $\tau_k = \gamma \iota^{m_k}$ with $\gamma > 0$, $\iota > 0$ and m_k is the smallest nonnegative integer m such that

$$\tau_k \|\nabla g_k(y_k) - \nabla g_k(z_k)\| \leq \sqrt{1-\kappa} \|y_k - z_k\|, \ \kappa \in (0,1).$$

We now give the convergence analysis of the algorithm (8).

Theorem 3.1. *The sequence $\{x_k\}$ generated by algorithm (8) converges to $x^* \in \Gamma$ provided $\lambda_k, \mu_k \in (0,2)$.*

Proof. Let $\tau \in \Gamma$ and $s_q(Ax_k) \in \partial_{S_m r_q} q(Ax_k)$. From (7) and the fact $Q_0 \subseteq S_m$, we have

$$\langle s_q(Ax_k), A\tau - Ax_k \rangle + q(Ax_k) \leq q(A\tau) + \frac{r_q}{2} d^2_{S_m}(A\tau) - \frac{r_q}{2} d^2_{S_m}(Ax_k) \leq 0$$

for any $A\tau \in Q_0$. Thus, we have $A\tau \in Q_{k,0}$, or equivalently, $g_k(\tau) = 0$. In the same way, we get $f_k(\tau) = 0$.

According to the definition of G_{g_k}, we consider two cases. If $Ax_k \in Q_{k,0}$, one can show that

$$\langle G_{g_k}(x_k) - \tau, G_{g_k}(x_k) - x_k \rangle = \langle G_{g_k}(x_k) - \tau, x_k - x_k \rangle = 0.$$

If $Ax_k \notin Q_{k,0}$, by the fact $g_k(\tau) = 0$, we obtain from (7) and (2.3) that

$$\begin{aligned}\langle G_{g_k}(x_k) - \tau, G_{g_k}(x_k) - x_k\rangle =& \left\langle x_k - \tau, \frac{-g_k(x_k)}{\|\nabla g_k(x)\|^2}\nabla g_k(x)\right\rangle + \frac{g^2_{q,k}(x_k)}{\|\nabla g_k(x)\|^2}\\ =&\frac{g_k(x_k)}{\|\nabla g_k(x)\|^2}\langle \tau - x_k, \nabla g_k(x)\rangle + \frac{g^2_{q,k}(x_k)}{\|\nabla g_k(x)\|^2}\\ \leq&\frac{g_k(x_k)}{\|\nabla g_k(x)\|^2}(g_k(\tau) - g_k(x_k)) + \frac{g^2_{q,k}(x_k)}{\|\nabla g_k(x)\|^2}\\ =&0.\end{aligned}$$

Hence,

$$\langle G_{g_k}(x_k) - \tau, G_{g_k}(x_k) - x_k\rangle \leq 0. \tag{9}$$

Set $w_k = R_{\lambda_k g_k}(x_k)$, $v_k = y_k - \tau_k(\nabla g_k(y_k) - \nabla g_k(z_k))$ and $u_k = z_k - \tau_k\nabla g_k(z_k)$. Using the same argument as that of (9), we have

$$\langle G_{f_k}(w_k) - \tau, G_{f_k}(w_k) - w_k\rangle \leq 0, \tag{10}$$

$$\langle G_{S_n,c}(v_k) - \tau, G_{S_n,c}(v_k) - v_k\rangle \leq 0, \tag{11}$$

and

$$\langle G_{S_n,c}(u_k) - \tau, G_{S_n,c}(u_k) - u_k\rangle \leq 0,$$

respectively, and the last inequality can be rewritten as

$$\langle y_k - \tau, y_k - z_k\rangle \leq -\tau_k \langle y_k - \tau, \nabla g_k(z_k)\rangle\,. \tag{12}$$

Also using (9), we achieve

$$\begin{aligned}\|w_k - \tau\|^2 =&\|x_k - \tau\|^2 + 2\lambda_k\langle x_k - G_{g_k}(x_k), G_{g_k}(x_k) - x_k\rangle\\ &+ 2\lambda_k\langle G_{g_k}(x_k) - \tau, G_{g_k}(x_k) - x_k\rangle + \lambda_k^2\left\|G_{g_k}(x_k) - x_k\right\|^2\\ \leq&\|x_k - \tau\|^2 - \lambda_k(2-\lambda_k)\left\|G_{g_k}(x_k) - x_k\right\|^2.\end{aligned}$$

This together with (8) and (10) implies that

$$\begin{aligned}\|z_k - \tau\|^2 =&\|w_k - \tau\|^2 + 2\mu_k\langle w_k - G_{f_k}(w_k), G_{f_k}(w_k) - w_k\rangle\\ &+ 2\mu_k\langle G_{f_k}(w_k) - \tau, G_{f_k}(w_k) - w_k\rangle + \mu_k^2\left\|G_{f_k}(w_k) - w_k\right\|^2\\ \leq&\|w_k - \tau\|^2 - \mu_k(2-\mu_k)\left\|G_{f_k}(w_k) - w_k\right\|^2\\ \leq&\|x_k - \tau\|^2 - \lambda_k(2-\lambda_k)\left\|G_{f_k}(x_k) - x_k\right\|^2 - \mu_k(2-\mu_k)\left\|G_{f_k}(w_k) - w_k\right\|^2.\end{aligned} \tag{13}$$

By (11), it follows that

$$\begin{aligned}\|G_{S_n,c}(v_k) - \tau\|^2 =& \left\|v_k + G_{S_n,c}(v_k) - v_k - \tau\right\|^2\\ =&\|v_k - \tau\|^2 + 2\langle v_k - G_{S_n,c}(v_k), G_{S_n,c}(v_k) - v_k\rangle\\ &+ 2\langle G_{S_n,c}(v_k) - \tau, G_{S_n,c}(v_k) - v_k\rangle + \left\|G_{S_n,c}(v_k) - v_k\right\|^2\\ \leq&\|v_k - \tau\|^2 - \left\|G_{S_n,c}(v_k) - v_k\right\|^2\\ \leq&\|v_k - \tau\|^2.\end{aligned} \tag{14}$$

By (8), (3) and (14), we get

$$\begin{aligned}\|x_{k+1} - \tau\|^2 \leq&\|y_k - \tau\|^2 + \tau_k^2\|\nabla g_k(y_k) - \nabla g_k(z_k)\|^2\\ &- 2\tau_k\langle y_k - \tau, \nabla g_k(y_k) - \nabla g_k(z_k)\rangle\\ \leq&\|y_k - \tau\|^2 + (1-\kappa)\|y_k - z_k\|^2\\ &- 2\tau_k\langle y_k - \tau, \nabla g_k(y_k) - \nabla g_k(z_k)\rangle\,.\end{aligned} \tag{15}$$

Note that

$$\begin{aligned}\|y_k-\tau\|^2 =&\|z_k-\tau\|^2+\|y_k-z_k\|^2+2\langle z_k-\tau, y_k-z_k\rangle \\ =&\|z_k-\tau\|^2-\|y_k-z_k\|^2+2\langle y_k-\tau, y_k-z_k\rangle .\end{aligned} \tag{16}$$

Using (12) and (16) in (15), we have

$$\|x_{k+1}-\tau\|^2 \le \|z_k-\tau\|^2-\kappa\|y_k-z_k\|^2-2\tau_k\langle y_k-\tau, \nabla g_k(y_k)\rangle .$$

Since ∇g_k is monotone, it follows that

$$\langle \nabla g_k(y_k), y_k-\tau\rangle \ge \langle \nabla g_k(\tau), y_k-\tau\rangle = 0.$$

This together with (13) implies that

$$\begin{aligned}\|x_{k+1}-\tau\|^2 \le&\|x_k-\tau\|^2-\lambda_k(2-\lambda_k)\|G_{g_k}(x_k)-x_k\|^2 \\ &-\mu_k(2-\mu_k)\|G_{f_k}(w_k)-w_k\|^2 .\end{aligned}$$

By the assumptions $0<\lambda_k<2$ and $0<\mu_k<2$, we get

$$\|x_{k+1}-\tau\|^2 \le \|x_k-\tau\|^2$$

for all $k \ge 1$. This shows that $\{x_k\}$ is bounded due to its Fejer monotonicity. Moreover, we conclude that

$$\lim_{k\to\infty}\|G_{g_k}(x_k)-x_k\| = \lim_{k\to\infty}\|G_{f_k}(w_k)-w_k\| = 0. \tag{17}$$

Therefore,

$$\|G_{g_k}(x_k)-x_k\| = \left\|x_k+\frac{-g_k(x_k)}{\|\nabla g_k(x_k)\|^2}\nabla g_k(x_k)-x_k\right\| = \frac{g_k(x_k)}{\|\nabla g_k(x_k)\|}. \tag{18}$$

Observe that

$$\|\nabla g_k(x_k)\| = \|\nabla g_k(x_k)-\nabla g_k(\tau)\| \le \|A\|^2\|x_k-\tau\|.$$

This shows that $\{\nabla g_k(x_k)\}$ is bounded. We get from (17) and (18) that

$$\lim_{k\to\infty}\left\|Ax_k-P_{Q_{k,0}}(Ax_k)\right\| = 0.$$

The locally boundedness of ∂q is obtained from the locally Lipschitzian of q and hence ∂q is bounded on bounded sets and so is $I-P_{S_m}$. By Lemma 2.2, we get $\partial_{S_m r_q} q$ is bounded on bounded sets. Thus,

$$q(Ax_k) \le \left\langle s_q(Ax_k), Ax_k-P_{Q_{k,0}}(Ax_k)\right\rangle \le \eta\left\|Ax_k-P_{Q_{k,0}}(Ax_k)\right\| .$$

where $\eta>0$ such that $\|s_q(Ax_k)\| \le \eta$. Since $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_i}\} \subset \{x_k\}$ such that $x_{k_i} \to x^*$. The continuity of q yields

$$q(Ax^*) = \lim_{i\to\infty} q(Ax_{k_i}) \le 0,$$

which implies $Ax^* \in Q_0$.

Since $w_k = R_{\lambda_k g_k}(x_k)$, it follows from (17) that

$$\lim_{i\to\infty}\|w_{k_i}-x_{k_i}\| = 0.$$

So, $w_{k_i} \to x^*$.

Next, according to the definition of G_{f_k}, we consider two possible cases. If $w_{k_i} \in C_{k_i,0}$. Then $c(x^*) \le 0$ and hence $x^* \in C_0$. If $w_{k_i} \notin C_{k_i,0}$. Using the same argument as that of (18), we get $\lim_{i\to\infty}\left\|w_{k_i}-P_{C_{k_i,0}}(w_{k_i})\right\| = 0$. Consequently, we have $x^* \in C_0$.

Finally, we have $x^* \in C_0$ and $Ax^* \in Q_0$ and the proof is completed. □

4. **Conclusion**

The split feasibility problem in convex case has been studied extensively. In this paper, we devote to solve non-convex split feasibility problems in finite dimensional spaces. We suggest an iterative algorithm based on subgradient projection method for solving this split problem. The step-size of the proposed sequence is chosen according to Armijo-type line rule. We show that the sequence generated by the proposed algorithm converges to a solution of the split feasibility problem under some mild conditions.

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