ISOLATED AND PROPER EFFICIENCIES FOR SEMI-INFINITE MULTIOBJECTIVE FRACTIONAL PROBLEMS

by Vivek Singh[1], Anurag Jayswal[2], Ioan Stancu-Minasian[3] and Andreea Mădălina Rusu-Stancu[4]

In this paper, we establish necessary and sufficient optimality conditions for (local) strongly isolated and (local) positively properly efficient solutions of a nonsmooth semi-infinite multiobjective fractional optimization problem with infinite number of inequality constraints by employing some advanced tools of variational analysis and generalized differentiation. Some non-trivial examples to justify the existence of optimality theorems are provided.

Keywords: Limiting subdifferential, generalized convex function, strongly isolated solution, positively properly efficient solution, efficient solution, semi-infinite multiobjective fractional programming, optimality conditions.

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1. Introduction

The root of optimization theory is infiltrating under various other branches of applied sciences at a fast pace. The analysis of mathematical problems of optimizing several ratios of functions simultaneously are commonly known as multiobjective fractional programming problems. The importance of such types of problems is well known in optimization theory as they occur in enormous numbers of applications in science, economics and engineering. Over the last decade, much research has been conducted on necessary/sufficient optimality conditions and duality theorems for multiobjective fractional programming problems, which are not necessarily smooth. For more details, we refer the interested reader to [1, 3, 5, 8, 14, 18, 19, 20, 27, 28, 29, 30, 31, 33].

During the most recent two decades, there has been a vastly fast evolution in subdifferential calculus of nonsmooth analysis which is well-recognized for its numerous applications to optimization theory. The Mordukhovich subdifferential is a highly vital concept in nonsmooth analysis and closely related to optimality conditions of locally Lipschitzian functions of optimization theory (see, [16, 24, 32]). The Mordukhovich subdifferential is a closed subset of the Clarke subdifferential and these subdifferentials are in general nonconvex sets, unlike the well-known Clarke subdifferentials. Therefore, keeping the importance of optimization problems and its wide applications, the explanations of the optimality conditions

[1] Manipal University Jaipur, Rajashthan, India E-mails: viveksingh.25jun@gmail.com

[2] Indian Institute of Technology, (Indian School of Mines), Dhanbad-826 004, Jharkhand, India, E-mail: anurag_jais123@yahoo.com

[3] Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics, Romanian Academy, 13 Septembrie Street, No. 13, 050711 Bucharest, Romania, E-mails: stancu_minasian@yahoo.com

[4] Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics, Romanian Academy, 13 Septembrie Street, No. 13, 050711 Bucharest, Romania, E-mail: andreea_madalina_s@yahoo.com

and calculus rules in terms of Mordukhovich subdifferentials provide more sharp results than those given in terms of the Clarke generalized gradient (see e.g., [22]). Chuong and Kim [8] derived optimality conditions and duality relations that are expressed in terms of limiting/Mordukhovich subdifferentials for nonsmooth multiobjective fractional programming problems.

The specialty of a multiobjective fractional optimization problem is that its objective functions are generally not convex functions. Indeed under all the more restrictive concavity/convexity assumptions, multiobjective fractional optimization problems are generally nonconvex ones. While, the (approximate) extremal principle [22], which plays a central role in variational analysis and generalized differentiation, has been well-recognized as a variational counterpart of the separation theorem for nonconvex sets. Subsequently, utilizing the extremal principle and other advanced techniques of variational analysis and generalized differentiation to prove optimality conditions appears to be appropriate for nonconvex and nonsmooth optimization problems.

Despite phenomenal research advance in several areas of optimization problems, we observe that the field of optimization problems with finite number of variables and infinitely many constraints (called also semi-infinite optimization problems) seems to be still less explored compared to the mathematical programming problem with a finite number of constraints. Kanzi and Nobakhtian [17] introduced several kinds of constraint qualifications of a nonsmooth multiobjective semi-infinite programming problem and discussed the optimality conditions for efficient and weak efficient solutions of a nonsmooth multiobjective semi-infinite programming problem. Choung and Yao [9] established optimality and duality results for semi-infinite multiobjective optimization problem. Ardali and Nobakhtian [4] studied the Fritz John and strong Kuhn-Tucker conditions for properly efficient and isolated efficient solutions of a nonsmooth vector optimization problem. Sufficient conditions also discussed under pseudoconvex sublevel sets. Very recently, Chuong [6] have supplied the optimality conditions and studied duality relation for local (weakly) efficient solutions of a nonsmooth fractional semi-infinite multiobjective optimization problem.

In this communication, motivated by the earlier works, we use the nonsmooth version of Fermat's rule, the sum rule for the Fréchet subdifferentials, and the sum rule as well as the quotient rule for limiting/Mordukhovich subdifferentials given in [22, 23] to prove necessary optimality theorems for (local) strongly isolated solutions and (local) positively properly efficient solutions of a nonsmooth semi-infinite multiobjective fractional optimization problem. Thereafter, we also give sufficient optimality theorems for such solutions to the considered problem by assuming (local) convex functions and generalized convex functions. Even though numerous deliberations have been done on this topic, it still remains a very interesting and demanding area of research. There are several approaches developed in the literature, see [2, 6, 15, 19, 20, 22, 23, 25] and the references therein.

The summary of the paper is as follows. Section 2 contains some basic definitions from variational analysis and several auxiliary results, which will be needed later in the sequel. Section 3 is devoted to the optimality conditions for (local) strongly isolated solutions and (local) positively properly efficient solutions, respectively. The final Section 4 contains the concluding remarks and further developments.

2. Preliminaries

In this section, we recall a number of basic definitions, lemmas and some auxiliary results which will be helpful in proving our mains results in the sequel of the paper.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. Unless otherwise stated, all the spaces in this paper are required to be Asplund (i.e., Banach spaces whose separable subspaces have separable duals), whose norms are denoted by $\|.\|$. Given a space X, its dual is denoted by X^* and the canonical pairing between X and X^* is denoted by $\langle ., . \rangle$. The symbol B_X stands for the closed unit ball in X. As usual, the polar cone of a set $S \subset X$ is defined by $S^\circ = \{x^* \in X^* : \langle x^*, x\rangle \leq 0,\ \forall x \in S\}$ and the notation cl$S$ and int$S$ represent the closure and respectively, the interior of S.

Definition 2.1 (Mordukhovich [22]). *Given a multifunction $F : X \rightrightarrows X^*$ between a Banach space and its dual, the notation*

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) = \{x^* \in X^* : \exists \text{ sequences } x_n \to \bar{x} \text{ and } x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in F(x_n) \text{ for all } n \in \mathbb{N}\}$$

signifies the sequential Painlevé-Kuratowski upper/outer limit of F as $x \to \bar{x}$ with respect to the norm topology of X and the weak topology of X^*, where the notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^* and $\mathbb{N}$ denotes the set of all natural numbers.*

A set $S \subset X$ is locally closed if for each $\bar{x} \in S$, there is a neighborhood U of $\bar{x}$ such that $S \cap \text{cl}U$ is closed.

Definition 2.2 (Mordukhovich [22]). *Given a locally closed set S, define the set of normals to S at $\bar{x} \in S$ by*

$$\widehat{N}(\bar{x}, S) = \{x^* \in X^* : \operatorname*{Lim\,sup}_{x \xrightarrow{S} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0\}, \tag{2.1}$$

where $x \xrightarrow{S} \bar{x}$ means that $x \to \bar{x}$ with $x \in S$. The set $\widehat{N}(\bar{x}, S)$ in (2.1) is a cone called the Fréchet normal cone to S at $\bar{x}$. If $\bar{x} \notin S$, we put $\widehat{N}(\bar{x}, S) = \emptyset$.

Definition 2.3 (Mordukhovich [22]). *The limiting/Mordukhovich normal cone to S at $\bar{x} \in S$, denoted by $N(\bar{x}, S)$, is obtained from $\widehat{N}(x, S)$ by taking the sequential Painlevé-Kuratowski upper limits as*

$$N(\bar{x}, S) = \operatorname*{Lim\,sup}_{x \xrightarrow{S} \bar{x}} \widehat{N}(x, S) \tag{2.2}$$

If $\bar{x} \notin S$, we put $N(\bar{x}, S) = \emptyset$. Specially, when S is locally convex around $\bar{x}$, i.e., there is a neighborhood $U \subset X$ of $\bar{x}$ such that $S \cap U$ is convex, then it holds (see Mordukhovich [22], Theorem 1.5) that

$$N(\bar{x}, S) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0,\ \forall x \in S \cap U\}. \tag{2.3}$$

Definition 2.4 (Mordukhovich [22]). *The limiting/Mordukhovich subdifferential and the Fréchet subdifferentials of an extended real-valued function $\psi : X \to \bar{\mathbb{R}} = [-\infty, \infty]$, at $\bar{x} \in X$ with $|\psi(\bar{x})| < \infty$ are respectively defined by*

$$\partial\psi(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N((\bar{x}, \psi(\bar{x})), \text{epi}\psi)\} \tag{2.4}$$

and

$$\widehat{\partial}\psi(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in \widehat{N}((\bar{x}, \psi(\bar{x})), \text{epi}\psi)\}, \tag{2.5}$$

where epi$\psi = \{(x, \alpha) \in X \times \mathbb{R} : \alpha \geq \psi(x)\}$.

If $|\psi(\bar{x})| = \infty$, one puts $\partial\psi(\bar{x}) = \widehat{\partial}\psi(\bar{x}) = \emptyset$. We get by Definitions **2**.2, **2**.4 and **2**.5 that for any $\bar{x} \in X$, $\widehat{\partial}\psi(\bar{x}) \subset \partial\psi(\bar{x})$. It is known (cf. Mordukhovich [22]) that when ψ is a convex function, then the subdifferentials defined in (**2**.4) and (**2**.5) coincide with the subdifferentials in the sense of convex analysis (cf. Rockafellar [26]).

The relation between the Mordukhovich normal cone and the Mordukhovich subdifferential of the indicator function can be described as (see Mordukhovich [22], Proposition 1.79)

$$N(\bar{x}, S) = \partial\delta(\bar{x}, S), \ \forall \bar{x} \in S, \tag{2.6}$$

where $\delta(., S)$ is the indicator function associated with S given by

$$\delta(x, S) = \begin{cases} 0, \ x \in S \\ \infty, \ \text{otherwise}, \end{cases}$$

The nonsmooth version of Fermat's rule (see Mordukhovich [22], Proposition 1.114), which is an important fact for many applications, can be formulated as follows: If $\bar{x}$ is a local minimizer for $\psi : X \to \bar{\mathbb{R}}$, then

$$0 \in \widehat{\partial}\psi(\bar{x}) \subset \partial\psi(\bar{x}). \tag{2.7}$$

We also consider the Fréchet upper subdifferential of ψ at $\bar{x}$ with $|\psi(\bar{x})| < \infty$, which is defined by

$$\widehat{\partial}^{+}\psi(\bar{x}) = -\widehat{\partial}(-\psi)(\bar{x}). \tag{2.8}$$

The following Fréchet subdifferential sum rule is as follows.

Lemma 2.1 (Mordukhovich et al.[23], Theorem 3.1)**.** *Let $\psi_i : X \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in X$ for $i = 1, 2$. If $\widehat{\partial}^{+}\psi_2(\bar{x}) \neq \emptyset$, then*

$$\widehat{\partial}(\psi_1 + \psi_2)(\bar{x}) \subset \bigcap_{x^* \in \widehat{\partial}^{+}\psi_2(\bar{x})} [\widehat{\partial}\psi_1(\bar{x}) + x^*].$$

Also, we recall the limiting subdifferential sum rule and the formula for the basic subdifferential of maximum function.

Lemma 2.2 (Mordukhovich et al.[22], Theorem 3.36)**.** *Let $\psi_i : X \to \bar{\mathbb{R}}$, $i = 1, ..., n$, $n \geq 2$, be lower semicontinous around $\bar{x} \in X$, and let all these functions, except possibly one, be Lipschitz continuous around $\bar{x}$. Then one has*

$$\partial(\psi_1 + \psi_2 + ... + \psi_n)(\bar{x}) \subset \partial\psi_1(\bar{x}) + \partial\psi_2(\bar{x}) + ... + \partial\psi_n(\bar{x}). \tag{2.9}$$

Lemma 2.3 (Mordukhovich et al.[22], Theorem 3.46 (ii))**.** *Let $\psi_i : X \to \bar{\mathbb{R}}$ be lower semicontinous around $\bar{x} \in X$ for $i \in I(\bar{x})$. Assume that each ψ_i is Lipschitz continuous around $\bar{x}$. Then*

$$\partial(\max\ \psi_i)(\bar{x}) \subset \bigcup \left\{ \partial\Big(\sum_{i \in I(\bar{x})} \lambda_i \psi_i \Big)(\bar{x}) : (\lambda_1, ..., \lambda_n) \in \Pi(\bar{x}) \right\},$$

where the sets

$$I(\bar{x}) = \{i \in \{1, ..., n\} : \psi(\bar{x}) = (\max\ \psi_i)(\bar{x})\},$$

$$\Pi(\bar{x}) = \{(\lambda_1, ..., \lambda_n) : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i(\psi(\bar{x}) - (\max\ \psi_i)(\bar{x})) = 0\}.$$

The following lemma which establishes the relation between the limiting subdifferential sum rule and the quotient rule (cf. Mordukhovich[22], Corollary 1.111(ii)) will play an important role in our paper, for the proof.

Lemma 2.4. *Let $\psi_i : X \to \bar{\mathbb{R}},\ i = 1, 2,$ be Lipschitz continuous around $\bar{x}$. Assume that $\psi_2(\bar{x}) \neq 0$. Then one has*

$$\partial\left(\frac{\psi_1}{\psi_2}\right)(\bar{x}) \subset \frac{\partial\left(\psi_2(\bar{x})\psi_1\right)(\bar{x}) + \partial\left(-\psi_1(\bar{x})\psi_2\right)(\bar{x})}{[\psi_2(\bar{x})]^2}. \qquad (\mathbf{2}.10)$$

In the sequel of the paper, assume that S is a nonempty locally closed subset of X, and let J be an arbitrary (possibly infinite) index set.

The problem to be considered in the present analysis is the following semi-infinite multiobjective fractional programming problem of the form:

$$\text{(P)} \qquad \min_{\mathbb{R}^p_+} \left\{\theta(x) = \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right) : x \in \mathbb{F}\right\}.$$

Here, the constraint set is defined by

$$\mathbb{F} = \{x \in S : h_j(x) \leq 0,\ j \in J\}, \qquad (\mathbf{2}.11)$$

and the functions $f_i,\ g_i,\ i = 1, ..., p,$ and $h_j,\ j \in J$, are locally Lipschitz on X. For the purpose of convenience, we assume further that $g_i(x) > 0,\ i = 1, ..., p,$ for all $x \in S$, and that $f_i(\bar{x}) \leq 0,\ i = 1, ..., p,$ for the reference point $\bar{x} \in S$. Hereafter, we use the notation $h_J = (h_j)_{j\in J}$ and $\theta = (\theta_1, \theta_2, ..., \theta_p)$, where $\theta_i = \frac{f_i}{g_i},\ i = 1, ..., p.$

By keeping in view, the definitions of local strongly isolated solution and local positively properly efficient solution in multiobjective optimization, given by Ginchev et al. [12] and Göpfert et al. [13, p. 110], respectively, we present the following definitions.

Definition 2.5. (i) *A point $\bar{x} \in \mathbb{F}$ is called a local efficient solution of problem* (P) *iff there exists a neighborhood U of $\bar{x}$ such that*

$$\forall x \in U \cap \mathbb{F}, \qquad \theta(x) - \theta(\bar{x}) \notin -\mathbb{R}^p_+ \setminus \{0\}.$$

(ii) *A point $\bar{x} \in \mathbb{F}$ is called a local strongly isolated solution of problem* (P) *iff there exist a neighborhood U of $\bar{x}$ and a constant $\nu > 0$ such that*

$$\forall x \in U \cap \mathbb{F}, \qquad \max_{1\leq i\leq p}\{\theta_i(x) - \theta_i(\bar{x})\} \geq \nu \|x - \bar{x}\|.$$

(iii) *A point $\bar{x} \in \mathbb{F}$ is called a local positively properly efficient solution of problem* (P) *iff there exist a neighborhood U of $\bar{x}$ and $\lambda \in \text{int}\mathbb{R}^p_+$ such that*

$$\forall x \in U \cap \mathbb{F}, \qquad \langle\lambda, \theta(x)\rangle \geq \langle\lambda, \theta(\bar{x})\rangle.$$

The set of local efficient solutions, local strongly isolated solutions, and local positively properly efficient solutions of problem (P) are denoted by loc$E$(P), loc$E^{i\nu}$(P), and locE^p(P) respectively. If $U = X$, one has the concepts of efficient solution, strongly isolated solution, and positively properly efficient solution for problem (P), and in this case we denote these solution sets by E(P), $E^{i\nu}$(P), and E^p(P) respectively.

It is known (see e.g., [11, 12]) that for our framework the inclusions

$$\text{loc}E^{i\nu}(\text{P}) \subset \text{loc}E(\text{P}) \text{ and } \text{loc}E^p(\text{P}) \subset \text{loc}E(\text{P})$$

are always valid, and the converse inclusions do not hold in general.

Let $\mathbb{R}^{(J)}_+$ be the collection of all the functions $\mu : J \to \mathbb{R}$ taking positive values μ_j only at finitely many points of J, and equal to zero at other points. The set of active constraint multipliers at $\bar{x} \in S$ is defined by

$$\Lambda(\bar{x}) = \{\mu \in \mathbb{R}^{(J)}_+ : \mu_j h_j(\bar{x}) = 0,\ \forall j \in J\}. \qquad (\mathbf{2}.12)$$

Definition 2.6 (Chuong and Yao[9])**.** *Let* $\bar{x} \in \mathbb{F}$. *We say that the limiting constraint qualification* (LCQ) *is satisfied at* $\bar{x}$ *iff*

$$N(\bar{x},\mathbb{F}) \subset \bigcup_{\mu\in\Lambda(\bar{x})}\left[\sum_{j\in J}\mu_j\partial h_j(\bar{x})\right] + N(\bar{x},S).$$

If we consider $\bar{x} \in \mathbb{F}$, $S = X$, the above-defined (LCQ) is exactly the limiting constraint qualification introduced in [7] for fixed parameter. The reader is referred to [10] for some sufficient conditions ensuring the (LCQ) in the case when h_j is convex for all $j \in J$.

3. Optimality conditions

In this section, we derive necessary and sufficient conditions for local strongly isolated solutions and local positively properly efficient solutions of problem (P).

Firstly, we give a necessary condition for local strongly isolated solutions of the problem (P) under the fulfillment of the (LCQ).

Theorem 3.1. *Let the* (LCQ)*, defined in Definition* **2**.6*, be satisfied at* $\bar{x} \in \mathbb{F}$. *If* $\bar{x} \in \mathrm{loc}E^{i\nu}(\mathrm{P})$ *for some* $\nu > 0$*, then there exist* $\lambda = (\lambda_1, ..., \lambda_p) \in \mathrm{int}\mathbb{R}^p_+$ *and* $\mu \in \Lambda(\bar{x})$ *such that*

$$\nu B_{X^*} \subset \left\{\sum_{i=1}^{p}\frac{\lambda_i}{g_i(\bar{x})}\left(\partial f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})}\partial g_i(\bar{x})\right) + \sum_{j\in J}\mu_j\partial h_j(\bar{x}) : \lambda_i \geq 0, i = 1, ..., p, \right.$$
$$\left. \sum_{i=1}^{p}\lambda_i = 1, \mu \in \Lambda(\bar{x})\right\} + N(\bar{x},S). \qquad (\mathbf{3}.1)$$

Proof. Let $\bar{x} \in \mathrm{loc}E^{i\nu}(\mathrm{P})$. We define $\psi(x) = \max\limits_{1\leq i\leq p}\left\{\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right\} - \nu\,\|x-\bar{x}\|$, $x \in X$, and consider the following scalar problem

$$\min_{x\in\mathbb{F}}\ \psi(x) \qquad (\mathbf{3}.2)$$

As $\bar{x} \in \mathrm{loc}E^{i\nu}(\mathrm{P})$, there exists a neighborhood U of $\bar{x}$ such that

$$\psi(x) \geq 0 = \psi(\bar{x}), \qquad \forall x \in U \cap \mathbb{F}.$$

It means that $\bar{x}$ is a local minimizer of problem (**3**.2). Thus $\bar{x}$ is a local minimizer of the following unconstrained scalar optimization problem

$$\min_{x\in X}\ \psi(x) + \delta(x,\mathbb{F}).$$

By using (**2**.7), the above defined problem can be rewritten as

$$0 \in \widehat{\partial}(\psi + \delta(.,\mathbb{F}))(\bar{x}). \qquad (\mathbf{3}.3)$$

Set $\psi_1(x) = \max\limits_{1\leq i\leq p}\left\{\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right\} + \delta(x,\mathbb{F})$ and $\psi_2(x) = -\nu\,\|x-\bar{x}\|$. Then $\psi(x) + \delta(.,\mathbb{F}) = \psi_1(x) + \psi_2(x)$ and therefore, we get by (**3**.3) that

$$0 \in \widehat{\partial}(\psi_1 + \psi_2)(\bar{x}). \qquad (\mathbf{3}.4)$$

It is easy to see that $-\psi_2$ is convex function, thus

$$\widehat{\partial}^+\psi_2(\bar{x}) = -\widehat{\partial}^+(-\psi_2)(\bar{x}) = -\partial(\nu\,\|.-\bar{x}\|)(\bar{x}) = \nu B_{X^*} \neq \emptyset.$$

Now, invoking Lemma **2**.1 and taking (**3**.3) into account, we get

$$0 \in \bigcap_{x^* \in \nu B_{X^*}} [\widehat{\partial}\psi_1(\bar{x}) + x^*].$$

This entails that $\nu B_{X^*} \subset \partial\psi_1(\bar{x})$, and thus

$$\nu B_{X^*} \subset \partial\psi_1(\bar{x}) = \partial\left(\max_{1\le i\le p}\left\{\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right\} + \delta(.,\mathbb{F})\right)(\bar{x}). \qquad (\mathbf{3}.5)$$

As the function $\max\limits_{1\le i\le p}\left\{\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right\}$ is Lipschitz continuous around $\bar{x}$ and the function $\delta(.,\mathbb{F})$ is lower semicontinuous around this point, it follows from the sum rule (cf. Mordukhovich[22], Theorem 3.36) applied to (**3**.5) and from the relation in (**2**.6) that

$$\nu B_{X^*} \subset \partial\left(\max_{1\le i\le p}\left\{\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right\}\right)(\bar{x}) + N(\bar{x},\mathbb{F}). \qquad (\mathbf{3}.6)$$

On one hand, applying Lemma **2**.2 and Lemma **2**.3, we obtain

$$\partial\left(\max_{1\le i\le p}\left\{\frac{f_i}{g_i}(.) - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right\}\right)(\bar{x}) \subset \left\{\sum_{i=1}^{p}\lambda_i\partial\left(\frac{f_i}{g_i}\right)(\bar{x}) : \lambda_i \ge 0, i=1,...,p, \sum_{i=1}^{p}\lambda_i = 1\right\},$$

which by using (**2**.10), yields

$$\partial\left(\max_{1\le i\le p}\left\{\frac{f_i}{g_i}(.) - \frac{f_i(\bar{x})}{g_i(\bar{x})}\right\}\right)(\bar{x}) \subset \left\{\sum_{i=1}^{p}\lambda_i\frac{\partial(g_i(\bar{x})f_i)(\bar{x}) + \partial(-f_i(\bar{x})g_i)(\bar{x})}{[g_i(\bar{x})]^2} : \lambda_i \ge 0, i=1,...,p, \sum_{i=1}^{p}\lambda_i = 1\right\}$$

$$= \left\{\sum_{i=1}^{p}\lambda_i\frac{g_i(\bar{x})\partial f_i(\bar{x}) - f_i(\bar{x})\partial g_i(\bar{x})}{[g_i(\bar{x})]^2} : \lambda_i \ge 0, i=1,...,p, \sum_{i=1}^{p}\lambda_i = 1\right\}, \qquad (\mathbf{3}.7)$$

where the equality holds due to the fact that $-f_i(\bar{x}) \ge 0,\ g_i(\bar{x}) > 0$ for $i = 1,...,p$.
On the other hand, the (LCQ) being satisfied at $\bar{x}$ entails that

$$N(\bar{x},\mathbb{F}) \subset \bigcup_{\mu\in\Lambda(\bar{x})}\left[\sum_{j\in J}\mu_j\partial h_j(\bar{x})\right] + N(\bar{x},S), \qquad (\mathbf{3}.8)$$

where the set $\Lambda(\bar{x})$ is defined in (**2**.12). It follows from (**3**.6)-(**3**.8) that

$$\nu B_{X^*} \subset \left\{\sum_{i=1}^{p}\frac{\lambda_i}{g_i(\bar{x})}\left(\partial f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})}\partial g_i(\bar{x})\right) + \sum_{j\in J}\mu_j\partial h_j(\bar{x}) : \lambda_i \ge 0, i=1,...,p, \sum_{i=1}^{p}\lambda_i = 1, \mu\in\Lambda(\bar{x})\right\} + N(\bar{x},S),$$

which completes the proof. □

Now, we give an example to show that the conclusion of the above Theorem 3.1. may fail to hold if the (LCQ) is not satisfied at the point under consideration.

Example 3.1. *Let $\theta : \mathbb{R} \to \mathbb{R}^2$ be defined by $\theta(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}\right)$, where*

$$f_1(x) = x^3,\ f_2(x) = x - x^2,\ g_1(x) = g_2(x) = |x| + 1,\ x \in \mathbb{R},$$

and let $h_j : \mathbb{R} \to \mathbb{R}$ be given by

$$h_j(x) = -jx^2,\ x \in \mathbb{R},\ j \in J = (-\infty, 0).$$

We consider problem (P) *with $p = 2$ and $S = (-\infty, 0] \subset \mathbb{R}$. Then $\mathbb{F} = \{0\}$, and thus $\bar{x} = 0 \in \text{loc}E^{i\nu}(\text{P})$ for any arbitrary $\nu > 0$. Since $N(\bar{x}, S) = [0, \infty)$ and $\partial h_j(\bar{x}) = 0$ for all $j \in J$, we have*

$$\bigcup_{\mu \in \Lambda(\bar{x})} \left[\sum_{j \in J} \mu_j \partial h_j(\bar{x})\right] + N(\bar{x}, S) = [0, \infty),$$

while $N(\bar{x}, \mathbb{F}) = \mathbb{R}$. Hence, the (LCQ) *is not satisfied at $\bar{x}$. In fact, (**3**.1) fails to hold.*

Let $f = (f_1, ..., f_p)$ and $g = (g_1, ..., g_p)$. In the next theorem to formulate sufficient conditions for local strongly isolated solutions of problem (P), we need to define local convexity for a family of functions.

Definition 3.1. *We say that the family of functions $h_J = (h_j)_{j \in J}$ is locally convex at $\bar{x} \in S$ iff there exists a neighborhood U of $\bar{x}$ such that $S \cap U$ is a convex set and $h_j, j \in J$, are convex functions on $S \cap U$.*

Theorem 3.2. *Let $\bar{x} \in \mathbb{F}$ be a feasible solution satisfying (**3**.1) for some $\nu > 0$. Assume that (f, g, h_J) is locally convex at $\bar{x}$. Then $\bar{x} \in \text{loc}E^{i\nu}(\text{P})$.*

Proof. Let $\bar{x} \in \mathbb{F}$ be a feasible solution satisfying (**3**.1) for some $\nu > 0$. As (f, g, h_J) is locally convex at $\bar{x}$, it follows that there exist a neighborhood U of $\bar{x}$ such that $U \cap S$ it is a convex set and f_i, g_i, $i = 1, ..., p$, h_j, $j \in J$, are convex functions on $U \cap S$. Note that, for any $y \in X$, we have

$$\|y\| = \max_{y^* \in B_{X^*}} \langle y^*, y \rangle,$$

and thus there is $y^* \in B_{X^*}$ such that $\|y\| = \langle y^*, y \rangle$.

Now, we arbitrarily choose $x \in U \cap \mathbb{F}$. Then there exists $x^* \in B_{X^*}$ such that

$$\|x - \bar{x}\| = \langle x^*, x - \bar{x} \rangle. \qquad (\mathbf{3}.9)$$

Since $\bar{x} \in \mathbb{F}$ satisfies (**3**.1), there exist $\lambda_i \geq 0, i = 1, ..., p$ with $\sum_{i=1}^{p} \lambda_i = 1, \mu \in \Lambda(\bar{x})$ and $u_i^* \in \partial f_i(\bar{x}),\ v_i^* \in \partial g_i(\bar{x}), i = 1, ..., p,\ \xi_j^* \in \partial h_j(\bar{x}), j \in J$ such that

$$\nu x^* - \left[\sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(u_i^* - \frac{f_i(\bar{x})}{g_i(\bar{x})} v_i^*\right) + \sum_{j \in J} \mu_j \xi_j^*\right] \in N(\bar{x}, S).$$

It follows by (**2**.3) that

$$\nu \langle x^*, x - \bar{x} \rangle - \left[\sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(\langle u_i^*, x - \bar{x} \rangle - \frac{f_i(\bar{x})}{g_i(\bar{x})} \langle v_i^*, x - \bar{x} \rangle\right) + \sum_{j \in J} \mu_j \langle \xi_j^*, x - \bar{x} \rangle\right] \leq 0,$$

which by the local convexity of (f, g, h_J) at $\bar{x}$ and equality (**3**.9), yields

$$\begin{aligned}\nu\|x-\bar{x}\| &\leq \sum_{i=1}^{p}\frac{\lambda_i}{g_i(\bar{x})}\left[f_i(x)-f_i(\bar{x})-\frac{f_i(\bar{x})}{g_i(\bar{x})}\left(g_i(x)-g_i(\bar{x})\right)\right]+\sum_{j\in J}\mu_j(h_j(x)-h_j(\bar{x}))\\ &= \sum_{i=1}^{p}\frac{\lambda_i}{g_i(\bar{x})}\left(f_i(x)-\frac{f_i(\bar{x})}{g_i(\bar{x})}g_i(x)\right)+\sum_{j\in J}\mu_j(h_j(x)-h_j(\bar{x}))\\ &\leq \sum_{i=1}^{p}\frac{\lambda_i}{g_i(\bar{x})}\left(f_i(x)-\frac{f_i(\bar{x})}{g_i(\bar{x})}g_i(x)\right),\end{aligned}$$

Since $\mu_j h_j(\bar{x}) = 0$, and $\mu_j h_j(x) \leq 0$ for all $j \in J$, therefore

$$\begin{aligned}\nu\|x-\bar{x}\| &\leq \sum_{i=1}^{p}\frac{\lambda_i}{g_i(\bar{x})}\left(f_i(x)-\frac{f_i(\bar{x})}{g_i(\bar{x})}g_i(x)\right) \quad \leq \sum_{i=1}^{p}\lambda_i\max_{1\leq i\leq p}\left\{\frac{1}{g_i(\bar{x})}\left(f_i(x)-\frac{f_i(\bar{x})}{g_i(\bar{x})}g_i(x)\right)\right\}\\ &= \max_{1\leq i\leq p}\left\{\frac{g_i(x)}{g_i(\bar{x})}\left(\theta_i(x)-\theta_i(\bar{x})\right)\right\}.\end{aligned}$$

This implies that

$$\nu^{'}\|x-\bar{x}\| \leq \max_{1\leq i\leq p}\left\{(\theta_i(x)-\theta_i(\bar{x}))\right\},$$

where $\nu^{'} = \frac{\nu}{\gamma}$ and $\gamma = \max\left\{\frac{g_i(x)}{g_i(\bar{x})} : 1 \leq i \leq p,\ x \in U \cap \mathbb{F}\right\} > 0$. This shows that $\bar{x} \in \text{loc}E^{i\nu}(\text{P})$ because x was arbitrarily chosen in $U \cap \mathbb{F}$. □

The following example asserts the importance of the local convexity of the objective function (f, g) imposed in the above theorem. Namely, a feasible point $\bar{x}$ satisfying (**3**.1) is not necessarily a local strongly isolated solution of problem (P) if the local convexity of (f, g) at $\bar{x}$ is violated.

Example 3.2. *Let* $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by* $\theta(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}\right)$, *where*

$$f_1(x) = \max\{0, -|x_1|\},\ f_2(x) = -x_1^2 - |x_2|,$$

$$g_1(x) = g_2(x) = x_1^2 + x_2^2 + 1, x = (x_1, x_2) \in \mathbb{R}^2,$$

and let $h_j : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$h_j(x) = j(x_1^2 + x_2^2), x = (x_1, x_2) \in \mathbb{R}^2, j \in J = (-\infty, 0).$$

Let us consider problem (P) *with* $p = 2$, *and* $S = \mathbb{R}^2$. *Then* $\mathbb{F} = S$. *Note that* f_1, f_2, g_1, g_2 *are locally Lipschitz at* $\bar{x} = (\bar{x}_1, \bar{x}_2) = (0, 0) \in \mathbb{F}$, *and* $\partial f_1(\bar{x}) = [-1, 1] \times \{0\}$, $\partial f_2(\bar{x}) = \{0\} \times [-1, 1]$, $\partial g_1(\bar{x}) = \partial g_2(\bar{x}) = \{(0, 0)\}$, $N(\bar{x}, S) = \{(0, 0)\}$. *Thus, we have*

$$\nu B_{X^*} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq \nu^2\}$$

and

$$\left\{\sum_{i=1}^{p}\frac{\lambda_i}{g_i(\bar{x})}\left(\partial f_i(\bar{x})-\frac{f_i(\bar{x})}{g_i(\bar{x})}\partial g_i(\bar{x})\right)+\sum_{j\in J}\mu_j\partial h_j(\bar{x}) : \lambda_i \geq 0, i = 1, ..., p, \right.$$
$$\left. \sum_{i=1}^{p}\lambda_i = 1, \mu \in \Lambda(\bar{x})\right\} + N(\bar{x}, S)$$
$$= \{x \in \mathbb{R}^2 : |x_1 + x_2| \leq 1\},$$

*which implies that (**3**.1) holds for any* $\nu \in (0,1]$*. However,* $\bar{x} \notin \mathrm{loc}E^{i\nu}(\mathrm{P})$*. The reason is that* (f_1, f_2, g_1, g_2) *is not locally convex at* $\bar{x}$*.*

The next theorem provides a necessary condition for local positively properly efficient solutions of problem (P) under the fulfillment of the (LCQ) defined in Definition **2**.6.

Theorem 3.3. *Let the* (LCQ) *be satisfied at* $\bar{x} \in \mathbb{F}$*. If* $\bar{x} \in \mathrm{loc}E^p(\mathrm{P})$*, then there exist* $\lambda = (\lambda_1, ..., \lambda_p) \in \mathrm{int}\mathbb{R}^p_+$ *and* $\mu \in \Lambda(\bar{x})$ *such that*

$$0 \in \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(\partial f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})} \partial g_i(\bar{x}) \right) + \sum_{j \in J} \mu_j \partial h_j(\bar{x}) + N(\bar{x}, S). \tag{3.10}$$

Proof. Let $\bar{x} \in \mathrm{loc}E^p(\mathrm{P})$. Then there exists a neighborhood U of $\bar{x}$ and $\lambda = (\lambda_1, ..., \lambda_p) \in \mathrm{int}\mathbb{R}^p_+$ such that

$$\sum_{i=1}^{p} \lambda_i \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \right] \geq 0, \ \forall x \in U \cap \mathbb{F}.$$

It means that $\bar{x}$ is a local minimizer of the following scalar optimization problem

$$\min_{x \in \mathbb{F}} \phi(x),$$

where

$$\phi(x) = \sum_{i=1}^{p} \lambda_i \frac{f_i(x)}{g_i(x)}. \tag{3.11}$$

Thus $\bar{x}$ is a local minimizer of the following unconstrained optimization problem

$$\min_{x \in X} \phi(x) + \delta(x, \mathbb{F}).$$

By using (**2**.7), the above defined problem can be rewritten as

$$0 \in \widehat{\partial}(\phi + \delta(., \mathbb{F}))(\bar{x}). \tag{3.12}$$

As the function ϕ is Lipschitz continuous around $\bar{x}$ and the function $\delta(., \mathbb{F})$ is lower semicontinuous around this point, it follows from the sum rule (cf. Mordukhovich[22], Theorem 3.36) applied to (**3**.12) and from the relation in (**2**.6) that

$$0 \in \partial\phi(\bar{x}) + \partial\delta(\bar{x}, \mathbb{F}) = \partial\phi(\bar{x}) + N(\bar{x}, \mathbb{F}). \tag{3.13}$$

In addition, from Lemma **2**.2

$$\begin{aligned} \partial\phi(\bar{x}) &= \partial \left(\sum_{i=1}^{p} \lambda_i \frac{f_i}{g_i}(.) \right)(\bar{x}) \\ &\subset \sum_{i=1}^{p} \lambda_i \partial \left(\frac{f_i}{g_i} \right)(\bar{x}), \end{aligned}$$

which by using (**2**.10), yields

$$\begin{aligned} \partial\phi(\bar{x}) &\subset \sum_{i=1}^{p} \lambda_i \frac{\partial(g_i(\bar{x})f_i)(\bar{x}) + \partial(-f_i(\bar{x})g_i)(\bar{x})}{[g_i(\bar{x})]^2} \\ &= \sum_{i=1}^{p} \lambda_i \frac{g_i(\bar{x})\partial f_i(\bar{x}) - f_i(\bar{x})\partial g_i(\bar{x})}{[g_i(\bar{x})]^2}, \end{aligned} \tag{3.14}$$

where the equality holds due to the fact that $-f_i(\bar{x}) \geq 0,\ g_i(\bar{x}) > 0$ for $i = 1, ..., p$.
On the other hand, the (LCQ) being satisfied at $\bar{x}$ entails that

$$N(\bar{x}, \mathbb{F}) \subset \bigcup_{\mu \in \Lambda(\bar{x})} \left[\sum_{j \in J} \mu_j \partial h_j(\bar{x}) \right] + N(\bar{x}, S), \qquad (\mathbf{3}.15)$$

where $\Lambda(\bar{x})$ is the set defined in (**2**.12). It follows from (**3**.13)-(**3**.15) that

$$0 \in \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(\partial f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})} \partial g_i(\bar{x}) \right) + \sum_{j \in J} \mu_j \partial h_j(\bar{x}) + N(\bar{x}, S),$$

for some $\mu \in \Lambda(\bar{x})$, which completes the proof. □

Remark 3.1. *Similarly, as shown in Example* **3**.*1, the conclusion of Theorem* **3**.*3 might fail if the considered* (LCQ) *is not satisfied at the point under consideration.*

To establish sufficient conditions for (global) positively properly efficient solutions of problem (P), in the next theorem, we need concept of generalized convexity for a family of locally Lipschitz functions.

Definition 3.2 (Chuong [6])**.** *We say that (θ, h_J) is generalized convex on S at $\bar{x} \in S$ if for any $x \in S$, $u_i^* \in \partial f_i(\bar{x})$, $v_i^* \in \partial g_i(\bar{x})$, $i = 1, ..., p$, and $\xi_j^* \in \partial h_j(\bar{x})$, $j \in J$ there exists $\omega \in N(\bar{x}, S)^\circ$ such that*

$$f_i(x) - f_i(\bar{x}) \geq \langle u_i^*, \omega \rangle,\ i = 1, ..., p,$$
$$g_i(x) - g_i(\bar{x}) \geq \langle v_i^*, \omega \rangle,\ i = 1, ..., p,$$
$$h_j(x) - h_j(\bar{x}) \geq \langle \xi_j^*, \omega \rangle,\ j \in J.$$

Theorem 3.4. *Let $\bar{x} \in \mathbb{F}$, and let (θ, h_J) be generalized convex on S at $\bar{x}$. If $\bar{x}$ satisfies (**3**.10), then $\bar{x} \in \mathrm{loc}E^p(\mathrm{P})$.*

Proof. Suppose that there exist $\lambda = (\lambda_1, ..., \lambda_p) \in \mathrm{int}\mathbb{R}^p_+$ and $\mu \in \Lambda(\bar{x})$ such that (**3**.10) holds. Then there exist $u_i^* \in \partial f_i(\bar{x})$, $v_i^* \in \partial g_i(\bar{x}), i = 1, ..., p$, and $\xi_j^* \in \partial h_j(\bar{x}), j \in J$ such that

$$-\left[\sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(u_i^* - \frac{f_i(\bar{x})}{g_i(\bar{x})} v_i^* \right) + \sum_{j \in J} \mu_j \xi_j^* \right] \in N(\bar{x}, S),$$

which by the definition of the polar cone and the generalized convexity of (θ, h_J), it follows that for each $x \in S$, there is $\omega \in N(\bar{x}, S)^\circ$ such that

$$\begin{aligned}
0 &\leq \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(\langle u_i^*, \omega \rangle - \frac{f_i(\bar{x})}{g_i(\bar{x})} \langle v_i^*, \omega \rangle \right) + \sum_{j \in J} \mu_j \langle \xi_j^*, \omega \rangle \\
&\leq \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left[f_i(x) - f_i(\bar{x}) - \frac{f_i(\bar{x})}{g_i(\bar{x})} (g_i(x) - g_i(\bar{x})) \right] + \sum_{j \in J} \mu_j (h_j(x) - h_j(\bar{x})) \\
&= \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x) \right) + \sum_{j \in J} \mu_j (h_j(x) - h_j(\bar{x})).
\end{aligned}$$

Hence,

$$0 \leq \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x) \right) + \sum_{j \in J} \mu_j (h_j(x) - h_j(\bar{x})). \qquad (\mathbf{3}.16)$$

In addition, it holds that

$$\sum_{j\in J} \mu_j(h_j(x) - h_j(\bar{x})) \leq 0,$$

due to the fact that $\mu_j h_j(\bar{x}) = 0$, and $\mu_j h_j(x) \leq 0$ for all $j \in J$. So, we get by (**3**.16) that

$$0 \leq \sum_{i=1}^{p} \frac{\lambda_i}{g_i(\bar{x})} \left(f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x) \right).$$

The above inequality is equivalent to the following one

$$\sum_{i=1}^{p} \lambda_i \frac{f_i(\bar{x})}{g_i(\bar{x})} \leq \sum_{i=1}^{p} \lambda_i \frac{f_i(x)}{g_i(x)},$$

which shows that $\bar{x} \in \text{loc}E^p(\text{P})$. The proof is complete. □

A point satisfying (**3**.10) is not necessarily a global positively properly efficient solution of problem (P) even in the smooth case if the generalized convex on S at the reference point of (θ, h_J) has been dropped. It is illustrated by the following simple example.

Example 3.3. *Let* $\theta : \mathbb{R} \to \mathbb{R}^2$ *be defined by* $\theta(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)} \right)$, *where*

$$f_1(x) = f_2(x) = -\arctan|x|,\ g_1(x) = xe^x + 1, g_2(x) = x^2 + 1,\ x \in \mathbb{R},$$

and let $h_j : \mathbb{R} \to \mathbb{R}$ *be given by*

$$h_j(x) = jx^3,\ x \in \mathbb{R},\ j \in J = (-\infty, 0).$$

Consider problem (P) *with* $p = 2$ *and* $S = \mathbb{R}$. *Then* $\mathbb{F} = [0, \infty)$ *and thus,* $\bar{x} = 0 \in \mathbb{F}$. *Observe that* $\bar{x}$ *satisfies (*__3__*.10). However,* $\bar{x} \notin \text{loc}E^p(\text{P})$.

4. Conclusions

In this paper, we have established necessary optimality conditions for (local) strongly isolated solutions and (local) positively properly efficient solutions of a nonsmooth semi-infinite multiobjective fractional optimization problem. Sufficient optimality conditions for the existence of such solutions have also been discussed under the assumptions of (local) convexity or generalized convexity. A dual problem for the primal problem can be presented, and weak and strong duality relations under the generalized convex assumptions can be derived. We will extend the results established in the paper to a larger class of nonsmooth variational and nonsmooth control multiobjective optimization problems. This will orient the future research of the authors.

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