STUDYING SOME FINITE FRAMES

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The finite frame theory is an essential part of frame theory due to its significant relevance in various branches of mathematical applications. Studying controlled finite frames is the goal of the work. To this end, we introduce controlled frames in a finite-dimensional Hilbert space and study some properties of them. The main class of finite frames in frame-applied problems is Parseval frames. By viewpoint to this, a brief discussion about Parseval controlled frames is investigated. Afterward, the paper characterizes all operators that construct controlled finite frames. Furthermore, controlled finite frames are also considered as a proper subset of dual frames by the equivalency relation between frames.

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1. Introduction

Nowadays, frame theory is a crucial field in all branches of science and has diverse and exciting applications in different fields. The importance of frames for signal processing was first revealed in 1952, and their significance has only increased since then. Frames and their duals can be seen as the most natural generalization of the concept of an orthonormal basis. The reconstruction of elements in a Hilbert space is based on a given frame and its duals. In the Parseval case, the dual can be the frame itself. Today, frame theory has an ever-growing number of applications in both applied and pure mathematics. Many of these applications require frames in finite-dimensional spaces. For instance, Jamali et al. and Javanshiri et al. have obtained results that are attractive in applications of frames [13, 14]. Given the importance of finite frame theory, our focus is on the study of controlled finite frames. In [2], the authors showed that controlled frames and classical frames are equivalent.

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Also, they said that the controlled frames are a generalization for considering the frame conditions. In that study, the authors tried to enhance the numerical efficiency of iterative algorithms for inverting the frame operator. The duals of a frame and Equivalent frames have a critical role in application problems.

The manuscript is primarily divided into three sections that introduce and characterize controlled finite frames, particularly Parseval controlled finite frames, as a subset of dual frames.

The remainder of this section provides a review of various notions and properties of operators and frames in Hilbert spaces. Section 2 is dedicated to defining controlled finite frames and examining their properties. It also presents the operators that can be used to construct controlled frames. The Gramian matrix and its properties for controlled frames are also explored. In section 3, a brief discussion is included on Parseval controlled frames. The final section presents the study of controlled frames as a proper subset of dual frames through equivalent frames. Finally, a result concerning controlled Riesz basis in a finite-dimensional Hilbert space is derived.

Now, we recall a brief account of the properties of operators and finite frames in Hilbert spaces; we refer the reader to [7, 11] for further details.

If S and T are two bounded linear operators on a Hilbert space H, both of which are self-adjoint, and satisfy $\langle Tx, x\rangle \geqslant \langle Sx, x\rangle$, for all $x \in H$, we denote this relationship as $T \geqslant S$. An operator T is considered positive if $\langle Tx, x\rangle \geqslant 0$, and strictly positive if $\langle Tx, x\rangle > 0$, for every $x \in H$. It is a well-established fact that every positive operator in a complex Hilbert space is self-adjoint, but this does not hold true in real Hilbert spaces.

Let H^N be an N-dimensional Hilbert space. A sequence $\{f_k\}_{k=1}^M$ in the Hilbert space H^N is considered a frame if the following inequalities are satisfied for some $0 < A \leq B < \infty$:

$$A\|f\|^2 \leqslant \sum_{k=1}^{M} \mid \langle f, f_k\rangle \mid^2 \leqslant B\|f\|^2, \ \forall f \in H^N.$$

In the case where $A = B$, the frame is referred to as tight, and if $A = B = 1$, it is called a Parseval frame.

The frame operator S_F is defined as $S_F f = \sum_{k=1}^M \langle f, f_k\rangle f_k$ for a frame $\{f_k\}_{k=1}^M$ on H^N.

The sequence $\{g_k\}_{k=1}^M$ is defined as a dual for $\{f_k\}_{k=1}^M$ if $f = \sum_{k=1}^M \langle f, f_k\rangle g_k$ for all $f \in H^N$. In the duality relation, the frame $\{g_k\}_{k=1}^M = \{S_F^{-1} f_k\}_{k=1}^M$ is known as the canonical dual frame of $\{f_k\}_{k=1}^M$ and leads to the reconstruction formula.

In [2], controlled frames were introduced and examined. We apply this concept as follows. In this paper, the notations H and H^N are used to denote a Hilbert space and a finite-dimensional Hilbert space, respectively. The set $GL(H)$ denotes the group of all bounded linear operators with a bounded inverse, and $GL^+(H)$ is the set of positive operators in $GL(H)$.

2. Notes on controlled finite frames

By attention to the applications of finite frames, Cassaza and Kutyniok illustrated finite frames and some of their properties in detail [6]. In recent years, there has been an introduction of controlled frames to enhance the numerical efficiency of interactive algorithms for inverting the frame operator and for spherical wavelets [2, 4]. Hence, Considering and studying controlled finite frames is important and remarkable.

2.1. Controlled Finite Frames and Frame Operator

This subsection delves into the exploration of some properties of controlled frames in finite Hilbert spaces.

Definition 2.1. *A sequence $\{f_k\}_{k=1}^M$ in H^N is considered a U-controlled frame for an invertible operator U on H^N if there exist positive constants A and B such that*

$$A\|f\|^2 \leqslant \sum_{k=1}^{M} \langle f, f_k\rangle \langle U f_k, f\rangle \leqslant B\|f\|^2, \ \forall f \in H^N.$$

If $A = B = \lambda$, then $\{f_k\}_{k=1}^M$ is a λ-tight U-controlled frame. For $\lambda = 1$, $\{f_k\}_{k=1}^M$ is referred to as a Parseval U-controlled frame.

Similar to ordinary frames, the controlled frame operator (or frame operator) is defined for a controlled frame on H^N by $S_{UF} f = \sum_{k=1}^{M} \langle f, f_k\rangle U f_k$.

Likewise, just like ordinary frames, the controlled synthesis operator $T_{UF}^* : \ell^2(M) \to H^N$ is defined by $T_{UF}^*(\{\alpha_k\}_{k=1}^M) = \sum_{k=1}^{M} \alpha_k U f_k$. It is evident from the definition of S_{UF} that $S_{UF} = T_{UF}^* T_F$, where T_F is the analysis operator of $\{f_k\}_{k=1}^M$.

Example 2.1. Suppose that $\{(1, 0), (1, 1), (0, 1)\}$ in $\mathbb{R}^2$ and U is the rotation operator $45°$. By the following computing, we see that $\{(1, 0), (1, 1), (0, 1)\}$ is a U-controlled frame with bounds $\frac{1}{2}$ and 3 in $\mathbb{R}^2$. For $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}
&\langle (x, y), (1, 0)\rangle \langle U(1, 0), (x, y)\rangle + \langle (x, y), (1, 1)\rangle \langle U(1, 1), (x, y)\rangle \\
&\ + \langle (x, y), (0, 1)\rangle \langle U(0, 1), (x, y)\rangle \\
=&\langle (x, y), (1, 0)\rangle \langle (1, 1), (x, y)\rangle + \langle (x, y), (1, 1)\rangle \langle (0, 1), (x, y)\rangle \\
&\ + \langle (x, y), (0, 1)\rangle \langle (-1, 1), (x, y)\rangle \\
=&x^2 + y^2 + xy.
\end{aligned}$$

By utilizing the definition of a controlled frame and its frame operator, it can be deduced that S is a positive and invertible operator. Additionally, these properties for an operator frame corresponding to a sequence imply that the sequence is a controlled frame. The following proposition serves to illustrate this result.

Proposition 2.1. *Let $\{f_k\}_{k=1}^M$ be a sequence in H^N and $U \in GL(H^N)$. Then the following statements are equivalent.*

i. $\{f_k\}_{k=1}^M$ *is a U-controlled frame with bounds A and B.*

ii. $S_{UF}(f) = \sum_{k=1}^M \langle f, f_k\rangle U f_k$ *is a positive and invertible operator on H^N.*

Proof. The proof of $i \Rightarrow ii$ is clear. To prove $ii \Rightarrow i$, the given equality results $S_{UF} \leqslant \|S_{UF}\| I$ and so $S_{UF}^{-1} \leqslant \|S_{UF}^{-1}\| I$. Therefore,

$$\|S_{UF}^{-1}\|^{-1} I \leqslant S_{UF} \leqslant \|S_{UF}\| I, \quad \forall f \in H^N.$$

□

Proposition 2.1 and the properties of operators on H^N obtain the following result.

Theorem 2.1. *Let $\{f_k\}_{k=1}^M$ be a frame for H^N with the frame operator S_F. If $U \in GL^+(H^N)$ is a self-adjoint operator such that $US_F = S_F U$, then $\{f_k\}_{k=1}^M$ forms a U-controlled frame.*

In [2], the authors discuss necessary and sufficient conditions for a frame to result in a controlled frame, as well as vice versa. In this work, we focus on the necessity of these conditions for both real and complex Hilbert spaces. Specifically, we will recall Propositions 3.2 and 3.3 from [2], which apply to infinite-dimensional Hilbert spaces.

Proposition 2.2. *[In [2] as Proposition 3.2] Suppose $\{f_k\}_{k\in\mathbb{N}}$ is a $U-$controlled frame for H and $U \in GL(H)$. Then $\{f_k\}_{k\in\mathbb{N}}$ is a classical frame. Furthermore, $US_F = S_F U^*$, which implies*

$$\sum_{k\in\mathbb{N}} \langle f, f_k\rangle U f_k = \sum_{k\in\mathbb{N}} \langle f, U f_k\rangle f_k.$$

Proposition 2.3. *[In [2] as Proposition 3.3] Let $U \in GL(H)$ be self-adjoint. The family $\{f_k\}_{k\in\mathbb{N}}$ is a $U-$controlled frame for H if and only if it is a (classical) frame for H, and U is positive and commutes with the frame operator S_F.*

Here, we invent the readers to the following example.

Example 2.2. The frame $\{f_k\}_{k=1}^5$ for $\mathbb{R}^2$ and the operator U are considered as follow.

$$\{f_k\}_{k=1}^5 = \{(1,0),(1,0),(0,1),(0,1),(0,1)\},\ U = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

Based on the definition of the frame operator, we can derive $S_F = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. It is clear that U is positive and invertible and by the easy computations, it is given that $\{f_k\}_{k=1}^5$ is a U-controlled frame with bounds 1 and 6 whereas $US_F \neq S_F U$ and also $US_F \neq S_F U^*$.

Example 2.2 demonstrates that the conditions "$US_F = S_FU^*$" or "$US_F = S_FU$" are not satisfied in a finite-dimensional real Hilbert space. Specifically, the proof presented for Proposition 2.2 is only valid over complex Hilbert spaces.

Moreover, in Example 2.2, if we set $U = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, then the obtained example shows that the results is Proposition 2.3 are valid but "$US_F \neq S_FU$".

Now, the following proposition demonstrates the same outcome for a real or complex finite-dimensional Hilbert space, even if the conditions "$US_F = S_FU^*$" and "$US_F = S_FU$" are not satisfied.

Proposition 2.4. *If $\{f_k\}_{k=1}^M$ is a $U-$controlled frame for H^N with the frame operator S_{UF}, then $\{f_k\}_{k=1}^M$ is a frame for H^N with the frame operator $U^{-1}S_{UF}$.*

Proof. Firstly, we define the following operator as the frame operator for $\{f_k\}_{k=1}^M$:

$$S_F f = \sum_{k=1}^{M} \langle f, f_k\rangle f_k, \ \forall f \in H^N.$$

The operator S_F is well-defined and $S_F f = U^{-1}S_{UF}f$, so S_F is invertible. Therefore, for $f \in H^N$, we have

$$f = \sum_{k=1}^{M} \langle S_F^{-1} f, f_k\rangle f_k = \sum_{k=1}^{M} \langle f, (S_F^{-1})^* f_k\rangle f_k.$$

This equality shows that $\{f_k\}_{k=1}^M$ is a generator for H^N, and so $\{f_k\}_{k=1}^M$ is a frame for H^N with the frame operator $U^{-1}S_{UF}$. □

Hint. Assuming $\{f_k\}_{k=1}^M$ is a U-controlled frame for H^N, we can conclude that $\{Uf_k\}_{k=1}^M$ is a spanning set for H^N, and for $f \in H^N$, one obtains $f = \sum_{i=1}^M \langle S_{UF}^{-1} f, f_k\rangle U f_k$. Therefore, T_{UF}^* is surjective.

In the subsequent analysis, we aim to characterize U-controlled frames for an N-dimensional Hilbert space by a generator set. The following theorems will assist in determining an operator U that constructs a U-controlled frame.

Theorem 2.2. *Assuming $\{f_k\}_{k=1}^M$ is a U-controlled frame and the controlled frame operator S_{UF} is a normal operator such that $US_{UF} = S_{UF}U$, the operator U is uniquely determined and is positive.*

Proof. By Proposition 2.4, $\{f_k\}_{k=1}^M$ is a frame with the frame operator $S_F = U^{-1}S_{UF}$. Since $S_{UF}U = US_{UF}$, we have $S_FU = US_F$ and also $S_{UF}S_F = US_FS_F = S_FUS_F = S_FS_{UF}$. Now, the operators S_{UF} and S_F are diagonalizable and commute with each other, then there exists a set of common orthonormal eigenvectors of S_{UF} and S_F as $\{e_k\}_{k=1}^N$. Suppose that $\{\lambda_k\}_{k=1}^N$ and $\{\alpha_k\}_{k=1}^N$ are eigenvalues of operators S_{UF} and S_F, respectively. Now, for $k \in \{1, ..., N\}$, the linear operator U is defined by $Ue_k = (S_{UF}S_F^{-1})(e_k) = \alpha_k^{-1}\lambda_k e_k$, and then $Uf = \sum_{k=1}^N \alpha_k^{-1}\lambda_k\langle f, e_k\rangle e_k$on H^N. It is clear that U is a positive operator. □

In the previous theorem, we can substitute the normal condition of S_{UF} with the normal condition for U because if $US_{UF} = S_{UF}U$, then S_{UF} is normal if and only if U is normal. In [2], the authors show that for a given frame, we can construct different controlled frames by different controlled operators; it means that the controlled operators are not necessarily unique (See [2], Proposition 3.3). Here, we have the uniqueness property for controlled operators for some special class of controlled frames by Theorem 2.2 in the following.

Corollary 2.1. *For every Parseval U-controlled frame, the operator U is uniquely determined.*

Proposition 2.5. *Suppose $\{f_k\}_{k=1}^M$ is a frame for H^N with the frame operator S_F. Let $\{e_i\}_{i=1}^N$ and $\{\alpha_i\}_{i=1}^N$ be the set of orthonormal eigenvectors and the set of eigenvalues of S_F, respectively. Then, for any set $\{\lambda_i\}_{i=1}^N \subseteq (0, +\infty)$, $\{f_k\}_{k=1}^M$ is a U-controlled frame, where U is defined as $Ue_i = \lambda_i e_i$ for $i = 1, ..., N$.*

Proof. First, we demonstrate that U commutes with S_F. Assume $f \in H^N$. Then

$$\begin{aligned} US_F f = \sum_{i=1}^N \alpha_i \langle f, e_i \rangle U e_i &= \sum_{i=1}^N \alpha_i \langle f, e_i \rangle \lambda_i e_i \\ &= \sum_{i=1}^N \lambda_i \langle f, e_i \rangle S_F e_i \\ &= S_F (\sum_{i=1}^N \langle f, e_i \rangle U e_i) = S_F U f. \end{aligned}$$

Based on the definition of U, we can conclude that $U = U^*$, and since all eigenvalues of U are positive, this shows that U is positive and invertible. Furthermore, U and S_F commute with each other. So, US_F is invertible and positive that $US_F f = \sum_{i=1}^N \langle f, f_k \rangle U f_k$, on H^N, and it shows that $\{f_k\}_{k=1}^M$ is a U-controlled frame with the frame operator US_F. □

The subsequent area of focus in the study of controlled frames is dedicated to the investigation of operators that preserve controlled frames. The following theorem illustrates that certain invertible operators uphold the controlled frame property.

Theorem 2.3. *Let $\{f_k\}_{k=1}^M$ be a U-controlled frame with frame operator S_{UF} and $T \in GL(H^N)$ such that $TU = UT$. Then $\{Tf_k\}_{k=1}^M$ is a U-controlled frame with frame operator $TS_{UF}T^*$. Moreover, if T is also positive, then $\{T^r f_k\}_{k=1}^M$ is a U-controlled frame for any $r \in \mathbb{R}$ with frame operator $T^r S_{UF} (T^r)^*$.*

Proof. The operator S_{UTF} corresponding to $\{Tf_k\}_{k=1}^M$ is $S_{UTF} = TS_{UF}T^*$ and then S_{UTF} is invertible. Also, S_{UTF} is positive because S_{UF} is positive and

$$\langle TS_{UF}T^* f, f \rangle = \langle S_{UF}T^* f, T^* f \rangle \geqslant 0, \ \forall f \in H^N.$$

Hence $\{Tf_k\}_{k=1}^M$ is a U-controlled frame. It is well-know if T is positive, invertible, and $TU = UT$, then $T^rU = UT^r$, and so according to the first part of theorem $\{T^rf_k\}_{k=1}^M$ is a U-controlled frame with the frame operator $T^rS_{UF}(T^r)^*$. □

Corollary 2.3. *If $\{f_k\}_{k=1}^M$ is a U-controlled frame such that $US_F = S_FU$, then $\{S_F^{\frac{r-1}{2}} f_k\}_{k=1}^M$ is a U-controlled frame for any $r \in \mathbb{R}$, with frame operator US_F^r.*

2.2. Controlled Finite Frames and Grammian Matrix

The concept of the Gramian matrix or Gramian operator for a frame $\{f_k\}_{k=1}^M$ has been introduced in [6]. Furthermore, we have dedicated our efforts to introducing the Gramian operator for a U-controlled frame and examining its properties.

Definition 2.2. *Let $\{f_k\}_{k=1}^M$ be a U-controlled frame with analysis operator T_F and synthesis operator T_{UF}^*. The operator G_{UF} is defined as $G_{UF} = T_FT_{UF}^*$ and is referred to as the U-Gramian operator. The canonical matrix representation of the Gramian operator of a U-controlled frame $\{f_k\}_{k=1}^M$ is obtained as follows.*

$$\begin{bmatrix} \langle Uf_1, f_1\rangle & \langle Uf_2, f_1\rangle & \dots & \langle Uf_M, f_1\rangle \\ \vdots & \vdots & & \vdots \\ \langle Uf_1, f_M\rangle & \langle Uf_2, f_M\rangle & \dots & \langle Uf_M, f_M\rangle \end{bmatrix}_{M\times M}$$

Remark 2.4. If $\{f_k\}_{k=1}^M$ is a Parseval U-controlled frame with the Gramian operator G_{UF}, then G_{UF} is an idempotent operator. This can be seen by using the relation $I = S_{UF} = T_{UF}^*T_F$ and observing that $G_{UF}^2 = T_FT_{UF}^*T_FT_{UF}^* = T_FT_{UF}^* = G_{UF}$.

In [6], the authors demonstrated that the invertibility of the Gramian matrix of a given frame is connected to the number of elements in the primary frame. In the subsequent section, we explore the relationship between the Gramian matrix of a frame and a constructed controlled frame using a primary sequence.

Theorem 2.4. *Let $\{f_k\}_{k=1}^M$ be a U-controlled frame for H^N. Then the following conditions are equivalent:*

(i) *G_{UF} is invertible;*
(ii) *G_F invertible;*
(iii) *$M = N$.*

Proof. In [6], the equivalence of parts ii and iii has already been demonstrated. Now, to establish the equivalence of i and ii, we begin by assuming that G_{UF} is invertible. Then, T_{UF}^* is injective. Additionally, T_{UF}^* is surjective because $\{f_k\}_{k=1}^M$ is a U-controlled frame. Specifically, T_{UF}^* is an invertible operator between $\ell^2(M)$ and H^N, leading to the conclusion that $M = N$.

Now, suppose $M = N$. Proposition 2.4 asserts that the U-controlled frame $\{f_k\}_{k=1}^M$ is a frame, and therefore T_F is injective. Furthermore, the synthesis operator T^*_{UF} of the U-controlled frame $\{f_k\}_{k=1}^M$ is surjective. Therefore, under the assumption $M = N$, the operators T_F and T^*_{UF} are invertible, implying that G_{UF} is also invertible. □

In Theorem 2.3, it is revealed that certain operators can transform U-controlled frames into other U-controlled frames. The subsequent theorem delves into the analysis of the Gramian matrix of these transferred U-controlled frames.

Theorem 2.5. *Let $\{f_k\}_{k=1}^M$ be a U-controlled frame for H^N, and let T be a linear operator that commutes with U. Then T is unitary if and only if the U-Gramian matrix of $\{Tf_k\}_{k=1}^M$ is equal to G_{UF}.*

Proof. Assume T is unitary. Then we have

$$\begin{aligned} G_{U(TF)} = \{\langle UTf_k, Tf_j\rangle\}_{j,k} &= \{\langle TUf_k, Tf_j\rangle\}_{j,k} \\ &= \{\langle Uf_k, f_j\rangle\} = G_{UF}. \end{aligned}$$

Conversely, let $G_{UF} = G_{U(TF)}$. Then the equality $\langle UTf_k, Tf_j\rangle = \langle Uf_k, f_j\rangle$ holds and so

$$\langle T^*UTf_k - Uf_k, f_j\rangle = 0, \ \forall k, j \in \mathbb{N}.$$

We fix $k \in \mathbb{N}$ and the equality $\langle (T^*UT - U)f_k, f_j\rangle = 0$ holds for every $j \in \mathbb{N}$. Since $\{f_k\}_{k=1}^M$ is a complete sequence, we have $(T^*UT - U)f_k = 0$, for every $k \in \mathbb{N}$. Now, this result is obtained for every $f \in H^N$ by the reconstruction formula; it means that $T^*UT = U$. Therefore $T^*T = I$ by the invertibility of U and $UT = TU$. □

Remark 2.5. If $\{f_k\}_{k=1}^M$ is a U-controlled frame for H^N, and λ is a nonzero eigenvalue of S_{UF} with the corresponding eigenvector $f \in H$, then if $\{\alpha_i\}_{i=1}^M := T_F f$, where T_F is the analysis operator of $\{f_k\}_{k=1}^M$, then λ and $\{\alpha_i\}_{i=1}^M$ are the eigenvalue and eigenvector of G_{UF}, respectively. Because

$$G_{UF}(\{\alpha_i\}_{i=1}^M) = T_F T^*_{UF}(T_F f) = T_F(S_{UF} f) = T_F(\lambda f) = \lambda\{\alpha_i\}_{i=1}^M.$$

Please note that if $f \neq 0$, then $\{\alpha_i\}_{i=1}^M \neq 0$ due to the injectivity of T_F. The converse relation also holds. To demonstrate this, assume that λ is a nonzero eigenvalue of G_{UF} corresponding to the eigenvector $\{\alpha_i\}_{i=1}^M \in \ell^2(M)$. Similar to the previous equalities, it can be shown that λ and $f := T^*_{UF}(\{\alpha_i\}_{i=1}^M)$ are the eigenvalue and eigenvector for S_{UF}, respectively.

3. **Parseval controlled frames**

In applied problems, Parseval frames (tight frames) are of great importance compared to other types of frames because they are the closest family to orthonormal bases. The section delves into the study of some properties of Parseval-controlled frames in a finite-dimensional Hilbert space. Now, the concept of a Parseval controlled frame is being studied, followed by the presentation of several properties of this sequence family.

Proposition 3.1. *Let $\{f_k\}_{k=1}^M$ be a Parseval U-controlled frame for H^N. Then*

$$\sum_{k=1}^{M}\langle Uf_k, f_k\rangle = N.$$

Proof. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for H^N. By the hypotheses,

$$e_i = S_{UF}e_i = \sum_{k=1}^{M}\langle e_i, f_k\rangle Uf_k.$$

Then

$$N = \sum_{i=1}^{N}\|e_i\|^2 = \sum_{i=1}^{N}\sum_{k=1}^{M}\langle e_i, f_k\rangle\langle Uf_k, e_i\rangle = \sum_{k=1}^{M}\langle Uf_k, f_k\rangle.$$

□

To investigate operators that maintain the controlled frame property, such as those outlined in Theorem 2.3, orthogonal projections play a crucial role. In a finite-dimensional Hilbert space, orthogonal projections can preserve controlled frames. The following proposition demonstrates this.

Proposition 3.2. *Suppose $\{f_k\}_{k=1}^M$ constitutes a U-controlled frame for H^N, where W is a subspace of H^N and P represents an orthogonal projection of H^N onto W with the property $UP = PU$. Then $\{Pf_k\}_{k=1}^M$ is a U-controlled frame for W. Furthermore, if $\{f_k\}_{k=1}^M$ is a Parseval U-controlled frame for H^N, then $\{Pf_k\}_{k=1}^M$ is a Parseval U-controlled frame for W.*

Proof. Since P is orthogonal projection for every $f \in W$, one obtains that

$$A\|f\|^2 = A\|Pf\|^2 \leqslant \sum_{k=1}^{M}\langle Pf, f_k\rangle\langle Uf_k, Pf\rangle \leqslant B\|Pf\|^2 = B\|f\|^2,$$

By the assumption $UP = PU$, and the above equality, $f \in W$, we have

$$A\|f\|^2 \leqslant \sum_{k=1}^{M}\langle Pf, f_k\rangle\langle PUf_k, f\rangle = \sum_{k=1}^{M}\langle Pf, f_k\rangle\langle UPf_k, f\rangle \leqslant B\|f\|^2,$$

then $\{Pf_k\}_{k=1}^M$ is a U-controlled frame for W. Now, suppose $\{f_k\}_{k=1}^M$ is a Parseval U-controlled frame. Then for every $f \in W$,

$$S_{UPF}(f) = \sum_{k=1}^{M}\langle f, Pf_k\rangle UPf_k = P\sum_{k=1}^{M}\langle Pf, f_k\rangle Uf_k = P^2 f = f.$$

Therefore, it shows that $\{Pf_k\}_{k=1}^M$ is a Parseval U-controlled frame for W. □

Just as with original frames, for every U-controlled frame, there exists a corresponding Parseval controlled frame. The subsequent remark serves to illustrate this concept.

Remark 3.1. If $\{f_k\}_{k=1}^M$ is a U-controlled frame with the controlled frame operator S_{UF}, then $S_{UF} = US_F$ and so one obtains $f = \sum_{k=1}^M \langle f, f_k\rangle (S_{UF}^{-1}U)f$ for every $f \in H^N$. It shows that $\{f_k\}_{k=1}^M$ is a Parseval $S_{UF}^{-1}U$-controlled frame. More precisely, $\{S_{UF}^{-1}Uf_k\}_{k=1}^M = \{S_F^{-1}f_k\}_{k=1}^M$ is the canonical dual of $\{f_k\}_{k=1}^M$.

The following example constructs a Parseval U-controlled frame by a non-Parseval frame.

Example 3.1. *The presented sequence in Example 2.1 is not a Parseval frame because*

$$S_F(x,y) = (x-y, 2(x+y)), \quad \forall (x,y) \in \mathbb{R}^2.$$

Hence, by Remark 3.1, the sequence $\{(1,0),(1,1),(0,1)\}$ *is a Parseval* S_F^{-1}*-controlled frame.*

Now, all of the constructed tight controlled frames can be characterized by a frame. The following theorem serves to illustrate this characterization.

Theorem 3.1. *For a frame* $\{f_k\}_{k=1}^M$ *with the frame operator* S_F*, the collection of all constructed tight controlled frames by* $\{f_k\}_{k=1}^M$ *corresponds to* λS_F^{-1} *for every* $\lambda \in \mathbb{C}$*. This implies that every tight controlled frame* $\{f_k\}_{k=1}^M$ *is precisely a* λ*-tight* λS_F^{-1}*-controlled frame for* $\lambda \in \mathbb{C}$*.*

Proof. Assume that $\{f_k\}_{k=1}^M$ is a λ-tight U-controlled frame, for $\lambda \in \mathbb{C}$. Then for $f \in H^N$, $\lambda f = \sum_{k=1}^M \langle f, f_k\rangle Uf_k$, so $\lambda I = S_{UF} = US_F$, and then $U = \lambda S_F^{-1}$. It shows that $\{f_k\}_{k=1}^M$ is a λ-tight λS_F^{-1}-controlled frame. □

It is a well-known fact that an important property of an operator is its trace. In operator theory, the trace of an operator is typically determined by the orthogonal bases of the framework spaces. However, it can also be computed using frames and Parseval controlled frames. Subsequently, the focus is on investigating the trace properties of an operator. To do so, it is necessary to briefly recall the properties of the trace of linear operators on H^N and then consider the trace of an operator using controlled frames. The trace of a linear operator $T \in L(H^N)$ is defined as follows:

$$Tr(T) = \sum_{j=1}^{N} \langle Te_j, e_j\rangle,$$

where $\{e_j\}_{j=1}^N$ is an orthonormal basis for H^N. In [8], Coope demonstrated that if T_1 and T_2 are self-adjoint positive operators, then $0 \leqslant Tr(T_1T_2) \leqslant Tr(T_1).Tr(T_2)$.

Now, we are prepared to examine some propositions regarding the trace of operator controlled frames.

Proposition 3.3. *Let* $\{f_k\}_{k=1}^M$ *be a* U*-controlled frame such that* $U \in GL^+(H^N)$ *be a self-adjoint operator. Then*

$$Tr(S_{UF}) \leqslant Tr(U)\sum_{i=1}^{M}\|f_i\|^2.$$

Proof. Suppose $\{\lambda_i\}_{i=1}^N$ is the set of eigenvalues of the operator frame S_F. By Theorem 1.6 [6], one concludes that

$$Tr(S_{U_F}) = Tr(US_F) \leqslant Tr(U)Tr(S_F) = Tr(U)\sum_{i=1}^{N}\lambda_i = Tr(U)\sum_{i=1}^{M}\|f_i\|^2.$$

□

In a specific scenario, the trace of an operator can be accurately computed using Parseval controlled frames. This concept is demonstrated in the following proposition.

Proposition 3.4. *Let* $\{f_k\}_{k=1}^M$ *be a Parseval* U*-controlled frame for* H^N *and* F *be a linear operator on* H^N*. Then* $Tr(F) = \sum_{k=1}^M \langle FUf_k, f_k\rangle$.

Proof. For an orthonormal basis $\{e_j\}_{j=1}^N$, $Tr(F) = \sum_{j=1}^N \langle Fe_j, e_j\rangle$, and for $j \in \{1, 2, ...N\}$, we have $Fe_j = \sum_{k=1}^M \langle Fe_j, f_k\rangle Uf_k$, by $\{f_k\}_{k=1}^M$ is a Parseval U-controlled frame. Then

$$\begin{aligned}
Tr(F) &= \sum_{j=1}^{N}\langle \sum_{k=1}^{M}\langle Fe_j, f_k\rangle Uf_k, e_j\rangle \\
&= \sum_{j=1}^{N}\sum_{k=1}^{M}\langle e_j, F^* f_k\rangle\langle Uf_k, e_j\rangle \\
&= \sum_{k=1}^{M}\langle \sum_{j=1}^{N}\langle Uf_k, e_j\rangle e_j, F^* f_k\rangle \\
&= \sum_{k=1}^{M}\langle Uf_k, F^* f_k\rangle = \sum_{k=1}^{M}\langle FUf_k, f_k\rangle.
\end{aligned}$$

□

4. **Dual frames and controlled frames**

The key property of frames is the frame decomposition in the applied problems. Dual frames play a crucial and practical role. With attention to the relation between dual frames and controlled frames, also controlled frames are important. The key advantage of using controlled frames and induced dual frames lies in the flexibility they provide for signal reconstruction: The control operator U can be designed to introduce redundancy in a controlled and structured way. Think of U as a linear transformation that maps the signal into a higher-dimensional space before the frame analysis takes place. This redundancy is crucial for robust signal reconstruction in the presence of noise or erasures., for example, in compressed sensing, resilience to local distortions or missing data [3, 16]. To delve deeper into the study of controlled frames, we utilize dual frames and the equivalence relation between frames. Initially, we review the definition of equivalent frames, and then, using the presented relation, we investigate controlled frames as a subset of the duals of a given frame.

A frame $\{g_k\}_{k=1}^M$ is considered equivalent to a frame $\{f_k\}_{k=1}^M$ if there exists an invertible operator $\Lambda \in B(H^N)$ such that $g_k = \Lambda f_k$ for every $f \in H^N$.

Remark 4.1. Every λ-tight U-controlled frame $\{f_k\}_{k=1}^M$ induces a Parseval controlled frame because for $f \in H^N$, we have

$$\lambda f = \sum_{k=1}^{M} \langle f, f_k\rangle U f_k, \implies f = \sum_{k=1}^{M} \langle f, f_k\rangle (\lambda^{-1}U) f_k.$$

This shows that $\{f_k\}_{k=1}^M$ is a Parseval $(\lambda^{-1}U)$-controlled frame and is equivalent to $\{(\lambda^{-1}U)f_k\}_{k=1}^M$ and also $\{(\lambda^{-1}U)f_k\}_{k=1}^M$ is a dual for $\{f_k\}_{k=1}^M$. Then every λ-tight U-controlled frame of $\{f_k\}_{k=1}^M$ induces a dual frame for $\{f_k\}_{k=1}^M$ such that it is equivalent to $\{f_k\}_{k=1}^M$.

The above result applies to every controlled frame of $\{f_k\}_{k=1}^M$. This is illustrated by the following theorem.

Theorem 4.1. *Let $\{f_k\}_{k=1}^M$ be a U-controlled frame. Then $\{f_k\}_{k=1}^M$ has a dual frame that is equivalent to $\{f_k\}_{k=1}^M$.*

Proof. Assume S_{UF} is the frame operator of $\{f_k\}_{k=1}^M$. Then for $f \in H^N$, one obtains that $S_{UF} f = \sum_{k=1}^M \langle f, f_k\rangle U f_k$, and also $f = \sum_{k=1}^M \langle f, f_k\rangle (S_{UF}^{-1}U) f_k$. The aforementioned equality indicates that $\{f_k\}_{k=1}^M$ is a Parseval controlled frame, and the frame $\{S_{UF}^{-1}U f_k\}_{k=1}^M$ is a dual frame for $\{f_k\}_{k=1}^M$, which is equivalent to $\{f_k\}_{k=1}^M$. □

By the assumptions in Theorem 4.1, we assert that $\{S_{UF}^{-1}U f_k\}_{k=1}^M$ is induced dual by the U-controlled frame $\{f_k\}_{k=1}^M$.

Now, the question arises: "Is every dual of $\{f_k\}_{k=1}^M$ induced as a controlled frame for $\{f_k\}_{k=1}^M$?" This means that if $\{g_k\}_{k=1}^M$ is a dual of $\{f_k\}_{k=1}^M$, then "Does there exist an invertible operator $\Lambda \in B(H^N)$ such that $g_k = \Lambda f_k$ for every $k \in \{1, ..., M\}$ and $\{f_k\}_{k=1}^M$ is a Λ-controlled frame?"

In general, the answer to this question is not true. In the following example, we consider this statement.

Example 4.2. Consider two sequences $\{f_k\}_{k=1}^3$ and $\{g_k\}_{k=1}^3$ in $\mathbb{R}^2$ as follows.

$$\{f_k\}_{k=1}^3 = \{(1,0),(1,0),(1,-1)\}, \quad \textit{and} \quad \{g_k\}_{k=1}^3 = \{(0,0),(1,1),(0,-1)\}.$$

$\{g_k\}_{k=1}^3$ is a dual for $\{f_k\}_{k=1}^3$. However, there does not exist an invertible operator $U \in B(H^N)$ such that $g_k = Uf_k$ for $k \in \{1, 2, 3\}$, and $\{g_k\}_{k=1}^3$ cannot induce a controlled frame of $\{f_k\}_{k=1}^3$.

We have observed that not every dual of a given frame can induce a controlled frame by the primary frame. The following proposition presents some duals of a given frame with this property.

Proposition 4.1. *If $\{g_k\}_{k=1}^M$ is a dual of $\{f_k\}_{k=1}^M$ and is equivalent to $\{f_k\}_{k=1}^M$, then $\{g_k\}_{k=1}^M$ induces a Parseval controlled frame of $\{f_k\}_{k=1}^M$.*

Proof. Let $\{g_k\}_{k=1}^M$ be a dual of $\{f_k\}_{k=1}^M$ such that is equivalent to $\{f_k\}_{k=1}^M$. Then there exists the invertible operator U such that $g_k = Uf_k$ for every $k \in \{1, ..., M\}$ and $f = \sum_{k=1}^M \langle f, f_k\rangle Uf_k$ for $f \in H^N$. This relation shows that $\{f_k\}_{k=1}^M$ is a Parseval U-controlled frame. □

Next, we present a sample of a dual $\{g_k\}_{k=1}^M$ as in the previous proposition.

Example 4.3. Let $\{f_k\}_{k=1}^M$ be a frame with frame operator S. The canonical dual $\{S^{-1}f_k\}_{k=1}^M$ of $\{f_k\}_{k=1}^M$ is equivalent to $\{f_k\}_{k=1}^M$ and by the reconstruction formula, $\{f_k\}_{k=1}^M$ is a Parseval S^{-1}-controlled frame.

Finally, we present two results for controlled Riesz bases in a finite-dimensional Hilbert space. To see this, we first briefly discuss Riesz bases for a finite-dimensional Hilbert space.

Remark 4.4. If $\{f_k\}_{k=1}^M$ is a Riesz basis for H^N, then there exists $\Lambda \in GL(H^N)$ such that $\Lambda e_k = f_k,\ k \in \{1, ..., M\}$ for some orthonormal basis $\{e_k\}_{k=1}^N$, and so $M = N$. Also, every Riesz basis $\{f_k\}_{k=1}^N$ has the following properties.

(i) $\{f_k\}_{k=1}^N$ is a generator for H^N.
(ii) $\{f_k\}_{k=1}^N$ is linearly independent.

Then $\{f_k\}_{k=1}^N$ is a Schauder basis for H^N. But it cannot be concluded that $\{f_k\}_{k=1}^M$ is an orthonormal basis. A counterexample for this is the Schauder basis $\{(1, \frac{1}{2}), (\frac{1}{2}, 1)\}$ for $\mathbb{R}^2$, which is not an orthogonal set.

Finally, a result about U-controlled Riesz bases is obtained. Similar to the definition of controlled frames, U-controlled Riesz bases can be defined. If for a U-controlled frame $\{f_k\}_{k=1}^M$, the frame $\{f_k\}_{k=1}^M$ is a Riesz basis, then we say that $\{f_k\}_{k=1}^M$ is a U-controlled Riesz basis. Now, we are ready to present the final corollary.

Corollary 4.5. *Suppose $\{e_k\}_{k=1}^N$ is an orthonormal basis for H^N, and let V and U be invertible operators on H^N. Then the following statements hold true.*

(i) *For every nonzero scalar λ, the sequence $\{Ve_k\}_{k=1}^N$ forms a λ^2-tight $\lambda(VV^*)^{-1}$-controlled Riesz basis for H^N.*

(ii) *If $UV = VU$, then $\{Ve_k\}_{k=1}^N$ is a $\lambda(UU^*)$-controlled Riesz basis for H^N.*

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