SOME GENERALIZATION OF NON-UNIQUE FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN b-METRIC SPACES

Seddik Merdaci[1], Taieb Hamaizia[2], Abdelkrim Aliouche[3]

In this paper, we prove common fixed point theorem for multi-valued generalized contractive mappings in b-metric spaces. Our result extend and generalize some existing results in literature.

Keywords: Common fixed points, Multi-valued mappings, Contraction, b-metric spaces.

MSC2010: 35B40, 74F05, 74F20, 93D15, 93D20.

1. Introduction

In 1969, Nadler [23] introduced the notion of a multi-valued contractive mapping in complete metric space and also proved Banach's fixed point theorem for a multi-valued mapping in a complete metric space.
On the other hand, In the year 1989, Bakhtin [5] and Czerwik [9], introduced b-metric space as a sharp generalization of metric space and proved analogue of Banach contraction principle in b-metric space.
Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (we may refer to [3, 6, 8, 16, 17, 18, 19, 20, 21, 22, 25, 26, 27]). In this paper, we obtain some common fixed point theorem for multi-valued maps on complete b-metric space. In addition, the theorem can be regarded as a generalization of some previous results on complete metric spaces. An example is given to illustrate our main results.

2. Preliminaries

Before going towards our findings, we need the following definitions and lemma.

Definition 2.1. [5, 9] *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if for all $x, y, z \in X$ the following conditions hold:*

(1) $d(x, y) = 0$ *if and only if* $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

[2] Applied Mathematics Laboratory, Department of Mathematics, University of Kasdi Merbah, Ouargla, 3000, Algeria, e-mail: merdaciseddik@yahoo.fr

[2] Laboratory of dynamical systems and control, Department of Mathematics and Informatics, Larbi Ben M'Hidi University, Oum-El-Bouaghi, 04000, Algeria, e-mail: tayeb042000@yahoo.fr

[3] Laboratory of dynamical systems and control, Department of Mathematics and Informatics, Larbi Ben M'Hidi University, Oum-El-Bouaghi, 04000, Algeria.

Example 2.1. [12] *The space* l_p,($0 < p < 1$),

$$l_p = \left\{ x_n \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

together with the function

$$d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = x_n, y = y_n \in l_p$, *is a b-metric space with coefficient* $s = 2^{\frac{1}{p}}$. *By an elementary calculation we obtain that*

$$d(x,z) \leq 2^{\frac{1}{p}} \left[d(x,y) + d(y,z) \right].$$

Example 2.2. [11] *Let* $X = \mathbb{N} \cup \{\infty\}$. *We define a mapping* $d : X \times X \longrightarrow \mathbb{R}^+$ *as follows:*

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty \\ 5 & \text{if one of } m,n \text{ is odd and the other is odd or } \infty \\ 2 & \text{otherwise } m = n. \end{cases}$$

Then (X,d) *is a b-metric space with coefficient* $s = \frac{5}{2}$.

There are various examples of b-metrics which could be found in [11, 14, 15]).

Definition 2.2. [11] *Let* (X,d) *be a b-metric space and* $\{x_n\}$ *a sequence in* X. *We say that :*

1. $\{x_n\}$ *converges to* x *if* $d(x_n, x) \to 0$, *as* $n \to +\infty$,
2. $\{x_n\}$ *is Cauchy sequence if* $d(x_n, x_m) \to 0$, *as* $n, m \to +\infty$,
3. (X,d) *is complete if every Cauchy sequence in* X *is convergent.*

Let (X,d) be a complete b-metric space. In the sequel, we use the following notations (see [10]):
$CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$,
$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$,
$\delta(A,B) = \sup\{d(a,B) : a \in A\}$,
$\delta(B,A) = \sup\{d(b,A) : b \in B\}$,
$H(A,B) = \max\{\delta(A,B), \delta(B,A)\}$.
Notice that H is called the Hausdorff metric induced by the metric d.
Forward, we denote by $F(T)$ the set of all fixed points of a multi-valued mapping T, that is,

$$F(T) = \{p \in X : p \in Tp\}.$$

Definition 2.3. *A point of* $x_0 \in X$ *is said to be a fixed point of the multi-valued mappings* $T : X \longrightarrow CB(X)$ *if* $x_0 \in Tx_0$.

Lemma 2.1. *Let* (X,d) *be a complete b-metric space. For any* $A, B, C \in CB(X)$ *and any* $x, y \in X$, *one has the following:*

1. $d(x,B) \leq d(x,b)$, *for any* $b \in B$.
2. $\delta(A,B) \leq H(A,B)$.
3. $d(x,B) \leq H(A,B)$, *for any* $x \in A$.
4. $H(A,A) = 0$.
5. $H(A,B) = H(B,A)$.
6. $H(A,C) \leq s[H(A,B) + H(B,C)]$.
7. $d(x,A) \leq s[d(x,y) + d(y,A)]$.

(8) $D(A,B) \leq \delta(A,B)$.

Lemma 2.2. [11] *Let (X,d) be a complete b-metric space and let $\{x_n\}$ be a sequence in X such that*

$$d(x_{n+1},x_{n+2}) \leq \lambda d(x_n,x_{n+1}), \text{ foe all } n = 0,1,2,\ldots$$

where $0 \leq \lambda < 1$. Then $\{x_n\}$ is a Cauchy sequence in X.

3. Mains results

Before proving our main results, we need the following Lemma:

Lemma 3.1. *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$, α nonnegative reel number, and $S,T : X \to CB(X)$ be multi-valued maps satisfying, for all $x,y \in X$*

$$s^\alpha \delta(Sx,Ty) \leq N(x,y)M(x,y), \tag{1}$$

where

$$N(x,y) = \frac{\max\{d(x,y), D(x,Sx)+D(y,Ty), D(x,Ty)+D(y,Sx)\}}{\delta(x,Sx)+\delta(y,Ty)+1}, \tag{2}$$

and

$$M(x,y) = \max\left\{d(x,y), D(x,Sx), D(y,Ty), \frac{D(x,Ty)+D(y,Sx)}{2s}\right\}. \tag{3}$$

Then every fixed point of S is a fixed point of T, and conversely.

Proof. Suppose that p is a fixed point of S. Using (1) and the definition of δ,

$$D(p,Tp) \leq \delta(p,Tp) \leq \delta(Sp,Tp) \leq \frac{1}{s^\alpha} N(p,p)M(p,p). \tag{4}$$

Where,

$$\begin{aligned} N(p,p) &= \frac{\max\{d(p,p), D(p,Sp)+D(p,Tp), D(p,Tp)+D(p,Sp)\}}{\delta(p,Sp)+\delta(p,Tp)+1} \\ &\leq \frac{D(p,Tp)}{D(p,Tp)+1} = \beta < 1, \end{aligned}$$

and,

$$\begin{aligned} M(p,p) &= \max\left\{d(p,p), D(p,Sp), D(p,Tp), \frac{D(p,Tp)+D(p,Sp)}{2s}\right\} \\ &\leq D(p,Tp). \end{aligned}$$

From 4

$$D(p,Tp) \leq \frac{\beta}{s^\alpha} D(p,Tp),$$

since $\dfrac{\beta}{s^\alpha} < 1$, which implies that p is also a fixed point of T.

In a similar manner it can be shown that, if $p \in Tp$, then $p \in Sp$.

□

Now, we prove the main result in this section.

Theorem 3.1. *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$, α nonnegative reel number, and $S,T : X \to CB(X)$ be multi-valued maps satisfying (1), (2) and (3). Then*
(a) S and T have at least one common fixed point $p \in X$.
(b) For n even, $\{(ST)^{n/2}x\}$ and $\{T(ST)^{n/2}x\}$ converge to a common fixed point for each

$x \in X$.

(c) If p and q are distinct common fixed points of S and T, then

$$\frac{s^{\alpha}}{2} \leq d(p,q).$$

Proof. Part (a), let $x_0 \in X, x_1 \in Sx_0$ and define $\{x_n\}$ by

$$x_{2n+1} \in Sx_{2n},\ x_{2n+2} \in Tx_{2n+1},\ \text{for all } n \geq 0. \tag{5}$$

Without loss of generality, we may assume that $\delta(Sx_{2n}, Tx_{2n-1}) \neq 0$ and $\delta(Sx_{2n}, Tx_{2n+1}) \neq 0$ for each n. For, if there exist an n such that $\delta(Sx_{2n}, Tx_{2n-1}) = 0$, then $Sx_{2n} = Tx_{2n-1}$, which implies that $x_{2n} \in Sx_{2n}$, since $x_{2n} \in Tx_{2n-1}$, and x_{2n} is a fixed point of S, hence of T by Lemma 3.1. Similar remarks apply if there exists an n for which $\delta(Sx_{2n}, Tx_{2n+1}) = 0$. We may also assume that $x_n \neq x_{n+1}$ for each n. For, if there exists an n for which $x_{2n} \neq x_{2n+1}$, then, since $x_{2n+1} \in Sx_{2n}, x_{2n+1} \in F(S)$, and by Lemma 3.1, $x_{2n} \in F(T)$. Similarly, $x_{2n+1} = x_{2n+2}$ for any n implies that $x_{2n+1} \in F(T) \cap F(S)$.

First we to show that $\{x_n\}$ is a Cauchy sequence in X. For this, consider

$$d(x_{2n+1}, x_{2n}) \leq \delta(Sx_{2n}, Tx_{2n-1}). \tag{6}$$

Note that $d_{2n} = d(x_{2n+1}, x_{2n})$.

From (2)

$$\begin{aligned} N(x_{2n}, x_{2n-1}) &\leq \frac{\max\{d_{2n-1}, d_{2n} + d_{2n-1}, 0 + d(x_{2n-1}, x_{2n+1})\}}{d_{2n} + d_{2n-1} + 1} \\ &\leq \frac{\max\{d_{2n-1}, d_{2n} + d_{2n-1}, s\,[d_{2n-1} + d_{2n}]\}}{d_{2n} + d_{2n-1} + 1} \\ &= s\frac{d_{2n-1} + d_{2n}}{d_{2n-1} + d_{2n} + 1} = s\beta_{2n}, \end{aligned} \tag{7}$$

where $\beta_{2n} = \dfrac{d_{2n-1} + d_{2n}}{d_{2n-1} + d_{2n} + 1}$.

From (3)

$$\begin{aligned} M(x_{2n}, x_{2n-1}) &\leq \max\left\{d_{2n-1}, d_{2n}, d_{2n-1}, \frac{0 + d(x_{2n-1}, x_{2n+1})}{2s}\right\} \\ &\leq \max\left\{d_{2n-1}, d_{2n}, \frac{d_{2n-1} + d_{2n}}{2}\right\} \\ &= \max\{d_{2n-1}, d_{2n}\}. \end{aligned} \tag{8}$$

Using (1), (7) and (8) in (6) yields

$$d_{2n} \leq \delta(Sx_{2n}, Tx_{2n-1}) \leq \frac{\beta_{2n}}{s^{\alpha-1}} \max\{d_{2n-1}, d_{2n}\}.$$

Since each $x_n \neq x_{n+1}, d_{2n} > 0$, the above inequality implies that

$$d_{2n} \leq \frac{\beta_{2n}}{s^{\alpha-1}} d_{2n-1}. \tag{9}$$

A similar computation verifies that

$$d_{2n+1} \leq \frac{\beta_{2n+1}}{s^{\alpha-1}} d_{2n}. \tag{10}$$

From inequalities (9) and (10), for all $n > 0$,

$$d_{n+1} \leq \frac{\beta_{n+1}}{s^{\alpha-1}} d_n. \tag{11}$$

Let $\lambda = \dfrac{\beta_{n+1}}{s^{\alpha-1}}$. Then, we have that $\lambda \in [0,1)$. Hence, by Lemma 2.2, we obtain that $\{x_n\}$ is a Cauchy sequence in (X,d). By completeness of (X,d), there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$.
Next, to show that p is a fixed point of T. For this, using triangular inequality, we have

$$\begin{aligned} D(p,Tp) &\le s[d(p,x_{2n+1}) + D(x_{2n+1},Tp)] \\ &\le s[d(p,x_{2n+1}) + \delta(Sx_{2n},Tp)]. \end{aligned} \tag{12}$$

Using (2),

$$\begin{aligned} N(x_{2n},p) &= \frac{\max\{d(x_{2n},p), D(x_{2n},Sx_{2n}) + D(p,Tp), D(x_{2n},Tp) + D(p,Sx_{2n})\}}{\delta(x_{2n},Sx_{2n}) + \delta(p,Tp) + 1} \\ &\le \frac{\max\{d(x_{2n},p), d(x_{2n},x_{2n+1}) + d(p,Tp), d(x_{2n},Tp) + d(p,x_{2n+1})\}}{d(x_{2n},x_{2n+1}) + d(p,Tp) + 1}. \end{aligned} \tag{13}$$

From (3),

$$\begin{aligned} M(x_{2n},p) &= \max\left\{d(x_{2n},p), D(x_{2n},Sx_{2n}), D(p,Tp), \frac{D(x_{2n},Tp) + D(p,Sx_{2n})}{2s}\right\} \\ &\le \max\left\{d(x_{2n},p), d(x_{2n},x_{2n+1}), D(p,Tp), \frac{d(x_{2n},Tp) + d(p,x_{2n+1})}{2s}\right\}. \end{aligned} \tag{14}$$

Substituting (13) and (14) into (12), using (1), and taking the limit of both sides as $n \longrightarrow \infty$, one obtains

$$D(p,Tp) \le \frac{1}{s^{\alpha-1}} \frac{d(p,Tp)}{d(p,Tp)+1} D(p,Tp),$$

since $\dfrac{1}{s^{\alpha-1}} \dfrac{d(p,Tp)}{d(p,Tp)+1} < 1$, which implies that $D(p,Tp) = 0$, and hence that $p \in F(T)$. From Lemma 3.1, $p \in F(S)$.
To prove (b), merely observe that, from (5) and the fact that x_0 is arbitrary, we may write.

$$x_{n+1} \in (ST)^{n/2}x \text{ and } x_{n+2} \in T(ST)^{n/2}x.$$

(c) Suppose that p and q are distinct common fixed points of S and T.
Then

$$d(p,q) \le \delta(Sp,Tq). \tag{15}$$

Using (2),

$$\begin{aligned} N(p,q) &= \frac{\max\{d(p,q), 0, D(p,Tq) + D(q,Sp)\}}{\delta(p,Sp) + \delta(q,Tq) + 1} \\ &\le \frac{\max\{d(p,q), d(p,q) + d(q,p)\}}{d(p,Sp) + d(q,Tq) + 1} \\ &= 2d(p,q). \end{aligned}$$

Using (3),

$$\begin{aligned} M(p,q) &= \max\left\{d(p,q), 0, 0, \frac{D(p,Tq) + D(q,Sp)}{2s}\right\} \\ &= d(p,q). \end{aligned}$$

Using (1) and substituting it into (15) gives

$$d(p,q) \le \frac{2}{s^{\alpha}} d^2(p,q).$$

which yields the result. □

The following corollaries can be deduced as particular cases of the main theorem.

Corollary 3.1. *Let (X,d) be a complete b-metric space with a coefficient $s \geq 1$, α nonnegative reel number, and $T : X \to CB(X)$ be a multivalued map satisfying for all $x, y \in X$*

$$s^{\alpha}\delta(Tx,Ty) \leq N(x,y)M(x,y), \tag{16}$$

where

$$N(x,y) = \frac{\max\{d(x,y), D(x,Tx) + D(y,Ty), D(x,Ty) + D(y,Tx)\}}{\delta(x,Tx) + \delta(y,Ty) + 1}, \tag{17}$$

and

$$M(x,y) = \max\left\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2s}\right\}. \tag{18}$$

Then
(a) T has at least one fixed point.
(b) $\{T^n x\}$ converge to a fixed point of T.
(c) If p and q are distinct fixed points of T, then

$$\frac{s^{\alpha}}{2} \leq d(p,q).$$

Proof. Take $S = T$ in Theorem 3.1. □

Corollary 3.2. *Let (X,d) be a complete b-metric space and let T be a selfmap of X satisfying*

$$s^{\alpha}d(Tx,Ty) \leq \left(\frac{d(x,Ty) + d(y,Tx)}{d(x,Tx) + d(y,Ty) + 1}\right) d(x,y), \tag{19}$$

for all $x, y \in X$. Then
(a) T has at least one fixed point.
(b) $\{T^n x\}$ converge to a fixed point of T.
(c) If p and q are distinct fixed points of T, then $\frac{s^{\alpha}}{2} \leq d(p,q)$.

Proof. Take $S = T$ in (1), $N(x,y) = \dfrac{d(x,Ty) + d(y,Tx)}{d(x,Tx) + d(y,Ty) + 1}$ in (2) and $M(x,y) = d(x,y)$ in (3), from Theorem 3.1. □

Remark 3.1. *By choosing :*
$s = 1$ in Theorem 3.1, we get Theorem 2.6 and 2.1 of [24].
$s = 1$ in Corollary 3.2, we get Theorem 1 of [18].

Example 3.1. *Let $X = \{0, \frac{1}{2}, 1\}$ and let $d : X \longrightarrow \mathbb{R}^+$ defined by*

$$d(0,\frac{1}{2}) = 1, \quad d(0,1) = 10, \quad d(1,\frac{1}{2}) = 8,$$

$$d(0,0) = d(\frac{1}{2},\frac{1}{2}) = d(1,1) = 0,$$

$$d(x,y) = d(y,x), \quad \text{for all} \quad x,y \in X.$$

(X,d) is a complete b-metric space with coefficient $s = \dfrac{10}{9}$, and $\alpha = 1$. Let $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 0, & x = 0,1 \\ \frac{1}{2}, & x = \frac{1}{2} \end{cases}$$

Then, we have the following cases:
- *When* $x = 0$ *and* $y = \frac{1}{2}$ *then,*

$$\begin{aligned} d(T0, T\frac{1}{2}) &= d(0, \frac{1}{2}) = 1 \\ &\leq \frac{9}{10}\left(\frac{d(0, T\frac{1}{2}) + d(\frac{1}{2}, T0)}{d(0, T0) + d(\frac{1}{2}, T\frac{1}{2}) + 1}\right) d(0, \frac{1}{2}) \\ &= \frac{9}{5}. \end{aligned}$$

- *When* $x = 1$ *and* $y = \frac{1}{2}$ *then,*

$$\begin{aligned} d(T1, T\frac{1}{2}) &= d(0, \frac{1}{2}) = 1 \\ &\leq \frac{9}{10}\left(\frac{d(1, T\frac{1}{2}) + d(\frac{1}{2}, T1)}{d(1, T1) + d(\frac{1}{2}, T\frac{1}{2}) + 1}\right) d(1, \frac{1}{2}) \\ &= \frac{324}{55}. \end{aligned}$$

- *When* $x = 0$ *and* $y = 1$ *then,*

$$\begin{aligned} d(T0, T1) &= d(0, 0) = 0 \\ &\leq \frac{9}{10}\left(\frac{d(0, T1) + d(1, T0)}{d(0, T0) + d(1, T1) + 1}\right) d(0, 1) \\ &= \frac{90}{11}. \end{aligned}$$

- *When* $x = \frac{1}{2}$ *and* $y = 0$ *then,*

$$\begin{aligned} d(T\frac{1}{2}, T0) &= d(\frac{1}{2}, 0) = 1 \\ &\leq \frac{9}{10}\left(\frac{d(\frac{1}{2}, T0) + d(0, T\frac{1}{2})}{d(\frac{1}{2}, T\frac{1}{2}) + d(0, T0) + 1}\right) d(\frac{1}{2}, 0) \\ &= \frac{81}{10}. \end{aligned}$$

Thus all the cases are verified. Moreover, it can be shown that T *satisfies all the conditions of the corollary 3.2. Then* T *has two distinct fixed points* $\{0, \frac{1}{2}\}$ *and* $\frac{5}{9} \leq d(0, \frac{1}{2}) = 1$.

4. Conclusions

In this work, we introduced and studied a new generalized contraction for a pair of multivalued mappings in complete b-metric spaces. Based on this a new contraction, some exciting fixed and common fixed point results were obtained. Our results are modifications and improvements for many existing results in the literature. Finally, we show the novelty of our work by setting up an example.

Acknowledgement: The authors thank the reviewers and the editor for their valuable remarks and comments which have improved the quality of the paper.

REFERENCES

[1] *M. Abbas, N. Hussain and B. E. Rhoades*, Coincidence point theorems for multivalued f-weak contraction mappings and applications, RACSAM., **105**(2011), No. 2, 261-272.
[2] *A. A. N. Abdou*, Common fixed point results for multi-valued mappings with some examples, J. Nonlinear Sci. Appl., **9**(2016), 787-798.
[3] *MU. Ali, T. Kamran and M. Postolache*, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem. Nonlinear Anal. Modelling Control, **22**(2017), No. 1, 17-30.
[4] *H. Aydi, M. Abbas and C. Vector*, Partial Hausdorff mertic and Nadler's fixed point theorem on partial metric spaces, Topol. Appl., **159**(2012), 3234-3242.
[5] *I.A. Bakhtin*, The contraction mapping principle in quasimetric spaces (Russian), Func. An. Gos. Ped. Inst. Unianowsk., **30**(1989), 26-37.
[6] *M. Boricean, M. Bota and A. Petruşel*, Multivalued fractals in b-metric spaces. Cent. Eur. J. Math., **8**(2010), No. 2, 367-377.
[7] *M. Boriceanu*, Fixed point theory for multivalued generalized contraction on a set with two b -metric, studia, univ Babes, Bolya: Math, Liv(s), **3**(2009), 1-14.
[8] *C. Chifu and G. Petrusel*, Fixed points for multivalued contractions in b-metric spaces with applications to fractals, Taiwanese J. Math., **18**(2014), No. 5, 1365-1375.
[9] *S. Czerwik*, Contraction moppings in b-metric apaces, Acta Math. Inform. Univ. Ostraviensis, **1**(1993), No. 1, 5-11.
[10] *S. Czerwick*, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Semin. Mat. Fis. Univ. Modena., **46**(1998), No. 2, 263-276.
[11] *H. Huang, G. Deng and S. Radenović*, Fixed point theorems in b-metric spaces with applications to differential equations, Fixed Point Theory Appl., **2018**(2018), 24 pages.
[12] *H. Huang, G. Deng and S. Radenović*, Fixed point theorems for C-class functions in b-metric spaces and applications, J. Nonlinear Sci. Appl., **10**(2017), 5853-5868.
[13] *N. Hussain, Z. D. Mitrović*, On multi-valued weak quasi-contractions in b-metric spaces, J. Nonlinear Sci. Appl., **10**(2017), 3815-3823.
[14] *M. Jleli, B. Samet, C. Vetr and F. Vetro*, Fixed points for multivalued mappings in b-metric spaces, Abstr. Appl. Anal., **2015**(2015), 1-7.
[15] *J. M. Joseph, D. D. Roselin and M. Marudai*, Fixed point theorems on multi valued mappings in b-metric spaces, Springer Plus, **5**(2016), 1-8.
[16] *M. Jovanović, Z. Kadelburg and S. Radenović*, Common fixed point results in metric-type spaces, Fixed Point Theory Appl., **2010**(2010), 15 pages.
[17] *T. Kamran, M. Postolache, MU. Ali and Q. Kiran*, Feng and Liu type F-contraction in b-metric spaces with application to integral equations. J. Math. Anal. **7**(2016), No. 5, 18-27.
[18] *F. Khojasteh, M. Abbas and S. Costache*, Two new types of fixed point theorems in complete metric spaces, Abstr. Appl. Anal., **2014**(2014), 5 pages.
[19] *K. Kukic, W. Shatanawi and M. Gardasevic-Filipovic*, Khan and ciric contraction principles in almost b-metric spaces, U.P.B. Sci. Bull., Series A, **82**(2020), No. 1, 1223-7027.
[20] *F. Lael, N. Saleem and M. Abbas*, On the fixed points of multivalued mappings in b-metric spacesand their application to nonlinear systems, U.P.B. Sci. Bull., Series A, **82**(2020), No. 4, 1223-7027.
[21] *N. Makran, A. El Haddouchi and B. Marzouki*, A common fixed point of multivalued maps in b-metric space, U.P.B. Sci. Bull., **82**(2020), No. 1, 1223-7027.
[22] *S. Merdaci, T. Hamaizia*, Some fixed point theorems of rational type contraction in b-metric spaces, Moroccan J. of Pure and Appl. Anal.,**7**(2021), No. 3, 350-363.
[23] *J. Nadler*, Multi-valued contraction mappings, Pacific J. Math., **30**(1969),475-488.
[24] *B.E. Rhoades*, Two new fixed point theorems, Gen. Math. Notes, **27**(2015), No. 2, 123-132.
[25] *M. Samreen, T. Kamran and M. Postolache*, Extended b-metric space, extended b-comparison function and nonlinear contractions, U.P.B. Sci. Bull., Series A, **80**(2018), No. 4, 1223-7027.
[26] *W. Shatanawi, A. Pitea and R. Lazovic*, Contraction conditions using comparison functions on b-metric spaces, Fixed Point Theory Appl. **2014**(2014), No. 135, doi: https://doi.org/10.1186/1687-1812-2014-135.
[27] *W. Sintunavarat*, Fixed point results in b-metric spaces approach to the existence of a solution for nonlinear integral equations. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM, **110**, (2016), 585-600.