RAMANUJAN'S TAU FUNCTION AS SUMS OVER PARTITIONS

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Ramanujan's tau function arises as the coefficient of q^n in the q-expansion of the Dedekind eta function, which is a modular form of weight 1/2. This connection to modular forms gives the Ramanujan tau function many interesting properties and relationships with various areas of mathematics, including number theory, combinatorics, and algebraic geometry. This paper presents a collection of decompositions of the Ramanujan tau function expressed as sums over partitions.

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1. Introduction

Any positive integer n can be written as a sum of one or more positive integers, i.e.,

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_r.$$

When the order of integers λ_i does not matter, this representation is known as an integer partition [1] and can be rewritten as

$$n = t_1 + 2t_2 + \cdots + nt_n, \tag{1}$$

where each positive integer i appears t_i times in the partition. Clearly, $t_i \in \mathbb{N}_0$ for any positive integer i. As usual, we denote by $p(n)$ the number of integer partitions of n, i.e.,

$$p(n) := \sum_{t_1+2t_2+\cdots+nt_n=n} 1$$

Euler showed that the generating function of $p(n)$, can be expressed as an elegant infinite product:

$$\sum_{n=0}^{\infty} p(n)\, q^n = \frac{1}{(q;q)_\infty}, \qquad |q| < 1. \tag{2}$$

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Here and throughout this paper, we use the following customary q-series notation:

$$(a;q)_n = \begin{cases} 1, & \text{for } n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n>0, \end{cases}$$
$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

In this context, q represents a complex number with $|q| < 1$. Whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$. All definitions and identities may be understood in the sense of formal power series in q. Further, for non-negative integers n and k the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise} \end{cases}$$

Expanding the reciprocal of this product by direct multiplication, Euler found the pentagonal number theorem:

$$\sum_{n=0}^{\infty}(-1)^{\lceil n/2 \rceil} q^{\omega_n} = (q;q)_\infty, \tag{3}$$

where

$$\omega_k = \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil$$

is the sequence of the generalized pentagonal numbers.

The introduction of a variable $z \in \mathbb{C}$ in Euler's pentagonal number theorem (3) by putting $q = e^{2\pi i z}$, and appending a factor $q^{1/24}$ leads to the appearance of the eta function $\eta(z)$, i.e.,

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty}(1 - e^{2n\pi i z}) = q^{1/24}(q;q)_\infty.$$

If z belongs to the upper half plane of complex numbers with positive imaginary part then $|q| < 1$ so the product converges absolutely and is nonzero. The function $\eta(z)$ was first introduced and studied in 1877 by Richard Dedekind in [4]. Introducing the variable z opens the gateway to the realm of modular functions and modular forms: it becomes evident from the definition that $\eta(z+1) = e^{1/24}\,\eta(z)$.

In his groundbreaking paper on arithmetic functions published in 1916, Srinivasa Ramanujan [9] studied the coefficients of $\eta^{24}(z)$ and introduced a significant function now known as the Ramanujan tau function. This function, denoted by $\tau(n)$, is defined for all positive integers n by

$$\sum_{n=1}^{\infty} \tau(n)\, q^n = q\,(q;q)_\infty^{24}, \tag{4}$$

and the expansion starts as

$$\sum_{n=1}^{\infty} \tau(n)\, q^n = q - 24\,q^2 + 252\,q^3 - 1472\,q^4 + 4830\,q^5 - 6048\,q^6 - 16744\,q^7 + \cdots$$

Ramanujan's tau function plays a central role in the study of integer partitions and modular forms.

Based on the fact that the functions $\tau(n)$ and $p(n)$ are related by the infinite product $(q;q)_\infty$, Brent [2, Theorem 3.1] provided the following decomposition of $\tau(n)$ as a sum over all the partitions of n:

$$\tau(n+1) = \sum_{t_1+2t_2+\cdots+nt_n=n} (-24)^{t_1+t_2+\cdots+t_n} \frac{\sigma(1)^{t_1}\sigma(2)^{t_2}\cdots\sigma(n)^{t_n}}{1^{t_1}t_1!\,2^{t_2}t_2!\,\cdots\,n^{t_n}t_n!}, \tag{5}$$

where $\sigma(k)$ is the sum of the positive divisors of k, i.e.,

$$\sigma(k) = \sum_{d|k} d.$$

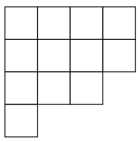
For example, the partition of 4 are:

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1. \tag{6}$$

According to (5), we can write

$$\begin{aligned}\tau(5) &= -24\,\frac{7}{4} + (-24)^2\,\frac{4}{3} + (-24)^2\,\frac{9}{8} + (-24)^3\,\frac{3}{4} + (-24)^4\,\frac{1}{24}\\ &= -42 + 768 + 648 - 10368 + 13824\\ &= 4830.\end{aligned}$$

The Young diagram of a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ is a diagram formed by left-justified rows of boxes such that the ith row form the top has λ_i boxes. For example, below is the Young diagram of $4+4+3+1$.



The *conjugate* of $\lambda_1+\lambda_2+\cdots+\lambda_r$, is the partition whose Young diagram is obtained from the Young diagram of $\lambda_1+\lambda_2+\cdots+\lambda_r$ by reflection along the main diagonal. For example, the conjugate of $4+4+3+1$ is $4+3+3+2$.

Considering that the right-hand side of equation (5) represents a summation over all the partitions of n, and conjugation acts as an involution on the set of partitions of n, we easily deduce the following equivalent version of Brent's decomposition of Ramanujan's tau function:

$$\tau(n+1) = \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=n\\ \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant 0}} \frac{(-24)^{\lambda_1}}{\lambda_1!} \prod_{i=2}^{n} \left(\frac{i-1}{i}\,\frac{\sigma(i)}{\sigma(i-1)}\right)^{\lambda_i} \binom{\lambda_{i-1}}{\lambda_i}.$$

In this paper, inspired by Brent's decomposition (5), we provide a collection of decompositions of Ramanujan's tau function as sums over partitions.

As we can see, the generating function of $\tau(n)$ is not an alternating series. The first result provide an alternating decomposition for the generating function of $\tau(n)$.

Theorem 1.1. *For* $|q|<1$,

$$\sum_{n=1}^{\infty} \tau(n)\,q^n = q\sum_{k=0}^{\infty} (-1)^k\,T_k(q)\,q^k,$$

where $T_0(q) = 1$ *and for* $n > 0$*,*

$$T_n(q) = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+t_2+\cdots+t_n}{t_1,t_2,\ldots,t_n} \binom{24}{t_1+t_2+\cdots+t_n} \prod_{i=1}^{n} \frac{q^{\binom{i}{2}t_i}}{(q;q)_i^{t_i}}.$$

Related to this result, we remark that for $n > 0$ all the coefficients of the series $T_n(q)$ are positive. As a consequence of Theorem 1.1, we remark the following alternating decomposition of Ramanujan's tau function.

Corollary 1.1. *Len* n *be a positive integer. Then*

$$\tau(n+1) = \sum_{k=1}^{n} (-1)^k \, [q^{n-k}] \, T_k(q).$$

For example, considering the partitions of 1, 2 and 3, i.e.,

$$1, \qquad 2 = 1+1 \qquad \text{and} \qquad 3 = 2+1 = 1+1+1,$$

we obtain:

$$\begin{aligned}
T_1(q) &= \frac{24}{1-q} \\
&= 24 + 24\,q + \mathbf{24}\,q^2 + 24\,q^3 + 24\,q^4 + 24\,q^5 + 24\,q^6 + \cdots, \\
T_2(q) &= \frac{24\,q}{(1-q)(1-q^2)} + \frac{276}{(1-q)^2} \\
&= 276 + \mathbf{576}\,q + 852\,q^2 + 1152\,q^3 + 1428\,q^4 + 1728\,q^5 + 2004\,q^6 + \cdots, \\
T_3(q) &= \frac{24\,q^3}{(1-q)(1-q^2)(1-q^3)} + \frac{552\,q}{(1-q)^2(1-q^2)} + \frac{2024}{(1-q)^3} \\
&= \mathbf{2024} + 6624\,q + 13248\,q^2 + 22472\,q^3 + 33696\,q^4 + 47520\,q^5 + \cdots,
\end{aligned}$$

Taking into account Corollary 1.1, we have:

$$\tau(4) = -[q^2]T_1(q) + [q^1]T_2(q) - [q^0]T_3(q) = -24 + 576 - 2024 = -1472.$$

The remainder of our paper is structured as follows. In the next section, we provide a finite version of Theorem 1.1. Thus Ramanujan's tau function becomes an alternative sum in which the terms are the coefficients of symmetric and unimodal polynomials. In Section 3, we consider the pentagonal partitions of n and obtain a decomposition for $\tau(n+1)$ in terms of multinomial coefficients. Our formula does not involves the sum of positive divisors function. In Section 4, we consider the triangular partitions of n and provide a new formula for $\tau(n+1)$ in terms of multinomial coefficients. The partitions of n with parts occurring at most 24 times are considered in Section 5. These partitions allow us to obtain a decomposition for $\tau(n+1)$ only in terms of binomial coefficients of the form $\binom{24}{k}$. In the last section, we consider the binary partitions of n and show that $\tau(n+1)$ can be expressed considering the coefficients of the McKay-Thompson series of class $2B$ for the Monster group [3]. This approach allows us to derive a new decomposition for $\tau(n+1)$ as a sum over all the partitions of n. This time, instead of employing the sum of positive divisors function, we utilize the 2-adic valuation.

2. A truncated form of Theorem 1.1

Theorem 1.1 can be interpreted as the limiting case $n \to \infty$ of the following result.

Theorem 2.1. *Let n be a positive integer. Then*

$$(q;q)_n^{24} = \sum_{k=0}^{24\,n} (-1)^k \, T_{n,k}(q)\, q^k,$$

where $T_{n,0}(q) = 1$ and for $k > 0$,

$$T_{n,k}(q) = \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \binom{24}{t_1+t_2+\cdots+t_k} \prod_{i=1}^{k} \left(q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \right)^{t_i}.$$

Proof. It is well known that the elementary symmetric functions [6]

$$e_k(x_1,x_2,\ldots,x_n) = \sum_{1\leqslant i_1<i_2<\cdots<i_n\leqslant n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

are characterized by the following formal power series identity in z:

$$\sum_{k=0}^{n} e_k(x_1,x_2,\ldots,x_n)\, z^k = \prod_{i=1}^{n} (1 + x_i\, z). \tag{7}$$

According to Merca [7, Corollary 1.2], we have

$$\begin{aligned}
&e_k(\underbrace{q,\ldots,q}_{24},\underbrace{q^2,\ldots,q^2}_{24},\ldots,\underbrace{q^n,\ldots,q^n}_{24}) \\
&= \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \binom{24}{t_1+t_2+\cdots+t_k} \prod_{i=1}^{k} e_i(q,q^2,\ldots,q^n)^{t_i} \\
&= q^k \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{t_1+t_2+\cdots+t_k}{t_1,t_2,\ldots,t_k} \binom{24}{t_1+t_2+\cdots+t_k} \prod_{i=1}^{k} \left(q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \right)^{t_i},
\end{aligned}$$

where we have invoked that

$$e_i(q,q^2,\ldots,q^n) = q^i\, e_i(1,q, \ldots,q^{n-1}) = q^{i+\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}.$$

Taking into account the equation (7) with x_i replaced by q^i and z replaced by -1, we can write

$$\prod_{i=1}^{n} (1-q^i)^{24} = \sum_{k=0}^{24\,n} (-1)^k\, e_k(\underbrace{q,\ldots,q}_{24},\underbrace{q^2,\ldots,q^2}_{24},\ldots,\underbrace{q^n,\ldots,q^n}_{24}).$$

This concludes the proof. □

It is well known that the gaussian polynomials $\begin{bmatrix} n \\ i \end{bmatrix}$ are symmetric and unimodal. Considering [5], we easily deduce that the polynomials $T_{n,1}(q), T_{n,2}(q), \ldots, T_{n,n}(q)$ are symmetric and unimodal. On the other hand, when $j \in \{0,1,\ldots,n\}$, it is an easy exercise to show that

$$[q^j]\, T_k = [q^j]\, T_{n,k}.$$

We remark the following consequence of Theorem 2.1.

Corollary 2.1. *Let m and n be positive integers such that $m \leqslant n$. Then*

$$\tau(m+1) = \sum_{k=1}^{m} (-1)^k \left[q^{m-k}\right] T_{n,k}(q).$$

For example, considering $n = 3$ and the partitions of 1, 2 and 3, we obtain:

$$\begin{aligned}
T_{3,1}(q) &= 24 \begin{bmatrix}3\\1\end{bmatrix} = 24 + 24\,q + 24\,q^2,\\
T_{3,2}(q) &= 24\,q \begin{bmatrix}3\\2\end{bmatrix} + 276 \begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = 276 + 576\,q + 852\,q^2 + 576\,q^3 + 276\,q^4,\\
T_{3,3}(q) &= 24\,q^3 \begin{bmatrix}3\\3\end{bmatrix} + 552\,q \begin{bmatrix}3\\2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} + 2024 \begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}\\
&= 2024 + 6624\,q + 13248\,q^2 + 15848\,q^3 + 13248\,q^4 + 6624\,q^5 + 2024\,q^6.
\end{aligned}$$

Taking into account Corollary 1.1, we can write

$$\begin{aligned}
\tau(2) &= -[q^0]T_{3,1}(q) = -24,\\
\tau(3) &= -[q^1]T_{3,1}(q) + [q^0]T_{3,2}(q) = -24 + 276 = 252,\\
\tau(4) &= -[q^2]T_{3,1}(q) + [q^1]T_{3,2}(q) - [q^0]T_{3,3}(q) = -24 + 576 - 2024 = -1472.
\end{aligned}$$

3. Pentagonal partitions into at most 24 parts

In this section, we consider the notion of *pentagonal partition.* A pentagonal partition is a regular partition in which all parts are positive generalized pentagonal numbers, that is, they belong to the set

$$\{\omega_n\}_{n\geqslant 1} = \{1,\ 2,\ 5,\ 7,\ 12,\ 15,\ 22,\ 26,\ 35,\ 40,\ 51,\ 57,\ 70,\ 77,\ 92,\ 100,\ 117,\ \ldots\}$$

Euler used (3) to derive the well known linear recurrence for the partition function $p(n)$:

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil}\, p\big(n-\omega_k\big) = \delta_{n,0},$$

where $\delta_{i,j}$ is Kronecker's delta. In 2014, Merca [8, Corollary 10] used Euler's results and showed that the generalized pentagonal numbers can be used to compute an isolated value of $p(n)$ without recursion:

$$p(n) = \sum_{\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n = n} (-1)^{t_3+t_4+t_7+t_8+\cdots} \binom{t_1+t_2+\cdots+t_n}{t_1, t_2, \ldots, t_n}. \tag{8}$$

We see that the partition function $p(n)$ was decomposed as a sum over all the pentagonal partitions of n. Our initial result provide a similar decomposition for Ramanujan's tau function.

Theorem 3.1. *Let n be a positive integer. Then*

$$\tau(n+1) = \sum_{\omega_1 t_1 + \cdots + \omega_n t_n = n} (-1)^{t_1+t_2+t_5+t_6+\cdots} \binom{t_1+t_2+\cdots+t_n}{t_1, t_2, \ldots, t_n} \binom{24}{t_1+t_2+\cdots+t_n}.$$

Proof. For any positive integer m and any non-negative integer n, the multinomial formula describes how a sum with m terms expands when raised to an arbitrary power n:

$$\left(\sum_{i=1}^{m} x_i\right)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1,k_2,\ldots,k_m} \prod_{i=1}^{m} x_i^{k_i},$$

where $k_1,k_2,\ldots,k_m$ are non-negative integers. In order to prove our identity, we consider the limiting case $m \to \infty$ of the multinomial formula as follows:

$$\begin{aligned}\left(\sum_{i=0}^{\infty} x_i\, q^i\right)^n &= \sum_{t_0+t_1+t_2+\cdots=n} \binom{n}{t_0,t_1,t_2,\ldots} \prod_{i=0}^{\infty} x_i^{t_i} q^{it_i}.\\ &= \sum_{k=0}^{\infty} q^k \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{n}{t_0,t_1,t_2,\ldots,t_k} \prod_{i=0}^{k} x_i^{t_i}\\ &= \sum_{k=0}^{\infty} q^k \sum_{t_1+2t_2+\cdots+kt_k=k} \binom{\sum_{i=1}^{k} t_i}{t_1,t_2,\ldots,t_k} \binom{n}{\sum_{i=1}^{k} t_i} \prod_{i=0}^{k} x_i^{t_i}, \qquad (9)\end{aligned}$$

where, in the last line, we have considered $t_0 = n - (t_1 + t_2 + \cdots + t_k)$.

According to Euler's pentagonal number theorem (3), we can write:

$$\begin{aligned}&\sum_{n=1}^{\infty} \tau(n)\, q^n = q \left(\sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{\omega_n}\right)^{24}\\ &= \sum_{n=0}^{\infty} q^{n+1} \sum_{\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n=n} \binom{\sum_{i=1}^{n} t_i}{t_1,t_2,\ldots,t_n} \binom{24}{\sum_{i=1}^{n} t_i} (-1)^{t_1+t_2+t_5+t_6+\cdots}.\end{aligned}$$

This concludes the proof. □

For example, the partitions of 4 in which all the parts are generalized pentagonal numbers are:

$$2+2 = 2+1+1 = 1+1+1+1.$$

According to Theorem 3.1, we have:

$$\begin{aligned}\tau(5) &= \binom{2}{2}\binom{24}{2} - \binom{3}{2,1}\binom{24}{3} + \binom{4}{4}\binom{24}{4}\\ &= 276 - 6072 + 10626 = 4830.\end{aligned}$$

The summation on the right-hand side of the equation given by Theorem 3.1 encompasses all pentagonal partitions of n, but not every term contributes because some are zero when $n > 24$. For $t_1 + t_2 + \cdots + t_n > 24$, the binomial coefficient $\binom{24}{t_1+t_2+\cdots+t_n}$ becomes zero, allowing us to focus solely on pentagonal partitions of n in which $t_1 + t_2 + \cdots + t_n \leqslant 24$.

4. **Triangular partitions into at most 8 parts**

A *triangular partition* is a regular partition in which all parts are positive triangular numbers, that is, they belong to the set

$$\begin{aligned}\{\theta_n\}_{n\geqslant 1} &= \{n(n+1)/2 \mid n \in \mathbb{N}\}\\ &= \{1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ 66,\ 78,\ 91,\ 105,\ 120,\ 136,\ \ldots\}.\end{aligned}$$

In this section, we consider the triangular partitions of n into at most 8 parts and provide the following decomposition for Ramanujan's tau function.

Theorem 4.1. *Let n be a positive integer. Then*

$$\tau(n+1) = \sum_{\theta_1 t_1+\theta_2 t_2+\cdots+\theta_n t_n=n} (-1)^{t_1+t_3+t_5+\cdots} \binom{\sum_{i=1}^n t_i}{t_1, t_2, \ldots, t_n} \binom{8}{\sum_{i=1}^n t_i} \prod_{i=1}^{n} (2i+1)^{t_i}.$$

Proof. Euler's pentagonal numbers theorem (3) has the following generalization which is well known as the Jacobi triple product identity [1]: for $x \neq 0$,

$$\sum_{n=-\infty}^{\infty} (-1)^n\, q^{n(n-1)/2}\, x^n = \prod_{n=1}^{\infty} (1-q^n)(1-x\,q^{n-1})(1-x^{-1}\,q^n).$$

Dividing this identity by $1-x$ and letting $x \to 1$ gives the Jacobi cubic analog of Euler's pentagonal number theorem

$$\sum_{n=0}^{\infty} (-1)^n\,(2n+1)\,q^{n(n+1)/2} = \prod_{n=1}^{\infty} (1-q^n)^3.$$

Thus we can write

$$\begin{aligned}\sum_{n=1}^{\infty} \tau(n)\,q^n &= q\left(\prod_{n=1}^{\infty}(1-q^n)^3\right)^8 = q\left(\sum_{n=0}^{\infty}(-1)^n\,(2n+1)\,q^{\theta_n}\right)^8\\ &= \sum_{n=0}^{\infty} q^{n+1} \sum_{\theta_1 t_1+\cdots+\theta_n t_n=n} \binom{\sum_{i=1}^n t_i}{t_1, t_2, \ldots, t_n} \binom{8}{\sum_{i=1}^n t_i} (-1)^{t_1+t_3+t_5+\cdots} \prod_{i=1}^{n} (2i+1)^{t_i},\end{aligned}$$

where we have invoked (9) with n replaced by 8 and

$$x_i = \begin{cases} (-1)^k\,(2k+1), & \text{if } i = \theta_k,\\ 0, & \text{otherwise.}\end{cases}$$

□

The summation on the right-hand side of the equation provided by Theorem 4.1 includes all triangular partitions of n, although not every term contributes due to some being zero when $n > 8$. When $t_1+t_2+\cdots+t_n > 8$, the binomial coefficient $\binom{8}{t_1+t_2+\cdots+t_n}$ becomes zero. Consequently, we can concentrate solely on triangular partitions of n where $t_1+t_2+\cdots+t_n \leqslant 8$.

For example, the partitions of 6 in which all the parts are positive triangular numbers are:

$$6 = 3+3 = 3+1+1+1 = 1+1+1+1+1+1.$$

According to Theorem 4.1, we have:

$$\begin{aligned}\tau(7) &= -\binom{1}{1}\binom{8}{1}\cdot 7 + \binom{2}{2}\binom{8}{2}\cdot 5^2 - \binom{4}{3,1}\binom{8}{4}\cdot 3^3\cdot 5 + \binom{6}{6}\binom{8}{6}\cdot 3^6\\ &= -56 + 700 - 37800 + 20412 = -16744.\end{aligned}$$

5. **Partitions with parts occurring at most 24 times**

The binomial theorem is a fundamental result in mathematics that describes the expansion of powers of binomials. It states that for any non-negative integer n and any real numbers a and b, the binomial expression $(a+b)^n$ can be expanded as the sum of $n+1$, each of the form $\binom{n}{k}\, a^{n-k}\, b^k$. The binomial theorem has wide-ranging applications in mathematics and its related fields. It is used extensively in calculus, combinatorics, probability theory, and algebra, among others. In this section, we consider a special case of the binomial theorem, i.e.,

$$(1-q)^{24} = \sum_{k=0}^{24}(-1)^k\binom{24}{k}q^k, \tag{10}$$

and provide a new decomposition for Ramanujan's tau function.

Theorem 5.1. *Let n be a positive integer. Then*

$$\tau(n+1) = \sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}\binom{24}{t_1}\binom{24}{t_2}\cdots\binom{24}{t_n}.$$

Proof. According to (10), we can write

$$\begin{aligned}\sum_{n=1}^{\infty}\tau(n)\,q^n &= q\prod_{n=1}^{\infty}(1-q^n)^{24} = q\prod_{n=1}^{\infty}\sum_{k=0}^{24}(-1)^k\binom{24}{k}q^{k\,n}\\ &= \sum_{n=0}^{\infty}q^{n+1}\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}\prod_{i=1}^{n}\binom{24}{t_i},\end{aligned}$$

where we have invoked the Cauchy multiplication of the power series. □

The summation on the right-hand side of Theorem 5.1 involves all partitions of n, yet not every term is significant since some become zero when a part has the multiplicity greater than 24. When $t_k > 24$, the binomial coefficient $\binom{24}{t_k}$ equals zero, enabling us to concentrate solely on partitions of n with parts occurring at most 24 times.

For example, the partitions of 5 are:

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

According to Theorem 5.1, we have:

$$\begin{aligned}\tau(6) &= -\binom{24}{1} + 2\binom{24}{1}^2 - 2\binom{24}{1}\binom{24}{2} + \binom{24}{1}\binom{24}{3} - \binom{24}{5}\\ &= -24 + 1152 - 13248 + 48576 - 42504 = -6048.\end{aligned}$$

6. Binary partitions and 2-adic valuation

A *binary partition* is a regular partition in which all parts are non-negative powers of two, that is, they belong to the set

$$\{2^n \mid n \in \mathbb{N}_0\} = \{1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512,\ 1024, \dots\}.$$

In this section, we show that Ramanujan's tau function $\tau(n+1)$ can be expressed as a sum over all binary partitions of n. To do this we consider the McKay-Thompson series of class $2B$ for the Monster group [3]:

$$\begin{aligned}\left(\frac{\eta(z)}{\eta(z^2)}\right)^{24} &= \frac{1}{q}\prod_{n=1}^{\infty}\frac{1}{(1+q^n)^{24}} = \sum_{n=0}^{\infty} a(n)\, q^{n-1}\\ &= q^{-1} - 24 + 276\,q - 2048\,q^2 + 11202\,q^3 - 49152\,q^4 + 184024\,q^5 - \cdots\end{aligned}$$

We have the following result.

Theorem 6.1. *Let n be a positive integer. Then*

$$\tau(n+1) = \sum_{2^0 t_0 + 2^1 t_1 + \cdots + 2^n t_n = n} a(t_0)\, a(t_1) \cdots a(t_n).$$

Proof. Considering the identity

$$1 = (1-q)\prod_{k=0}^{\infty}(1+q^{2^k}), \qquad |q|<1,$$

we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\tau(n)\, q^n &= q\prod_{n=1}^{\infty}\prod_{k=0}^{\infty}\frac{1}{(1+q^{2^k\cdot n})^{24}} = q\prod_{k=0}^{\infty}\sum_{n=0}^{\infty} a(n)\, q^{2^k\cdot n}\\ &= \sum_{n=0}^{\infty} q^{n+1} \sum_{2^0\, t_0 + 2^1\, t_1 + \cdots + 2^n\, t_n = n}\ \prod_{i=0}^{n} a(t_i),\end{aligned}$$

where we have invoked the Cauchy multiplication of power series. This concludes the proof. □

For example, the binary partitions of 5 are:

$$4+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1+1.$$

According to Theorem 6.1, we have:

$$\begin{aligned}\tau(6) &= a(1)\, a(1) + a(2)\, a(1) + a(1)\, a(3) + a(5)\\ &= (-24)^2 + 276\cdot(-24) + (-24)(-2048) + (-49152) = -6048.\end{aligned}$$

In this context, we remark that the coefficients of McKay-Thompson series of class $2B$ for the Monster group can be expressed as sums over partitions.

Theorem 6.2. *Let n be a positive integer. Then*

1. $a(n) = \displaystyle\sum_{t_1 + 2t_2 + \cdots + n t_n = n} (-1)^{t_1+t_2+\cdots+t_n} \binom{23+t_1}{t_1}\binom{23+t_2}{t_2}\cdots\binom{23+t_n}{t_n}.$
2. $a(n) = \displaystyle\sum_{t_1 + 3t_2 + \cdots + (2\lceil n/2\rceil - 1)\, t_{\lceil n/2\rceil} = n} (-1)^{t_1+t_2+\cdots+t_{\lceil n/2\rceil}} \binom{24}{t_1}\binom{24}{t_2}\cdots\binom{24}{t_{\lceil n/2\rceil}}.$

Proof. 1. Considering the generating function for $a(n)$, we can write

$$\sum_{n=0}^{\infty} a(n)\, q^{n-1} = \frac{1}{q} \prod_{n=1}^{\infty} \frac{1}{(1+q^n)^{24}} = \frac{1}{q} \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{23+k}{k} (-q^n)^k$$
$$= \sum_{n=0}^{\infty} q^{n-1} \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+t_2+\cdots+t_n} \prod_{i=1}^{n} \binom{23+t_i}{t_i},$$

where we have invoked the Cauchy multiplication of power series. This concludes the proof.

2. Take into account the Euler's identity

$$\prod_{n=1}^{\infty} (1+q^n) = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}},$$

we can write

$$\sum_{n=0}^{\infty} a(n)\, q^{n-1} = \frac{1}{q} \prod_{n=1}^{\infty} \frac{1}{(1+q^n)^{24}} = \frac{1}{q} \prod_{n=1}^{\infty} (1-q^{2n-1})^{24}$$
$$= \frac{1}{q} \prod_{n=1}^{\infty} \sum_{k=0}^{24} (-1)^k \binom{24}{k} q^{(2n-1)\,k}$$
$$= \sum_{n=0}^{\infty} q^{n-1} \sum_{t_1+3t_2+\cdots+(2\lceil n/2\rceil-1)t_{\lceil n/2\rceil}=n} (-1)^{t_1+t_2+\cdots+t_{\lceil n/2\rceil}} \prod_{i=1}^{\lceil n/2\rceil} \binom{24}{t_i},$$

where we have invoked the Cauchy multiplication of power series. This concludes the proof. □

For example, the partitions of 4 are:

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1+1.$$

According to the first equation of Theorem 6.2, we have:

$$a(4) = -\binom{24}{1} + \binom{24}{1}\binom{24}{1} + \binom{25}{2} - \binom{24}{1}\binom{25}{2} + \binom{27}{4}$$
$$= -24 + 576 + 300 - 7200 + 17550 = 11202$$

Considering the partition of 4 into odd parts, i.e.,

$$3+1 = 1+1+1+1,$$

and the second equation of Theorem 6.2, we have:

$$a(4) = \binom{24}{1}\binom{24}{1} + \binom{24}{4} = 576 + 10626 = 11202$$

The 2-adic valuation of an integer n is the exponent of the highest power of 2 that divides n and is denoted by $\nu_2(n)$. We prove that Ramanujan's tau function can be expressed in terms of $\nu_2(n)$.

Theorem 6.3. *Let n be a positive integer. Then*

$$\tau(n+1) = \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+t_2+\cdots+t_n} \prod_{i=1}^{n} \binom{24\,\nu_2(i)+23+t_i}{t_i}.$$

Proof. Considering the identity (10), we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\tau(n)\,q^n &= q\prod_{n=1}^{\infty}\prod_{k=0}^{\infty}\frac{1}{(1+q^{2^k\cdot n})^{24}}\\ &= q\prod_{n=1}^{\infty}\frac{1}{(1+q^n)^{24\,\nu_2(2n)}}\\ &= q\prod_{n=1}^{\infty}\sum_{k=0}^{\infty}\binom{24\,\nu_2(2k)-1+k}{k}(-q^n)^k\\ &= \sum_{n=0}^{\infty}q^{n+1}\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}\prod_{i=1}^{n}\binom{24\,\nu_2(2i)-1+t_i}{t_i},\end{aligned}$$

where we have invoked the Cauchy multiplication of power series. This concludes the proof. □

For example, considering the partitions of 4 and Theorem 6.3, we have:

$$\begin{aligned}\tau(5) = -\binom{71+1}{1}+\binom{23+1}{1}^2+\binom{48+1}{2}-\binom{23+2}{2}\binom{47+1}{1}+\binom{23+4}{4}\\ = -72+576+1176-14400+17550=4830.\end{aligned}$$

While the decomposition of $\tau(n+1)$ provided by Theorem 6.3 involves a sum over all the partitions of n, it is evident that this is not an equivalent version of Brent's decomposition (5).

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