A NOTE ON "ORDERED SEMIGROUPS CHARACTERIZED BY THEIR $(\in, \in \vee q)$-FUZZY BI-IDEALS"

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This paper is a short note on the paper [1, Y.B. Jun et al., Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals, Bull. Malays. Math. Sci. (2) 32(3) (2008) 391-408]. *We show that Theorem 4.3 does not hold in general. Then, we discuss more update result than Theorem 4.3.*

Keywords: Ordered semigroup, $(\in, \in \vee q)$-fuzzy bi-ideal, regular ordered semigroup

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1. Introduction

Jun et al., [1] introduced the concept of $(\in, \in \vee q)$-fuzzy bi-ideals of ordered semigroup, and gave some characterization theorems. In [1, Theorem 4.3], they proved that an ordered semigroup S is regular if and only if for every $(\in, \in \vee q)$-fuzzy bi-ideal λ of S we have $\lambda \circ_{0.5} 1 \circ_{0.5} \lambda = \lambda$. We note that Theorem 4.3 in [1] is not true for all $(\in, \in \vee q)$-fuzzy bi-ideals λ of S.

The first aim of this paper is to show that [1, Theorem 4.3] is not true by giving an Example. The second purpose of this article is to verify Theorem 4.3 in [1] by an Example. We discuss more updated results than Theorem 4.3 in [1].

2. Preliminaries

By an *ordered semigroup* we mean an ordered set S at the same time a semigroup such that $x, y \in S$, $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all $z \in S$. Let S be an ordered semigroup. Then, for $A \subseteq S$, we denote$(A] := \{a \in S : a \leq b \text{ for some } b \in A\}$.

For $A, B \subseteq S$, we denote, $AB := \{ab : a \in A,\ b \in B\}$. Let $A, B \subseteq S$. Then, $A \subseteq (A]$, $(A](B] \subseteq (AB]$, and $((A]] = (A]$. Let S be an ordered semigroup and $\oslash \neq B \subseteq S$. Then, B is called a *subsemigroup* of S if for all $x, y \in B$ implies that $xy \in B$ or $B^2 \subseteq B$. A subsemigroup B of an ordered semigroup S is called a *bi-ideal* of S if

$$(1),\ BSB \subseteq B\ \ (2)\ \ (\forall x \in S)\,(\forall y \in B),\,(x \leq y \Rightarrow x \in B).$$

An element x of an ordered semigroup S is called *regular* if there exists $a \in S$ such that $x \leq xax$. An ordered semigroup is called *regular* if every element of S is regular.

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Equivalently, if $(\forall x \in S)$ $(x \in (xSx])$ or $(\forall A \subseteq S)$ $(A \subseteq (ASA])$. For $a \in S$, define $A_a := \{(y,z) \in S_1 \times S_2 : a \leq yz\}$.

For a set S, let $\mathcal{F}(S) := \{\lambda : \lambda : S \longrightarrow [0,1]\}$. Elements of $\mathcal{F}(S)$ are called *fuzzy subsets of* S. For any $\lambda \in \mathcal{F}(S)$ and $t \in [0,1]$, the set $U(\lambda;t) = \{x \in S : \lambda(x) \geq t\}$ is called a level subset of λ. The product $\lambda \circ_{0.5} \mu$ of two fuzzy subsets λ and μ defined as:

$$(\lambda \circ_{0.5} \mu)(1) = \begin{cases} \bigvee\limits_{(y,z)\in A_x} \{(\lambda)(y) \wedge \mu(z) \wedge 0.5\} & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset \end{cases}$$

Given an element $x \in S$, consider a mapping

$$\lambda : S \longrightarrow [0,1], \qquad y \longrightarrow \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if y} \neq x. \end{cases}$$

Then, $\lambda \in \mathcal{F}(S)$, and it is said to be a *fuzzy point* with support x, and value t, is denoted by x_t.

For fuzzy point x_t and fuzzy subset λ of a set S, Pu and Liu [2] introduced the symbol $x_t \alpha \lambda$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. To say that $x_t \in \lambda$ (resp. $x_t q \lambda$), we mean $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$), and this case, x_t is said to be *belong to* (resp. *be quasi-coincident with*) a fuzzy subset λ. To say $x_t \in \vee q \lambda$ (resp. $x_t \in \wedge q \lambda$), we mean $x_t \in \lambda$ or $x_t q \lambda$ (resp. $x_t \in \lambda$ and $x_t q \lambda$), and $x_t \overline{\in \vee q} \lambda$ mean that $x_t \in \vee q \lambda$ does not hold.

A fuzzy subset λ of an ordered semigroup S is called a *fuzzy subsemigroup* of S if for all $x, y \in S$, $(B1)\, \lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$.

A fuzzy subsemigroup λ of an ordered semigroup S is called a *fuzzy bi-ideal* of S if for all $x, y, z \in S$, $(B2)$ $(\forall x \in S)(x \leq y \Rightarrow \lambda(x) \geq \lambda(y))$, and $(B3)$ $\lambda(xyz) \geq \min\{\lambda(x), \lambda(z)\}$.

A fuzzy subset λ of an ordered semigroup S is called an $(\in, \in \vee q)$-*fuzzy bi-ideal* of S if it satisfies:

(1) $(\forall x, y \in S)(x \leq y,\ y_t \in \lambda \Rightarrow x_t \in \vee q \lambda)$,
(2) $(\forall x, y \in S)$ $(x_{t_1}, y_{t_2} \in \lambda \Rightarrow (xy)_{t_1 \wedge t_2} \in \vee q \lambda)$,
(3) $(\forall x, y, z \in S)$ $(x_{t_1}, z_{t_2} \in \lambda \Rightarrow (xyz)_{t_1 \wedge t_2} \in \vee q \lambda)$.

3. **Note on " Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals"**

In what follows, let S denote an ordered semigroup. For any fuzzy subset λ and $t \in (0,1]$, we consider two subsets: $Q(\lambda;t) = \{x \in S : x_t q \lambda\}$ and $[\lambda]_t = \{x \in S : x_t \in \vee q \lambda\}$. It is clear that $[\lambda]_t = U(\lambda;t) \cup Q(\lambda : t)$. In [1], Jun et al. discussed the following theorem.

Theorem 3.1. [1, Theorem 4.3] *An ordered semigroup S is regular if and only if for every $(\in, \in \vee q)$-fuzzy bi-ideal λ of S we have $\lambda \circ_{0.5} 1 \circ_{0.5} \lambda = \lambda$.*

Example 3.1. *Consider a set* $S = \{0,1,2,3,4\}$ *with the following multiplication "." and ordered relation "$\leq$":*

$*$	0	1	2	3	4
0	0	3	0	3	3
1	0	1	0	3	3
2	0	3	2	3	4
3	0	3	0	3	3
4	0	3	2	3	4

$\leq := \{(0,0),(0,2),(0,3),(0,4),(1,1),(1,3),(1,4),(2,2),(2,4),(3,3),(3,4)(4,4)\}$.

Then, $(S,*,\leq)$ *is a regular ordered semigroup, i.e* $0 \leq 0*1*0$, $1 \leq 1*0*1$, $2 \leq 2*4*2$, $3 \leq 3*0*3$ *and* $4 \leq 4*2*4$. *We define a fuzzy subset* λ *of* S *by* $\lambda(0) = 0.9$, $\lambda(1) = 0.8$, $\lambda(4) = 0.6$, $\lambda(3) = 0.8$, $\lambda(2) = 0.6$. *Then,* $U(\lambda;t) = \begin{cases} S & \text{if } t \in (0,0.6], \\ \{0,1,3\} & \text{if } t \in (0.6,0.8], \\ \{0\} & \text{if } t \in (0.8,0.9], \\ \emptyset & \text{if } t \in (0.9,1]. \end{cases}$ *Clearly,* λ *is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of* S.
Now, we check Theorem 3.1 and let $1 \in S$. *Then, the only possible case is* $1 = 1*1$.

$$(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(1) = \bigvee_{1=1*1} \{(\lambda \circ_{0.5} 1)(1) \wedge \lambda(1) \wedge 0.5\}$$

$$= \bigvee_{1=1*1} \left\{ \bigvee_{1=1*1} \{\lambda(1) \wedge 1(1) \wedge 0.5\} \wedge \lambda(1) \wedge 0.5 \right\} = \bigvee_{1=1*1} \{\{0.9 \wedge 1 \wedge 0.5\} \wedge 0.9 \wedge 0.5\}$$

$$= \bigvee_{1=1*1} \{0.5 \wedge 0.9 \wedge 0.5\} = 0.5\,(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(1) < \lambda(1).$$

So, $\lambda \circ_{0.5} 1 \circ_{0.5} \lambda \neq \lambda$. *This Example shows that Theorem 3.1, does not hold.*

Definition 3.1. [3] *Let* λ *be a fuzzy subset of an ordered semigroup* S. *We define the upper part* λ^+ *and the lower part* λ^- *of* λ *as follows,* $\lambda^+(x) = \lambda(x) \vee 0.5$ *and* $\lambda^-(x) = \lambda(x) \wedge 0.5$.

Definition 3.2. [3] *Let* A *be a non-empty subset of an ordered semigroup* S. *Then, the lower and upper parts of the characteristic function are* $C_A^-(x) = \begin{cases} 0.5 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ *and* $C_A^+(x) = \begin{cases} 1 & \text{if } x \in A \\ 0.5 & \text{if } x \notin A \end{cases}$

Lemma 3.1. [3] *Let* A *and* B *be non-empty subsets of an ordered semigroup* S. *Then, the following properties hold.*

(1) $(C_A \wedge C_B)^- = C^-_{A\cap B}$, (2) $(C_A \vee C_B)^- = C^-_{A\cup B}$,
(3) $(C_A \circ C_B)^- = C^-_{(AB]}$ *or* $(C_A \circ_{0.5} C_B) = C^-_{(AB]}$.

Lemma 3.2. *The lower part of characteristic function* C_B^- *is an* $(\in, \in \vee q)$*-fuzzy bi-ideal of* S *if and only if* B *is a bi-ideal of* S.

We present a new statement which is more general than [1, Theorem 4.3].

Theorem 3.2. *An ordered semigroup* S *is regular if and only if for every* $(\in, \in \vee q)$*-fuzzy bi-ideal* λ *of* S, *we have* $\lambda \circ_{0.5} 1 \circ_{0.5} \lambda = \lambda^-$ *or* $(\lambda \circ 1 \circ \lambda)^- = \lambda^-$.

Proof. Assume that S is a regular ordered semigroup and let $x \in S$. Since S is regular, so there exists $a \in S$ such that $x \leq xax$. Then, $(xa, x) \in A_x$,

$$\begin{aligned}
(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(x) &= \bigvee_{(y,z)\in A_x} \min\{(\lambda \circ_{0.5} 1)(y), \lambda(z), 0.5\} \\
&\geq \min\{(\lambda \circ_{0.5} 1)(xa), \lambda(x), 0.5\} \\
&= \min\left\{\bigvee_{(p,q)\in A_{xa}} \min\{\lambda(p), 1(q), 0.5\}, \lambda(x), 0.5\right\} \\
&\geq \min\{\min\{\lambda(x), 1(a), 0.5\}, \lambda(x), 0.5\} \\
&= \min\{\min\{\lambda(x), 1, 0.5\}, \lambda(x), 0.5\} = \min\{\lambda(x), 0.5\} \\
&= \lambda(x) \wedge 0.5 = \lambda^-(x) (\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(x) \geq \lambda^-(x)
\end{aligned}$$

Thus, $\lambda \circ_{0.5} 1 \circ_{0.5} \lambda \geq \lambda^-$. Since λ is an $(\in, \in \vee q)$-fuzzy bi-ideal λ of S, so, we have

$$\begin{aligned}
(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(x) &= \bigvee_{(y,z)\in A_x} \min\{(\lambda \circ_{0.5} 1)(y), \lambda(z), 0.5\} \\
&= \bigvee_{(y,z)\in A_x} \min\left\{\bigvee_{(p,q)\in A_y} \min\{\lambda(p), 1(q)\}, \lambda(z), 0.5\right\} \\
&= \bigvee_{(y,z)\in A_x} \min\left\{\bigvee_{(p,q)\in A_y} \min\{\lambda(p), 1\}, \lambda(z), 0.5\right\} \\
&= \bigvee_{(pq,z)\in A_x} \min\{\lambda(p), \lambda(z), 0.5\} = \bigvee_{(pq,z)\in A_x} \min\{\lambda(p), \lambda(z), 0.5, 0.5\} \\
&= \bigvee_{(pq,z)\in A_x} \min\{\lambda(p), \lambda(z), 0.5, \} \wedge 0.5 \\
&\leq \bigvee_{(pq,z)\in A_x} \min\{\lambda(pqz)\} \wedge 0.5 \quad \begin{pmatrix} \text{because } \lambda \text{ is an } (\in, \in \vee q) \\ \text{-fuzzy bi-ideal of } S. \end{pmatrix} \\
&= \lambda(x) \wedge 0.5 = \lambda^-(x) (\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(x) \leq \lambda^-(x).
\end{aligned}$$

Thus, $(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda) \leq \lambda^-$. Therefore $(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda) = \lambda^-$.

Conversely, Let B be a bi-ideal of S. Then $(BSB] \subseteq B$. Let $x \in B$. Then, by Lemma 3.2, C_B^- is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. So, we have$C_B^-(x) = \left(C_B^- \circ_{0.5} 1 \circ_{0.5} C_B^-\right)(x) = C_{(BSB]}^-(x)$ by Lemma 3.1 (3). So, $x \in B$. Then, $C_B^-(x) = 0.5 \Rightarrow C_{(BSB]}^-(x) = 0.5 \Rightarrow x \in (BSB]$. Thus, $B \subseteq (BSB]$. Therefore, $B = (BSB]$, so S is regular. This completes the proof. □

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