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POPOV-BELEVICH-HAUTUS THEOREM FOR LINEAR MULTITIME AUTONOMOUS DYNAMICAL SYSTEMS

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Această lucrare studiaza sistemele liniare multitemporale comandate. Se demonstrează că un sistem liniar multitemporal staționar este izomorf cu un nou sistem, mai simplu, punîndu-se în evidență un subsistem complet controlabil pentru acest nou sistem. În final se demonstrează o Teoremă de tip Popov-Belevich-Hautus pentru sisteme multitemporale, scopul acestui articol.

In this paper we are concerned with multitime linear dynamical system. First, we identify a certain isomorphism between the initial system and another simpler one, using as a key tool, a completely controllable subsystem. Then, we present a Popov-Belevich-Hautus type Theorem for multitime systems.

Keywords: controllability, control matrix, isomorphic systems, Popov theorems.
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1. Introduction

Multitime control theory has been developed in many directions. In Romania, one of them was initiated and well developed by C. Udrişte: multitime dynamical systems; multitime maximum principle ([1] – [8]), and another by V. Prepeliță: the class of hybrid systems: discrete and continuous [9], [10]. In Germany, there is a research group led by S. Pickenhain ([11], [12], [13]) who studied Dieudonné – Rashevsky type problems employing advanced techniques from the distribution theory. In USA, a research group led by J.A. Ball has obtained different results via discrete and robust control (see [14]).

The present paper follows the direction proposed by the paper [1]. More precisely, we prove some new theorems concerning the multitime control systems. These results may be regarded as non-trivial extensions of some well-known results in the single-time theory (see [15]).

2. Controllability of linear multitime autonomous dynamical sistems

In this section we recall some mathematical ingredients from the paper [1]. Consider the autonomous control system

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$$\frac{\partial x}{\partial t^{\alpha}} = M_{\alpha} x(t) + N_{\alpha} u_{\alpha}(t), \qquad \forall \alpha = \overline{1,m}, \tag{1}$$

where

$t = \left(t^1, t^2, \ldots, t^m\right) \in \mathbb{R}^m$, $x = \left(x^1, x^2, \ldots, x^n\right)^T : \mathbb{R}^m \to \mathbb{R}^n = M_{n,1}\left(\mathbb{R}\right)$,

$M_{\alpha} \in M_n\left(\mathbb{R}\right)$, $N_{\alpha} \in M_{n,k}\left(\mathbb{R}\right)$, M_{α} and N_{α} are constant matrices

and $u_{\alpha} : \mathbb{R}^m \to \mathbb{R}^k = M_{k,1}\left(\mathbb{R}\right)$ are C^1 - control functions.

The system (1) is said to be *completely integrable* if $\forall \left(t_0, x_0\right) \in \mathbb{R}^m \times \mathbb{R}^n$, there exist an open set $D_0 \subseteq \mathbb{R}^m$, $t_0 \in D_0$, and a differentiable function $x : D_0 \to \mathbb{R}^n$ such that $x(\cdot)$ verifies the system (1) on D_0 and $x\left(t_0\right) = x_0$. In this case, $x(\cdot)$ will be called a *solution* for (1).

We reformulate a result in [1] for autonomous systems.

Theorem 1. *The system* (1) *is completely integrable if and only if the following relations hold*

$$M_{\alpha} M_{\beta} = M_{\beta} M_{\alpha}, \qquad \forall \alpha, \beta = \overline{1,m}, \tag{2}$$

$$M_{\alpha} N_{\beta} u_{\beta}(t) + N_{\alpha} \frac{\partial u_{\alpha}}{\partial t^{\beta}}(t) = M_{\beta} N_{\alpha} u_{\alpha}(t) + N_{\beta} \frac{\partial u_{\beta}}{\partial t^{\alpha}}(t), \tag{3}$$

$\forall t \in \mathbb{R}^m$, $\forall \alpha, \beta = \overline{1,m}$.

In these conditions, any solution $x(\cdot)$ *will be a* C^2 *- solution and it can be uniquely extended to a global solution* $\left(x : \mathbb{R}^m \to \mathbb{R}^n\right)$. *Moreover, if two solutions coincide at a point, then they will coincide on the whole space* $\mathbb{R}^m$.

Definition 1. *Suppose that the matrices* $M_1, M_2, \ldots, M_n$ *verify* (2) *for all* $\alpha, \beta = \overline{1,m}$. *Then the vector space*

$$\Big\{ u = \left(u_{\alpha}\right)_{\alpha = \overline{1,m}} \mid u_{\alpha} : \mathbb{R}^m \to \mathbb{R}^k = M_{k,1}\left(\mathbb{R}\right) \text{ is of class } C^1, \forall \alpha = \overline{1,m}$$
$$\text{and verifies (3) for all } \alpha, \beta \Big\}$$

is called the admissible controls space associated to the system (1).

So, if $M_{\alpha} M_{\beta} = M_{\beta} M_{\alpha}$, $\forall \alpha, \beta = \overline{1,m}$, then the system (1) is completely integrable if and only if $\left(u_{\alpha}\right)_{\alpha = \overline{1,m}}$ is a control function. Thereafter, by a solution we mean a global solution.

Definition 2. *Let us consider the system* (1), *with the matrices* M_α *verifying the relations* (2).

a) *For* $s \in \mathbb{R}^m$ *and* $y \in \mathbb{R}^n$, *the pair* (s, y) *is called phase of system (1), with condition* $x(t_0) = x_0$, *if* $x(s) = y$, *where* $x(\cdot)$ *is the unique solution of this problem.*

b) *For* $(t_0, x_0), (s, y) \in \mathbb{R}^m \times \mathbb{R}^n$, *we say that the phase* (t_0, x_0) *is transferred to the phase* (s, y) *if the problems* $\{(1), x(t_0) = x_0\}$ *and* $\{(1), x(s) = y\}$ *have the same solution (with the same control* $u(\cdot)$ *for both); or, equivalently, the solution* $x(\cdot)$ *for problem* $\{(1), x(t_0) = x_0\}$ *verifies* $x(s) = y$. *We say that* $u(\cdot)$ *transfers the phase* (t_0, x_0) *to the phase* (s, y).

c) *A phase* (t, x) *is called reachable if there exists* $t_0 \in \mathbb{R}^m$, *with* $t_0^\alpha < t^\alpha$, $\forall \alpha$, *and there exists a control* $u(\cdot)$ *which transfers the phase* $(t_0, 0)$ *to the phase* (t, x).

d) *A phase* (t, x) *is called controllable if there exist* $s \in \mathbb{R}^m$, *with* $s^\alpha > t^\alpha$, $\forall \alpha$, *and a control* $u(\cdot)$ *which transfers the phase* (t, x) *to the phase* $(s, 0)$.

e) *Let* $t_0, t \in \mathbb{R}^m$, *with* $t_0^\alpha < t^\alpha, \forall \alpha$. *The system* (1) *is called completely reachable from* t_0 *to* t, *if for all* $x \in \mathbb{R}^n$, *the phase* $(t_0, 0)$ *is transferred to the phase* (t, x) *; i.e., for all* x, *the phase* (t, x) *is reachable for the same* t_0.

f) *Let* $t \in \mathbb{R}^m$. *The system* (1) *is called completely reachable at the moment* t, *if for all* $t_0 \in \mathbb{R}^m$, *with* $t_0^\alpha < t^\alpha$, $\forall \alpha$, *and for all* $x \in \mathbb{R}^n$, *the phase* $(t_0, 0)$ *is transferred to the phase* (t, x).

g) *Let* $t_0, t \in \mathbb{R}^m$, *with* $t_0^\alpha < t^\alpha, \forall \alpha$. *The system* (1) *is called completely controllable from* t_0 *to* t, *if for all* $x \in \mathbb{R}^n$, *the phase* (t_0, x) *is transferred to the phase* $(t, 0)$; *i.e., for all* x, *the phase* (t_0, x) *is controllable for the same* t.

h) *Let* $t_0 \in \mathbb{R}^m$. *The system* (1) *is called completely controllable at the moment* t_0, *if for all* $t \in \mathbb{R}^m$, *with* $t^\alpha > t_0^\alpha$, $\forall \alpha$, *and for all* $x \in \mathbb{R}^n$, *the phase* (t_0, x) *is tranferred to the phase* $(t, 0)$.

i) *The system* (1) *is called completely reachable if it is completely reachable at any moment in* $\mathbb{R}^m$.
The system (1) *is called completely controllable if it is completely controllable at any moment in* $\mathbb{R}^m$.

Definition 3. *Let us consider that the matrices* $M_\alpha \in M_n(\mathbb{R})$, $\forall \alpha = \overline{1,m}$ *verify the relations* (2). *For each* $\alpha = \overline{1,m}$, *we define the matrix*

$$G_\alpha := \begin{pmatrix} N_\alpha & M_1 N_\alpha & M_2 N_\alpha & \dots & M_m N_\alpha & \dots & M_1^{k_1} \cdot M_2^{k_2} \cdot \dots \cdot M_m^{k_m} N_\alpha & \dots \end{pmatrix}$$

The matrix G_α *contains all block matrices of the form*

$$M_1^{k_1} \cdot M_2^{k_2} \cdot \dots \cdot M_m^{k_m} N_\alpha ,$$

with $0 \le k_1; k_2; \dots; k_m \le n-1$.
Further, the order of the block matrices $M_1^{k_1} \cdot M_2^{k_2} \cdot \dots \cdot M_m^{k_m} N_\alpha$ *in* G_α *has to be specified. In this way, the matrix* G_α *will be well (unique) defined.*
Let us define an order relation on the set $\{(k_1; k_2; \dots; k_m) \mid 0 \le k_1; k_2; \dots; k_m \le n-1\}$, *denoted by* $\prec$:

$$(k_1; k_2; \dots; k_m) \prec (q_1; q_2; \dots; q_m) \text{ iff}$$
$$k_1 + k_2 + \dots + k_m < q_1 + q_2 + \dots + q_m$$
$$\text{or}$$
$$k_1 + k_2 + \dots + k_m = q_1 + q_2 + \dots + q_m \text{ and}$$

$k_1 > q_1$ *or there exists* $j = \overline{2,m}$ *such that* $k_1 = q_1$, $k_2 = q_2$, ..., $k_{j-1} = q_{j-1}, k_j > q_j$, *or* $(k_1; k_2; \dots; k_m) = (q_1; q_2; \dots; q_m)$.

This means that the block matrices $M_1^{k_1} \cdot M_2^{k_2} \cdot \dots \cdot M_m^{k_m} N_\alpha$ *are written in the increasing order of the sum* $k_1 + k_2 + \dots + k_m$ *in* G_α; *in the case when two such sums are equal, the block matrices are written in the decreasing lexicographical order of* $(k_1; k_2; \dots; k_m)$.
The matrix

$$G := \begin{pmatrix} G_1 & G_2 & \dots & G_m \end{pmatrix}$$

is called the controllability matrix of system (1).

Notations: Let K be a field and $A \in M_{m,n}(K)$. We denote by $\mathrm{Im}(A)$ and by $\mathrm{Ker}(A)$ the image and the kernel, respectively, of linear map

$$f : K^n = M_{n,1}(K) \to K^m = M_{m,1}(K), \quad f(x) = Ax.$$

One can readily notice that $\mathrm{Im}(A)$ is the subspace of $K^m = M_{m,1}(K)$ generated by the columns of A.

Theorem 2 ([1]**).** *Let us consider the system* (1), *with the matrices* M_α *verifying the relations* (2). *Moreover, we suppose that*

$$M_\alpha N_\beta N_\beta^T + N_\beta N_\beta^T M_\alpha^T = M_\beta N_\alpha N_\alpha^T + N_\alpha N_\alpha^T M_\beta^T, \qquad \forall \alpha, \beta = \overline{1,m}. \tag{4}$$

i) If $t^\alpha > t_0^\alpha$, $\forall \alpha$ *(or* $t^\alpha < t_0^\alpha$, $\forall \alpha$*), then the phase* (t_0, x_0) *is transferred to the phase* (t, y) *if and only if*

$$x_0 - e^{M_\alpha (t_0^\alpha - t^\alpha)} y \in \mathrm{Im}(G),$$

equivalently,

$$y - e^{M_\alpha (t^\alpha - t_0^\alpha)} x_0 \in \mathrm{Im}(G).$$

ii) The phase (t_0, x_0) *is controllable if and only if* $x_0 \in \mathrm{Im}(G)$. *If there exists* t_0 *such that the phase* (t_0, x_0) *is controllable, then for all* t, *the phases* (t, x_0) *are controllable.*

iii) The phase (t, y) *is reachable if and only if* $y \in \mathrm{Im}(G)$. *If there exists* t *such that the phase* (t, y) *is reachable, then for all* s, *the phases* (s, y) *are reachable.*

iv) If the phase (t_0, x_0) *is controllable (or reachable), then for all* t, *the phases* (t, x_0) *are controllable and reachable.*

v) Let $t_0^\alpha, t^\alpha \in \mathbb{R}^m$, *with* $t_0^\alpha < t^\alpha$, $\forall \alpha$. *The system is completely controllable from* t_0 *to* t *if and only if* rank $G = n$.

vi) Let $t_0^\alpha, t^\alpha \in \mathbb{R}^m$, *cu* $t_0^\alpha < t^\alpha$, $\forall \alpha$. *The system is completely reachable from* t_0 *to* t *if and only if* rank $G = n$.

vii) If there exist $t_0^\alpha, t^\alpha \in \mathbb{R}^m$, *with* $t_0^\alpha < t^\alpha$, $\forall \alpha$, *and if the system is completely controllable (or completely reachable) from* t_0 *to* t, *then the system is completely controllable and completely reachable (equivalent* rank $G = n$*).*

3. Isomorphic systems. The Popov - Belevich - Hautus Theorem

In this section we present the main results of our paper. Recall that M_α, $\tilde{M}_\alpha \in M_n(\mathbb{R})$, N_α, $\tilde{N}_\alpha \in M_{n,k}(\mathbb{R})$ are constant matrices ($\alpha = \overline{1,m}$).

Definition 4. *Let us denote by*

$$\Sigma := \left((M_\alpha)_\alpha ; (N_\alpha)_\alpha \right) \quad \textit{and} \quad \tilde{\Sigma} := \left((\tilde{M}_\alpha)_\alpha ; (\tilde{N}_\alpha)_\alpha \right),$$

respectively, the following autonomous systems

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha x(t) + N_\alpha u_\alpha(t), \qquad \forall \alpha = \overline{1,m}, \tag{Σ}$$

$$\frac{\partial x}{\partial t^\alpha} = \tilde{M}_\alpha x(t) + \tilde{N}_\alpha u_\alpha(t), \qquad \forall \alpha = \overline{1,m}. \tag{$\tilde{\Sigma}$}$$

The systems Σ *and* $\tilde{\Sigma}$ *are isomorphic if there exists an invertible matrix* $T \in M_n(\mathbb{R})$ *such that*

$$\tilde{M}_\alpha = T^{-1} M_\alpha T \quad \textit{and} \quad \tilde{N}_\alpha = T^{-1} N_\alpha, \quad \forall \alpha = \overline{1,m}.$$

In this case, we say that the matrix T *determines an isomorphism between* Σ *and* $\tilde{\Sigma}$.

It is noteworthy that if such T does exist, then the matrices M_α verify relations (2) iff the matrices $\tilde{M}_\alpha$ verify relations (2); similarly, the matrices M_α, N_α verify (4) iff the matrices $\tilde{M}_\alpha$, $\tilde{N}_\alpha$ verify (4).

The isomorphism of autonomous systems is an equivalence relation.

The next proposition contains some straightforward properties regarding the systems Σ and $\tilde{\Sigma}$.

Proposition 1. *Consider the isomorphic systems* Σ *and* $\tilde{\Sigma}$*, via the matrix* T*. Suppose that* Σ *(and hence* $\tilde{\Sigma}$*) verifies relations* (2).

a) The function $(u_\alpha)_{\alpha=\overline{1,m}}$ *is a control for* Σ *if and only if* $(u_\alpha)_{\alpha=\overline{1,m}}$ *is a control for* $\tilde{\Sigma}$*. In other words,* $(u_\alpha)_{\alpha=\overline{1,m}}$ *verifies* (3) *if and only if it also verifies the corresponding relations* (3) *for* $\tilde{\Sigma}$*. So two isomorphic systems have the same control space.*

b) The function $x(\cdot)$ *is the solution of problem* $\{\Sigma,\ x(t_0) = x_0\}$ *if and only if* $\tilde{x}(\cdot) = T^{-1}x(\cdot)$ *is the solution of problem* $\{\tilde{\Sigma},\ \tilde{x}(t_0) = T^{-1}x_0\}$.

c) If G *and* $\tilde{G}$ *are the contollability matrices of the systems* Σ *and* $\tilde{\Sigma}$ *respectively, then* $\tilde{G} = T^{-1}G$*. So, if the relations* (4) *hold, we deducem from Theorem 2 that the system* Σ *is completely controllable if and only if the system* $\tilde{\Sigma}$ *is completely controllable (The last statement remains valid even if relations* (4) *do not hold).*

Proposition 2. *If the matrices* M_α *verify* (2) *and* G *is the contollability matrix of system (1), then* $M_\alpha \operatorname{Im}(G) \subseteq \operatorname{Im}(G)$*, i.e.,* $\operatorname{Im}(G)$ *is an* M_α *invariant subspace (*$\forall \alpha = \overline{1,m}$*).*

Proof. Let $N_{j\beta}$ be the j-th column of the matrix N_β. The set $\operatorname{Im}(G)$ is the subspace of $\mathbb{R}^n = M_{n,1}(\mathbb{R})$ generated by the columns of G, i.e.,

$$M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_m^{k_m} N_{j\beta}, \quad 0 \le k_1; k_2; \ldots; k_m \le n-1, \quad 0 \le j \le k .$$

We have $M_\alpha \cdot \left(M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_m^{k_m} N_{j\beta}\right) = M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_\alpha^{k_\alpha+1} \cdot \ldots \cdot M_m^{k_m} N_{j\beta}$. For $k_\alpha \le n-2$, the last matrix is in $\operatorname{Im}(G)$.

It rests to study the case $k_\alpha = n-1$. Using Hamilton – Cayley Theorem, there exist $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ such that $M_\alpha^n = \sum_{p=0}^{n-1} a_p M_\alpha^p$. So, for $k_\alpha = n-1$, we have

$$M_\alpha \cdot \left(M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_m^{k_m} N_{j\beta}\right) = \sum_{p=0}^{n-1} a_p M_1^{k_1} \cdot M_2^{k_2} \cdot \ldots \cdot M_\alpha^{p} \cdot \ldots \cdot M_m^{k_m} N_{j\beta} \in \operatorname{Im}(G).$$

Theorem 3. *Let us assume that the matrices* $M_\alpha \in M_n(\mathbb{R})$, $N_\alpha \in M_{n,k}(\mathbb{R})$, $\forall \alpha = \overline{1,m}$ *verify the relations* (2) *and* (4). *Let* $\Sigma := \left((M_\alpha)_\alpha ; (N_\alpha)_\alpha\right)$ *be the system*

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha x(t) + N_\alpha u_\alpha(t), \qquad \forall \alpha = \overline{1,m}, \qquad (\Sigma)$$

whose controllability matrix G *has* $\operatorname{rank} G = r$. *Suppose* Σ *is not completely controllable, hence* $r < n$.

If $r = 0$*, then* $N_\alpha = 0$*,* $\forall \alpha$*, hence* $\Sigma := \left((M_\alpha)_\alpha ; (O_{n,k})_\alpha\right)$.

If $1 \le r < n$*, then :*

(1) there exists a system $\tilde{\Sigma} := \left((\tilde{M}_\alpha)_\alpha ; (\tilde{N}_\alpha)_\alpha\right)$*, which is isomorphic with* Σ*, where*

$$\tilde{M}_\alpha = \begin{pmatrix} L_{\alpha,1} & L_{\alpha,2} \\ O_{n-r,r} & L_{\alpha,3} \end{pmatrix}, \quad \tilde{N}_\alpha = \begin{pmatrix} K_{\alpha,1} \\ O_{n-r,k} \end{pmatrix},$$

and $L_{\alpha,1} \in M_r(\mathbb{R})$, $L_{\alpha,2} \in M_{r,n-r}(\mathbb{R})$, $L_{\alpha,3} \in M_{n-r}(\mathbb{R})$, $K_{\alpha,1} \in M_{r,k}(\mathbb{R})$ *;* $O_{n-r,r}$ *and* $O_{n-r,k}$ *are null matrices;*

(2) the system $\Sigma_1 := \left(\left(L_{\alpha,1}\right)_\alpha ; \left(K_{\alpha,1}\right)_\alpha\right)$ *is completely controllable (the system* Σ_1 *verify also the relations* (2) *şi* (4) *).*

In fact, if $\tilde{G}$ *and* G_1 *are the controllability matrices of systems* $\tilde{\Sigma}$ *and* Σ_1 *respectively, then* $\tilde{G} = \begin{pmatrix} G_1 \\ O \end{pmatrix}$. *Also* $L_{\alpha,3}L_{\beta,3} = L_{\beta,3}L_{\alpha,3}$, $\forall \alpha, \beta = \overline{1,m}$.

Proof. It is sufficient to prove the case $r \geq 1$.

We have $\dim(\mathrm{Im}(G)) = r$. Let $\{v_1, v_2, ..., v_r\}$ be a basis in $\mathrm{Im}(G)$. We complete it to a basis $B = \{v_1, v_2, ..., v_r, v_{r+1}, ..., v_n\}$ of $\mathbb{R}^n = M_{n,1}(\mathbb{R})$. Let T be the matrix which has the vector columns $v_1, v_2, ..., v_r, v_{r+1}, ..., v_n$. The matrix T is the switching matrix from the canonical basis to the basis B. We consider the linear application

$$\varphi : \mathbb{R}^n = M_{n,1}(\mathbb{R}) \to \mathbb{R}^n = M_{n,1}(\mathbb{R}), \quad \varphi(x) = M_\alpha x .$$

According to Proposition 2, the set $\mathrm{Im}(G)$ is a φ invariant subspace. It follows that the matrix of φ with respect to the basis B, has the form:

$$\begin{pmatrix} L_{\alpha,1} & L_{\alpha,2} \\ O_{n-r,r} & L_{\alpha,3} \end{pmatrix}, \tag{5}$$

where $L_{\alpha,1} \in M_r(\mathbb{R})$, $L_{\alpha,2} \in M_{r,n-r}(\mathbb{R})$, $L_{\alpha,3} \in M_{n-r}(\mathbb{R})$. On the other hand, the associated matrix of φ , with respect to the canonical basis, is M_α. The relation between the two matrices is

$$T^{-1}M_\alpha T = \begin{pmatrix} L_{\alpha,1} & L_{\alpha,2} \\ O_{n-r,r} & L_{\alpha,3} \end{pmatrix}. \tag{6}$$

We choose $\tilde{M}_\alpha$ as the matrix (5).

Let g be the number of columns for G .We consider the linear application

$$\psi : \mathbb{R}^g = M_{g,1}(\mathbb{R}) \to \mathbb{R}^n = M_{n,1}(\mathbb{R}), \quad \psi(y) = G\,y .$$

Denote by $M_{can,B}(\psi)$ the matrix associated to ψ, as linear application from $\mathbb{R}^g$ with the canonical basis, to $\mathbb{R}^n$ with the basis B. Denote by $M_{can,can}(\psi)$ the matrix associated to ψ, as linear application from $\mathbb{R}^g$ with canonical basis (in $\mathbb{R}^g$), to $\mathbb{R}^n$ with canonical basis (in $\mathbb{R}^n$). Denote by $M_{B,can}(Id)$ the matrix associated to the identity on $\mathbb{R}^n$ with basis B, to $\mathbb{R}^n$ with canonical basis.

$$\left(\mathbb{R}^g, can\right) \xrightarrow{\psi} \left(\mathbb{R}^n, B\right) \xrightarrow{Id} \left(\mathbb{R}^n, can\right)$$

From the equality $Id \circ \psi = \psi$, passing to the associated matrices we find

$$M_{B,can}\left(Id\right) \cdot M_{can,B}\left(\psi\right) = M_{can,can}\left(\psi\right). \tag{7}$$

But $M_{can,can}\left(\psi\right) = G$ and $M_{B,can}\left(Id\right) = T$.

Since $\psi(y) = Gy \in \operatorname{Im}\left(G\right) = Sp\left\{v_1, v_2, ..., v_r\right\}$, the matrix $M_{can,B}\left(\psi\right)$ is of the form $M_{can,B}\left(\psi\right) = \begin{pmatrix} K \\ O_{n-r,g} \end{pmatrix}$, with $K \in M_{r,g}\left(\mathbb{R}\right)$. The relation (7) becomes

$$T\begin{pmatrix} K \\ O_{n-r,g} \end{pmatrix} = G \quad \text{or} \quad T^{-1}G = \begin{pmatrix} K \\ O_{n-r,g} \end{pmatrix}. \tag{8}$$

If we multiply, at the left, the columns of N_α (which are columns of the matrix G), by the relation (8), we obtain

$$T^{-1}N_\alpha = \begin{pmatrix} K_{\alpha,1} \\ O_{n-r,k} \end{pmatrix}, \quad \text{with} \quad K_{\alpha,1} \in M_{r,k}\left(\mathbb{R}\right). \tag{9}$$

We choose as $\tilde{N}_\alpha = \begin{pmatrix} K_{\alpha,1} \\ O_{n-r,k} \end{pmatrix}$. From (6) şi (9) it follows that T determines an isomorphism between systems Σ and $\tilde{\Sigma}$. Hence the matrices $\tilde{M}_\alpha$, $\tilde{N}_\alpha$ verify the relations (2) şi (4) corresponding to the system $\tilde{\Sigma}$ (see the remarks after the Definition 4).

We have $\tilde{M}_\beta \tilde{N}_\alpha = \begin{pmatrix} L_{\beta,1}K_{\alpha,1} \\ O_{n-r,k} \end{pmatrix}$ and

$$\tilde{M}_1^{k_1} \cdot \tilde{M}_2^{k_2} \cdot ... \cdot \tilde{M}_m^{k_m} \cdot \tilde{N}_\alpha = \begin{pmatrix} L_{1,1}^{k_1} \cdot L_{2,1}^{k_2} \cdot ... \cdot L_{m,1}^{k_m} \cdot K_{\alpha,1} \\ O_{n-r,k} \end{pmatrix}, \quad \forall k_1; k_2; ...; k_m \geq 0 .$$

It follows $\tilde{G} = \begin{pmatrix} G_1 \\ O_{n-r,g} \end{pmatrix}$. The relations (2) for $\tilde{\Sigma}$ become

$$\tilde{M}_\alpha \tilde{M}_\beta = \tilde{M}_\beta \tilde{M}_\alpha \text{ or } \begin{pmatrix} L_{\alpha,1}L_{\beta,1} & ... \\ O_{n-r,r} & L_{\alpha,3}L_{\beta,3} \end{pmatrix} = \begin{pmatrix} L_{\beta,1}L_{\alpha,1} & ... \\ O_{n-r,r} & L_{\beta,3}L_{\alpha,3} \end{pmatrix}.$$

Hence, the relations (2) are true for Σ_1, too. And it follows the equalities $L_{\alpha,3}L_{\beta,3} = L_{\beta,3}L_{\alpha,3}$.

The relations (4) for $\tilde{\Sigma}$ are

$$\tilde{M}_\alpha \tilde{N}_\beta \tilde{N}_\beta^T + \tilde{N}_\beta \tilde{N}_\beta^T \tilde{M}_\alpha^T = \tilde{M}_\beta \tilde{N}_\alpha \tilde{N}_\alpha^T + \tilde{N}_\alpha \tilde{N}_\alpha^T \tilde{M}_\beta^T$$

or

$$\begin{pmatrix} L_{\alpha,1} K_{\beta,1} K_{\beta,1}{}^T + K_{\beta,1} K_{\beta,1}{}^T L_{\alpha,1}^T \\ O_{n-r,k} \end{pmatrix} = \begin{pmatrix} L_{\beta,1} K_{\alpha,1} K_{\alpha,1}{}^T + K_{\alpha,1} K_{\alpha,1}{}^T L_{\beta,1}{}^T \\ O_{n-r,k} \end{pmatrix},$$

so the relations (4) are true for system Σ_1, too.

The system Σ_1 is completely controllable because

$$r = \text{rank } G = \text{rank } \tilde{G} = \text{rank } G_1 .$$

Lemma. *Let $F \neq \varnothing$, $F \subseteq M_n(\mathbb{C})$, such that each pair in the set F commutes under multiplication. Then there is $w \neq 0$, $w \in \mathbb{C}^n = M_{n,1}(\mathbb{C})$, which is an eigenvector of every matrix in F* (see [16]).

Theorem 4. *Let us consider that the matrices $M_\alpha \in M_n(\mathbb{R})$, $N_\alpha \in M_{n,k}(\mathbb{R})$, $\forall \alpha = \overline{1,m}$ verify the relations* (2) *and* (4). *Let $\Sigma := \left((M_\alpha)_\alpha ; (N_\alpha)_\alpha \right)$ be the PDE system*

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha x(t) + N_\alpha u_\alpha(t), \qquad \forall \alpha = \overline{1,m}. \tag{Σ}$$

Then the system Σ is completely controllable if and only if does not exist $v \in \mathbb{C}^n = M_{n,1}(\mathbb{C})$, $v \neq 0$, with the properties

i) v^T *is a left eigenvector of each matrix* M_α, $\forall \alpha = \overline{1,m}$;

ii) $v^T N_\alpha = 0$, $\forall \alpha = \overline{1,m}$.

Proof. Let Σ completely controllable system. Suppose, by contradiction that there exists $v \neq 0$ with the properties $i)$ and $ii)$; i.e., there exist $\lambda_\alpha \in \mathbb{C}$ with $v^T M_\alpha = \lambda_\alpha v^T$, $\forall \alpha = \overline{1,m}$. So, for any $k_1, k_2, ..., k_m \geq 0$, we have

$$v^T M_1^{k_1} \cdot M_2^{k_2} \cdot ... \cdot M_m^{k_m} \cdot N_\alpha = \lambda_1^{k_1} \cdot \lambda_2^{k_2} \cdot ... \cdot \lambda_m^{k_m} \cdot v^T N_\alpha = 0,$$

hence $v^T G = 0$. Since $v \neq 0$, it follows $\text{rank}\, G < n$, contradiction with Σ completely controllable (where G is the controllability matrix of the system Σ).

Conversely, suppose that $v \neq 0$, with the properties $i)$ and $ii)$, does not exist. Suppose, by contradiction that Σ is not completely controllable, that is $r := \text{rank}\, G < n$. We apply the Theorem 3 to the case $r > 0$. Let T be a matrix which determines an isomorphism between Σ and $\tilde{\Sigma}$ (where $\tilde{\Sigma}$ is the system from Theorem 3).

Consider the set $F=\left\{L_{1,3}^T, L_{2,3}^T, \ldots, L_{m,3}^T\right\}$. Since

$$L_{\alpha,3} L_{\beta,3}=L_{\beta,3} L_{\alpha,3}, \quad \forall \alpha=\overline{1, m},$$

we may apply the above Lemma. So there exist $w \neq 0$, $w \in \mathbb{C}^{n-r}=M_{n-r,1}(\mathbb{C})$ and $\lambda_\alpha \in \mathbb{C}$ such that

$$L_{\alpha,3}^T w=\lambda_\alpha w \Leftrightarrow w^T L_{\alpha,3}=\lambda_\alpha w^T, \quad \forall \alpha=\overline{1, m}.$$

Let us choose $v=\left(T^{-1}\right)^T\begin{pmatrix} O_{r,1} \\ w \end{pmatrix} \in M_{n,1}(\mathbb{C})=\mathbb{C}^n$. Clearly, v is nonzero and

$$v^T M_\alpha=\begin{pmatrix} O_{1,r} & w^T \end{pmatrix} T^{-1} \cdot T \tilde{M}_\alpha T^{-1}=$$

$$=\begin{pmatrix} O_{1,r} & w^T \end{pmatrix}\begin{pmatrix} L_{\alpha,1} & L_{\alpha,2} \\ O_{n-r,r} & L_{\alpha,3} \end{pmatrix} T^{-1}=\begin{pmatrix} O_{1,r} & w^T L_{\alpha,3} \end{pmatrix} T^{-1}=$$

$$=\begin{pmatrix} O_{1,r} & \lambda_\alpha w^T \end{pmatrix} T^{-1}=\lambda_\alpha\begin{pmatrix} O_{1,r} & w^T \end{pmatrix} T^{-1}=\lambda_\alpha v^T.$$

$$v^T N_\alpha=\begin{pmatrix} O_{1,r} & w^T \end{pmatrix} T^{-1} \cdot T \tilde{N}_\alpha=\begin{pmatrix} O_{1,r} & w^T \end{pmatrix}\begin{pmatrix} K_{\alpha,1} \\ O_{n-r,k} \end{pmatrix}=0.$$

Hence v satisfies $i)$ and $ii)$, that is a contradiction.

The case $r=0$ can be similarly treated. It is sufficient to apply the above Lemma to the matrices M_α in order to obtain a contradiction.

Theorem 5. (Popov - Belevich - Hautus Theorem)

Let us consider that the matrices $M_\alpha \in M_n(\mathbb{R})$, $N_\alpha \in M_{n,k}(\mathbb{R})$, $\forall \alpha=\overline{1, m}$ *verify relations* (2) *and* (4). *Let* $\Sigma:=\left(\left(M_\alpha\right)_\alpha ;\left(N_\alpha\right)_\alpha\right)$ *be the system*

$$\frac{\partial x}{\partial t^\alpha}=M_\alpha x(t)+N_\alpha u_\alpha(t), \qquad \forall \alpha=\overline{1, m}. \tag{Σ}$$

Then, the system Σ *is completely controllable if and only if the following matrix*

$$\begin{pmatrix} M_1-s_1 I_n & N_1 & \ldots & M_\alpha-s_\alpha I_n & N_\alpha & \ldots & M_m-s_m I_n & N_m \end{pmatrix} \tag{10}$$

has rank n, $\forall s_1, s_2, \ldots, s_m \in \mathbb{C}$ *(we remark that it is sufficient to take* s_α *an eigenvalue for* M_α*).*

Proof. Let Σ be a completely controllable system. Assume that there exist $s_1, s_2, \ldots, s_m$ such that the rank of the matrix in (10) is $<n$. Then, there is a $v \in M_{n,1}(\mathbb{C})$, $v \neq 0$, such that v^T vanishes the matrix in (10), at the left. This implies that v verifies $i)$ and $ii)$. According to Theorem 4, we obtain that Σ is not completely controllable, which false.

Conversely, if the rank of the matrices in (10) is n, $\forall s_\alpha \in \mathbb{C}$, let us suppose that Σ is not completely controllable. Then, due to Theorem 4, there exist $v \neq 0$ and some eigenvalues λ_α of M_α such that $v^T\left(M_\alpha - \lambda_\alpha I_n\right) = 0$ and $v^T N_\alpha = 0$, $\forall \alpha = 1,2,...,m$; so v^T vanishes the matrix in (10) at the left, whenever $s_\alpha = \lambda_\alpha$, $\forall \alpha = 1,2,...,m$. Since $v \neq 0$, we deduce that the rank of this matrix is $< n$, which is false.

4. Remarks and conclusions

Let us notice that the **Popov - Belevich - Hautus Theorem** is not valid if relations (4) do not hold, even if relations (2) are satisfied. For example, for $m = 3$, $n = 3$, $k = 1$,

$$M_1 = M_2 = M_3 \overset{not}{=} M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ şi } N_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ N_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ N_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

They satisfy $MN_1 = N_2$, $MN_2 = N_3$, $MN_3 = N_1$.

It is easy to see that relations (2) are verified. Let us determine the control space. The vector matrix $u = \left(u_1, u_2, u_3\right)$ is a control if and only if the relations (3) hold. If, in (3) we take $\alpha = 2$, $\beta = 3$, then

$$M_2 N_3 u_3(t) + N_2 \frac{\partial u_2}{\partial t^3}(t) = M_3 N_2 u_2(t) + N_3 \frac{\partial u_3}{\partial t^2}(t), \quad \forall t \in \mathbb{R}^3,$$

or

$$u_3(t) N_1 + \frac{\partial u_2}{\partial t^3}(t) N_2 = u_2(t) N_3 + \frac{\partial u_3}{\partial t^2}(t) N_3.$$

Since N_1, N_2, N_3 are linearly independent, we get that $u_3(t) = 0$ and

$$u_2(t) + \frac{\partial u_3}{\partial t^2}(t) = 0, \quad \text{that is,} \quad u_2(t) = u_3(t) = 0, \quad \forall t \in \mathbb{R}^3.$$

Similarly, if in (3) we take $\alpha = 1$, $\beta = 2$, then we obtain $N_1 \frac{\partial u_1}{\partial t^2}(t) = N_2 u_1(t)$, so $u_1(t) = 0$, $\forall t \in \mathbb{R}^3$.

We conclude that the control space is null. The only one solution of the system (1) which vanishes at some point is therefore the null solution. We deduce that there are no controllable states $\left(t_0, x_0\right)$, with $x_0 \neq 0$.

The matrices (10) have rank $3(=n)$, $\forall s_1, s_2, ..., s_m \in \mathbb{C}$, so Theorem 5 is not applicable in this case, because relations (4) do not hold.

Our original contributions are contained in Theorem 3, Theorem 4 and Theorem 5 (**Popov - Belevich – Hautus Theorem**). In proving these results for the multitime case, we have used special techniques from the theory of m – flow type systems as well as geometric interpretations of the m – dimensional evolutions. Therefore our results complete the theory of the papers [1] – [8].

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