EXIT TIMES FOR GEOMETRIC BROWNIAN MOTION

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Geometric Brownian motion(GBM) process has been applied to numerous fields including finance, insurance, engineering and so on. In this paper, the surplus process of an insurance company are modeled by GBM. Some exit times for the process is studied, and the Laplace-Stieltjes transform (LST) of the exit times is obtained. Then, numerical examples are given to illustrate the applications of the LST of some exit times.

Keywords: exit time, strong Markov property, geometric Brownian motion, martingale.
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1. Introduction

Geometric Brownian motion(GBM) process was initially proposed by Fischer Black and Myron Scholes(1973)[2]. Since then, the model has been applied to numerous fields including finance, insurance, engineering and has been further studied by many authors. For example, Postali and Picchetti(2006)[14] showed that GBM process performs well as a proxy for the movement of oil prices and for a state variable to evaluate oil deposits. Gao and Yin(2008)[7] considered the perturbed classical compound Poisson risk model compounded by GBM with a constant dividend barrier strategy. Vajargah and Shoghi(2015) [15] studied the simulation of stochastic differential equation of GBM by quasi-Monte Carlo method and its application in prediction of total index of stock market and value at risk. The surplus process $\{X(t), t \geq 0\}$ of an insurance company are modeled by GBM as follows

$$\begin{cases} \mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}B(t), \\ X(0) = x, \end{cases} \tag{1}$$

where μ be the drift in which represents deterministic trends, $\sigma > 0$ be the volatility refer to the influence of unpredictable events, x is the initial surplus of an insurance company, and $\{B(t), t \geq 0\}$ is a standard Brownian motion in which the mean change in the value of the variable is zero and the variance of change equal to one per unit time.

For any interval $[b, a]$, where $b < u < a$, define the first hitting time of the upper barrier a and the first hitting time of the lower barrier b for the risk process $\{X(t), t \geq 0\}$ to be

$$T_a = \begin{cases} \inf\{t \geq 0, X(t) = a\}, \\ \infty, \quad \text{if} X(t) \neq a \text{ for all } t \geq 0, \end{cases} \quad T_b = \begin{cases} \inf\{t \geq 0, X(t) = b\}, \\ \infty, \quad \text{if } X(t) \neq b \text{ for all } t \geq 0. \end{cases}$$

Then $T_{a,b} = T_a \wedge T_b := \min(T_a, T_b)$ is the first exit time of the process $\{X(t), t \geq 0\}$ from the interval (b, a). For a boundary a and a fixed $\alpha > 0$, let the Laplace-Stieltjes transforms

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(LST) of T_a is $E_x[e^{-\alpha T_a}] = E[e^{-\alpha T_a}|X(0) = x]$. Then the probability of $T_a < \infty$ can be defined as follows

$$P_x(T_a < \infty) = P(T_a < \infty|X(0) = x) = \lim_{\alpha\to 0} E_x[e^{-\alpha T_a}]. \tag{2}$$

The mathematical expectation of T_a can be obtained by the formula

$$E_x[T_a I(T_a < \infty)] = E[T_a I(T_a < \infty)|X(0) = x] = -\lim_{\alpha\to 0} \frac{\mathrm{d}(E_x[e^{-\alpha T_a}])}{\mathrm{d}\alpha}, \tag{3}$$

where $I(\cdot)$ is the indicator function. In particular, when $a = 0$, $P_x(T_0 < \infty)$ is the probability of ultimate ruin and $E_x[T_0 I(T_0 < \infty)]$ is the average of the ruin time. For an interval $[b, a]$, let the LST of T_a for $T_a < T_b$ is $E_x[e^{-\alpha T_a} I(T_a < T_b)]$ and the LST of T_b for $T_b < T_a$ is $E_x[e^{-\alpha T_b} I(T_b < T_a)]$. In particular, when $b = 0$, the former can be represented as the expected present value of a payment of 1 due at the time when the surplus reaches the level a for the first time, provided that ruin has not occurred in the meantime, which plays an important role in dividend problems with barrier strategies, the latter can be represented as the LST of the time of ruin before reaching the upper barrier a, which plays a key role in the problems of negative duration and occupation time. Then the probability of $T_a < T_b$ and $T_b < T_a$ can be defined as follows

$$\begin{aligned} P_x(T_a < T_b) &= \lim_{\alpha\to 0} E_x[e^{-\alpha T_a} I(T_a < T_b)], \\ P_x(T_b < T_a) &= \lim_{\alpha\to 0} E_x[e^{-\alpha T_b} I(T_b < T_a)]. \end{aligned} \tag{4}$$

The mathematical expectation of the first exit time from the upper barrier a and the lower barrier b can be derived by the formulas

$$\begin{aligned} E_x[T_a I(T_a < T_b)] &= -\lim_{\alpha\to 0} \frac{\mathrm{d}(E_x[e^{-\alpha T_a} I(T_a<T_b)])}{\mathrm{d}\alpha}, \\ E_x[T_b I(T_b < T_a)] &= -\lim_{\alpha\to 0} \frac{\mathrm{d}(E_x[e^{-\alpha T_b} I(T_b<T_a)])}{\mathrm{d}\alpha}. \end{aligned} \tag{5}$$

Correspondingly, the relation between LST of the exit times T_a, T_b and $T_{a,b}$ as follows

$$E_x[e^{-\alpha T_{a,b}}] = E_x[e^{-\alpha T_a} I(T_a < T_b)] + E_x[e^{-\alpha T_b} I(T_b < T_a)].$$

This paper investigates the LST of some exit times for GBM process. The exit times have been studied by many authors in some risk models, such as Gerber(1990)[9] and Alfredo and Dos Reis(1993)[6] studied some stopping times of the classical risk process. Kella and Stadje(2001)[12] and Perry and Stadje(2001)[13] considered some exit times of the compound Poisson risk process. Chiu and Yin (2002, 2002, 2005)[3][4][5] investigated some passage times of the reserve-dependent risk process and the spectrally negative Lévy process. Alili, Patie and Pedersen(2005)[1] considered the first hitting time of an Ornstein-Uhlenbeck process. Jacobsen and Jensen(2007)[10] considered the exit times of the piecewise exponential Markov processes with two-sided jumps.

The remainder of the paper is organized as follows. In *section* 2, some preliminaries of GBM process are given. In *section* 3, using infinitesimal operators and martingale, the LST of some exit times are obtained. Then, several numerical examples are discussed in order to illustrate the influence of the upper barrier and lower barrier of an interval and the initial surplus on the exit times in *section* 4.

2. **Preliminaries**

The model (1) is a time-homogeneous Markov process (see Karatzas and Shreve(1991)[11]) taking values in $\mathbb{R}$ with generator $\mathscr{A}$ that satisfies

$$\mathscr{A} f(z) = \frac{\sigma^2}{2} z^2 f''(z) + \mu z f'(z),$$

where f belongs to the domain $\mathscr{D}(\mathscr{A})$ of the generator $\mathscr{A}$ of $\{X(t), t \geq 0\}$. Furthermore $(X(t), t)$ is also Markovian with generator $\mathscr{B}$ that satisfies

$$\mathscr{B}h(z,t) = \mathscr{A}h(z,t) + \tfrac{\partial}{\partial t}h(z,t), \tag{6}$$

provided that $h(z,\cdot)$ has a continuous first derivative for each z and that, for each t, $h(\cdot,t)$ is in the domain of $\mathscr{A}$, then $h(z,t) \in \mathscr{D}(\mathscr{B})$. Denote by $\mathscr{F}_t = \sigma\{X(s), 0 < s \leq t\}$ the natural filtration. For later use, we give the following Lemma.

Lemma 2.1. *If $h(z,t)$ is a twice continuously differentiable in z and once in t function with bounded first derivative in z, then $h(z,t) \in \mathscr{D}(\mathscr{B})$ and furthermore*

$$M_h(t) = h(X(t),t) - \textstyle\int_0^t \mathscr{B}h(X(s),s)ds$$

is a martingale.

In order to obtain the LST of some exit times, it is required to find a solution of the equation as follows

$$\mathscr{A}f(z) = \alpha f(z), \quad \text{for } \alpha > 0, \tag{7}$$

that is

$$\tfrac{\sigma^2}{2}z^2 f''(z) + \mu z f'(z) = \alpha f(z).$$

The above equation is a second-order linear differential equation, the general solution is a linear combination of the form $C_1h_1(z)+C_2h_2(z)$, where C_1, C_2 are arbitrary constants and the two positive independent solutions h_1, h_2 as follows

$$h_1(z) = z^{\frac{-2\sqrt{2}\mu+\sqrt{2}\sigma^2-\sqrt{8\mu^2+16\alpha\sigma^2-8\mu\sigma^2+2\sigma^4}}{2\sqrt{2}\sigma^2}},$$
$$h_2(z) = z^{\frac{-2\sqrt{2}\mu+\sqrt{2}\sigma^2+\sqrt{8\mu^2+16\alpha\sigma^2-8\mu\sigma^2+2\sigma^4}}{2\sqrt{2}\sigma^2}}.$$

It is easy to verify that h_1 is strictly decreasing, h_2 is strictly increasing and $h_1(z) \to 0$ as $z \to +\infty$.

3. The LST of some exit times

Theorem 3.1. *Given that the initial surplus $0 < a < x$, the LST of the time to hit a boundary a is given by*

$$E_x[e^{-\alpha T_a}] = \tfrac{f_1(x)}{f_1(a)}, \tag{8}$$

where $f_1(z) = h_1(z)$.

Proof. Let $h(z,t)$ takes the form $h(z,t) = \mathrm{e}^{-\alpha t}f_1(z)$, it is clear that $h(z,t)$ is in the domain of $\mathscr{B}$. It follows from (6) and (7) that

$$\mathscr{B}h(z,t) = \mathscr{A}h(z,t) + \tfrac{\partial}{\partial t}h(z,t) = 0, \quad \text{for all } z \text{ and } t > 0.$$

By *Lemma* 2.1, one can get $E_x[e^{-\alpha t}f_1(X(t)] = f_1(x)$. Thus, for every stopping time T_a and initial surplus x, one have

$$E_x[e^{-\alpha(t\wedge T_a)}f_1(X(t\wedge T_a))] = f_1(x). \tag{9}$$

Because $f_1(z)$ is bounded on the range of possible values of $\{X(t\wedge T_a), t \geq 0\}$, letting $t \to \infty$ in (9), dominated convergence theorem yields

$$E_x[e^{-\alpha T_a}f_1(X(T_a))] = f_1(a)E_x[e^{-\alpha T_a}] = f_1(x).$$

So that

$$E_x[e^{-\alpha T_a}] = \frac{f_1(x)}{f_1(a)}.$$

This completes the proof.

Theorem 3.2. *For $0 < b < x < a$, the LST of the first exit time from the upper barrier a is given by*

$$E_x[e^{-\alpha T_a} I(T_a < T_b)] = \tfrac{f_2(x)}{f_2(a)}, \tag{10}$$

where $f_2(z) = C_1 h_1(z) + C_2 h_2(z)$ and C_1, C_2 satisfies $f_2(b) = 0$.

Proof. Let $h(z,t)$ takes the form $h(z,t) = \mathrm{e}^{-\alpha t} f_2(z)$, it is clear that $h(z,t)$ is in the domain of $\mathscr{B}$. It follows from (6) and (7) that

$$\mathscr{B}h(z,t) = \mathscr{A}h(z,t) + \tfrac{\partial}{\partial t}h(z,t) = 0, \quad \text{for all } z \text{ and } t > 0.$$

By *Lemma* 2.1, one can get $E_x[e^{-\alpha t} f_2(X(t)] = f_2(x)$. Thus, for every stopping time $T_{a,b}$ and initial surplus x, one have

$$E_x[e^{-\alpha(t\wedge T_{a,b})} f_2(X(t \wedge T_{a,b}))] = f_2(x). \tag{11}$$

Letting $t \to \infty$ in (11), dominated convergence theorem yields

$$E_x[e^{-\alpha T_{a,b}} f_2(X(T_{a,b}))] = f_2(x).$$

By the definitions of $T_{a,b}$, one have

$$E_x[e^{-\alpha T_a} f_2(X(T_a)) I(T_a < T_b)] + E_x[e^{-\alpha T_b} f_2(X(T_b)) I(T_b < T_a)] = f_2(x).$$

It follows from $f_2(b) = 0$ that

$$E_x[e^{-\alpha T_a} f_2(X(T_a)) I(T_a < T_b)] = f_2(a) E_x[e^{-\alpha T_a} I(T_a < T_b)] = f_2(x).$$

Thus,

$$E_x[e^{-\alpha T_a} I(T_a < T_b)] = \frac{f_2(x)}{f_2(a)}.$$

This completes the proof.

Theorem 3.3. *For $0 < b < x < a$, the LST of the first exit time from the lower barrier b is given by*

$$E_x[e^{-\alpha T_b} I(T_b < T_a)] = \tfrac{f_3(x)}{f_3(b)}, \tag{12}$$

where $f_3(z) = C_1 h_1(z) + C_2 h_2(z)$ and C_1, C_2 satisfies $f_3(a) = 0$.

Proof. Let $h(z,t)$ takes the form $h(z,t) = \mathrm{e}^{-\alpha t} f_3(z)$, it is clear that $h(z,t)$ is in the domain of $\mathscr{B}$. It follows from (6) and (7) that

$$\mathscr{B}h(z,t) = \mathscr{A}h(z,t) + \tfrac{\partial}{\partial t}h(z,t) = 0, \quad \text{for all } z \text{ and } t > 0.$$

By *Lemma* 2.1, one can get $E_x[e^{-\alpha t} f_3(X(t)] = f_3(x)$. Thus, for every stopping time $T_{a,b}$ and initial surplus x, one have

$$E_x[e^{-\alpha(t\wedge T_{a,b})} f_3(X(t \wedge T_{a,b}))] = f_3(x). \tag{13}$$

Letting $t \to \infty$ in (13), dominated convergence theorem yields

$$E_x[e^{-\alpha T_{a,b}} f_3(X(T_{a,b}))] = f_3(x).$$

By the definitions of $T_{a,b}$, one have

$$E_x[e^{-\alpha T_a} f_3(X(T_a)) I(T_a < T_b)] + E_x[e^{-\alpha T_b} f_3(X(T_b)) I(T_b < T_a)] = f_3(x).$$

It follows from $f_3(a) = 0$ that

$$E_x[e^{-\alpha T_b} f_3(X(T_b)) I(T_b < T_a)] = f_3(b) E_x[e^{-\alpha T_b} I(T_b < T_a)] = f_3(x).$$

Thus,

$$E_x[e^{-\alpha T_b} I(T_b < T_a)] = \frac{f_3(u)}{f_3(b)}.$$

This completes the proof.

4. **Numerical examples**

In this section, numerical examples are presented to illustrate the application of the exit times for GBM process. Gao(2010)[8] used GMB to describe stock price indices and estimated $\mu = 0.53512, \sigma = 0.30758$ by means of the Shanghai Composite Index (000001) closing index from December 1, 2008 to November 30, 2009. Wang(2007)[16]used GMB to depict the price of oil and obtained $\mu = 0.1, \sigma = 0.11$ with the aid of an oil project in the Gulf of Mexico from 1970 to 1997. In describing oil prices and some stock prices with GBM, the drift coefficient and volatility coefficient are roughly equivalent. Thus, we take the drift coefficient and volatility coefficient as one unit to study. All illustrations will be based on the parameters $\mu = 1$ and $\sigma = 1$.

By *Theorem* 3.1, we study the LST of exit time T_a. The LST of T_a are plotted in *Figure* 1 for different initial surplus and different boundaries. It follows from (2), (3) that the numerical characteristics of T_a are obtained in *Table* 1.

FIGURE 1. The LST of T_a

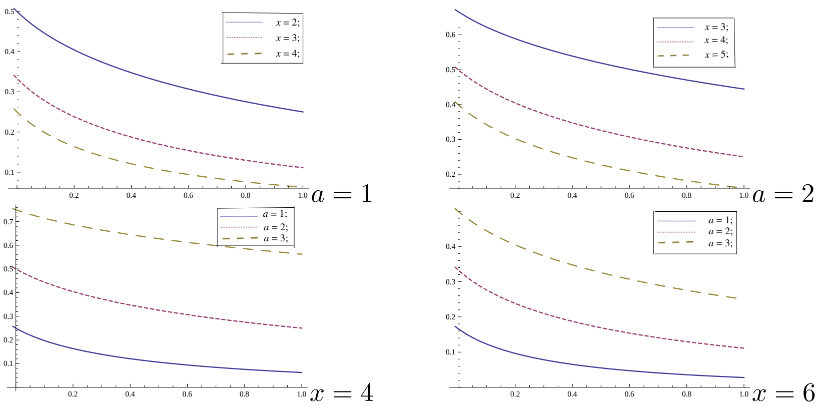

TABLE 1. The numerical characteristics of T_a

a	b	x	$P_x[T_a < \infty]$	$E_x[T_a]$
1	b	3	0.3333333333333333	0.7324081924454064‘
1	b	4	0.25	0.6931471805599453‘
2	b	3	0.6666666666666666	0.5406201441442191‘
2	b	4	0.5	0.6931471805599453‘
2	b	5	0.4	0.7330325854993242‘
3	b	4	0.75	0.4315231086776712‘
3	b	5	0.6	0.6129907485191889‘

The results in *Figure* 1 and *Table* 1 show that, the LST of exit time T_a and the probability of $T_a < \infty$ trend to decreasing along with the initial surplus increased, the mathematical expectation of T_a trends to increasing along with the initial surplus increased.

Note that, on one hand, $E_x[e^{-\alpha T_a}]$ is decreasing as α increases and increasing as a increases. On the other hand, $E_x[e^{-\alpha T_a}]$ and $P_x(T_a < \infty)$ are increasing as a increases, and $E_x[T_a]$ is decreasing as a increases.

By *Theorem* 3.2 and *Theorem* 3.3, we study the LST of the first exit times from an interval $[b, a]$. The LST of T_a for $T_a < T_b$ are plotted in *Figure* 2 for different upper barriers of an interval. The LST of T_b for $T_b < T_a$ are plotted in *Figure* 3 for different lower barriers of an interval. It follows from (4) and (5) that the numerical characteristics of T_a and T_b are obtained in *Table* 2.

Figure 2. The LST of T_a for $T_a < T_b$

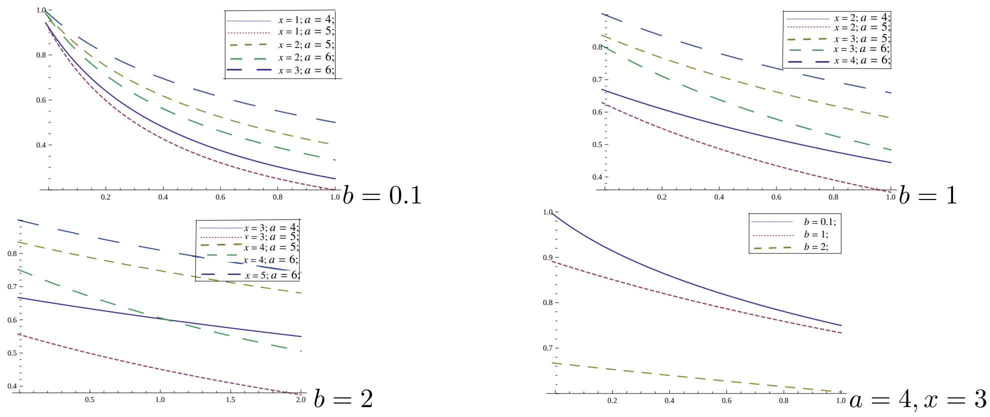

Figure 3. The LST of T_b for $T_b < T_a$

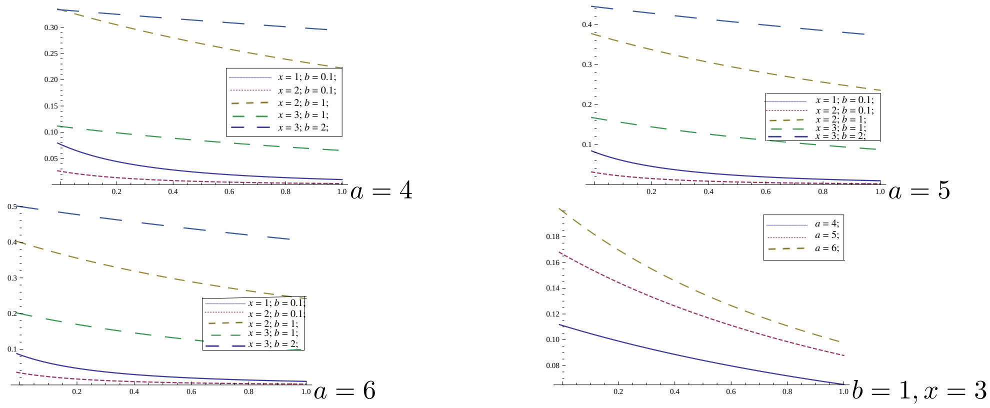

Table 2. The numerical characteristics of T_a and T_b

b	x	a	$P_x[T_a < T_b]$	$P_x[T_b < T_a]$	$E_x[T_a I(T_a < T_b)]$	$E_x[T_b I(T_b < T_a)]$
0.1	1	5	0.9183673469387754	0.08163265306122448	2.309559326812696	0.27061886451621503
0.1	2	5	0.9693877551020407	0.03061224489795918	1.474680930389038	0.11838892078203708
0.1	2	6	0.9661016949152542	0.033898305084745756	1.7816140086446755	0.13802788650699355
0.1	3	6	0.9830508474576269	0.016949152542372878	1.1744964314976283	0.07300658852998436
1	2	5	0.625	0.375	0.4183941587141433‘	0.2071088707085913‘
1	3	5	0.8333333333333334	0.1666666666666666	0.3615538188582179‘	0.12361812452906257‘
1	3	6	0.8	0.2	0.49798188733289184‘	0.17160868609577648‘
2	3	4	0.6666666666666667	0.3333333333333333	0.06948800151868384‘	0.04377802301158035‘
2	3	5	0.5555555555555556	0.4444444444444444	0.12298463018393169‘	0.08418596679324386‘
2	4	6	0.75	0.25	0.17667455348457528‘	0.08494951839769871‘

The results in $Figure\ 2-3$ and $Table$ 2 show that the LST of T_a for $T_a < T_b$ and the LST of T_b for $T_b < T_a$ is decreasing as α increases. It Follows from $Figure$ 2 and $Table$ 2 that, the LST of T_a for $T_a < T_b$ and the probability of $T_a < T_b$ trend to increasing along with the initial surplus increased, the mathematical expectation of the first exit time from the upper barrier a trends to decreasing along with the initial surplus increased. Note that, on one hand, $E_x[e^{-\alpha T_a} I(T_a < T_b)]$ and $P_x(T_a < T_b)$ are decreasing as a or b increases. On the other hand, $E_x[T_a I(T_a < T_b)]$ is increasing as a increases and decreasing as b increases. It Follows from $Figure$ 3 that, the situation is converse, the LST of T_b for $T_b < T_a$, the probability of $T_b < T_a$ and the mathematical expectation of the first exit time from the lower barrier b trend to decreasing along with the initial surplus increased. Note that, $E_x[e^{-\alpha T_b} I(T_b < T_a)]$, $P_x(T_b < T_a)$ and $E_x[T_b I(T_b < T_a)]$ are increasing as a or b increases. It means that the change of a and b has a great influence on its value.

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