RANDOM VARIABLES ASSOCIATED TO A SYSTEM OF n POINTS IN THE SPACE M_2

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In această lucrare se consideră acoperirea convexă I_n a unui sistem de n puncte aleatoare, independente şi repartizate uniform în domeniul convex $D \subset M_2$ cu frontiera netedă şi se determină valorile medii ale variabilelor aleatoare X_n, Y_n, Z_n, care reprezintă numărul de laturi, lungimea şi respectiv aria înfăşurătoarei I_n. Lucrarea reprezintă o extindere a rezultatelor obţinute în spaţiul euclidian E_2 de către A. Renyi şi R. Sulanke în [7], *privind acoperirea convexă a n puncte aleatoare situate într-un domeniu convex.*

The goal of this paper is to determine the mean values of the random variables X_n, Y_n, Z_n, which represent the number of sides, the length and the area of the convex envelope I_n of a system of n random, independent and uniformly distributed points in a convex domain $D \subset M_2$ with smooth boundary. In this paper we present an extension of the results obtained in the Euclidian space E_2 by A. Renyi and R. Sulanke in [7], *concerning the convex envelope of n random points in a convex domain.*

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1. Introduction

Throughout this paper we suppose that it is known the differential geometry of the Minkowski space M_2 ([1],[2],[3]). The Minkowski norm is defined relative to a smooth, convex, closed, central-symmetric relative to the origin $\mathcal{O}$ of the space M_2 and without fixed points curve. The Minkowski frames will be affine frames in relation to which the area of the indicatrix is π.

We parametrize the indicatrix U by equation [1] $t = t(\varphi)$, $0 \leq \varphi \leq 2\pi$, where φ is the double of the area of the sector $(\mathcal{O}, t_0, t)$ and t_0 corresponds to a fixed point on U.

The curve $T: \ x = -n(\varphi)$, where $n(\varphi) = \dfrac{dt}{d\varphi}$ is the isoperimetric of the space M_2.

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The interpretation of the parameter φ implies that $[t,n]=1$, where by $[x,y]$ we denote the determinant constituted of the vectors $x,y\in M_2$ reported to a Minkowski base.

The Minkowski curvature of the curve T is denoted by $\lambda(\varphi)$ and satisfies the relation

$$\frac{dn}{d\varphi}=-\lambda^{-1}(\varphi)t,\ \text{ where }\ \lambda(\varphi)>0.$$

2. Measures for sets of systems of n points in the space M_2

Let $\{P_1,\ldots,P_n\}\subset M_2$ be a system of points. We denote by $(x_i,y_i),\ i=\overline{1,n}$ the coordinates of the points $P_i,\ \ i=\overline{1,n}$ relative to a fixed Minkowski frame in M_2.

The elementary measure of the set of systems of n points is

$$dP_1\ldots dP_n=|dx_1\wedge dy_1\wedge\ldots dx_n\wedge dy_n| \tag{1}$$

Let $Q\subset M_2$ be a set of systems of n points to which it corresponds a set Q^* in the space of parameters $(x_1,y_1,\ldots,x_n,y_n)$.

The measure of the set Q is

$$\mu(Q)=\int_{Q^*}dP_1\ldots dP_n \tag{2}$$

If $Q_0\subset Q$ is a subset of systems of n points $\{P_1,\ldots,P_n\}$ with a certain property and to which it corresponds the set Q^* in the space of parameters, then the probability of the subset Q_0 is

$$P(Q_0)=\frac{\int_{Q_0^*}dP_1\ldots dP_n}{\int_{Q^*}dP_1\ldots dP_n} \tag{3}$$

Let $V=V(P_1,\ldots,P_n)$ be a random variable associated to the subset of systems of points $Q_0\subset Q$. Then the mean value for V is

$$E(V)=\frac{\int_{Q_0^*}V(P_1,\ldots,P_n)dP_1\ldots dP_n}{\int_{Q^*}dP_1\ldots dP_n} \tag{4}$$

3. Systems of n random points in a convex domain $K\subset M_2$ with smooth boundary

Let $\{P_1,\ldots,P_n\}$ be a system of n random, independent and uniformly distributed points in the convex domain $K\subset M_2$ with smooth boundary denoted by ∂K with the equation

$$x=-H(\varphi)n+H'(\varphi)t.$$

The function H is positive, periodic of period 2π and of class $\mathcal{C}^2$ and it is called support function for the curve ∂K.

We denote by L, respectively S, the Minkowski length of the boundary ∂K, respectively the area of the domain K.

Let I_n be the convex envelope of the system of points $\{P_1, \ldots, P_n\}$ in K and let ∂K be the boundary of I_n.

We associate to the system of points $\{P_1, \ldots, P_n\}$ the random variables X_n, Y_n, Z_n representing the number of sides, the Minkowski length and respectively the area of the convex envelope I_n.

We denote

$$\varepsilon_{ij} = \begin{cases} 1, & \overline{P_iP_j} \in \partial I_n, i \neq j \\ 0, & \text{în rest} \end{cases}$$

for all $i, j = \overline{1, n}$.

Hence the random variables X_n, Y_n, Z_n have the forms

$$X_n = \sum_{1 \le i < j \le n} \varepsilon_{ij}, \quad Y_n = \sum_{1 \le i < j \le n} Y_{ij}, \quad Z_n = \sum_{1 \le i < j \le n} Z_{ij},$$

where $Y_{ij} = \|\overline{P_iP_j}\|\varepsilon_{ij}$, $Z_{ij} = \Delta_{ij}\varepsilon_{ij}$ and Δ_{ij} is the Minkowski area of the triangle $\mathcal{O}P_iP_j$.

We suppose that the origin of the space M_2 is in the interior of the domain K. Therefore, we have

$$\mathcal{O}P_i = -Hn + \rho_i t, \quad \mathcal{O}P_j = -Hn + \rho_j t, \quad \rho_i, \rho_j \in \mathbb{R},$$

$$\Delta_{ij} = \frac{1}{2}|[\mathcal{O}P_i, \mathcal{O}P_j]| = \frac{1}{2}H\|P_iP_j\|,$$

where $H = H(P_i, P_j)$ is the support function of the straight line determined by the points P_i and P_j and $\|\,,\,\|$ is the Minkowski norm.

4. **The computation of the mean values** $E(X_n), E(Y_n), E(Z_n)$

Each random variable X_n, Y_n, Z_n is a sum of C_n^2 random variables. We have

$$E(X_n) = C_n^2 E(\varepsilon_{ij})$$
$$E(Y_n) = C_n^2 E(Y_{ij})$$
$$E(Z_n) = C_n^2 E(Z_{ij})$$

where $i < j$, $i, j = \overline{1, n}$.

Since the random points $\{P_1, \ldots, P_n\}$ are uniformly distributed and independent, the mean values $E(\varepsilon_{ij}), E(Y_{ij}), E(Z_{ij})$ are independent of $i, j = \overline{1, n}$ and they depend only on n.

Let g be a straight line determined by the points P_i and P_j, $i \neq j$, situated in K. The straight line g separates the convex domain K in two subsets K_1 and K_2 such that $K_1 \cup K_2 = K$.

We suppose that the Minkowski area for K_1 denoted by s_k satisfies the condition $s_k \le \dfrac{S}{2}$. The remaining points of the system are $\{P_3, \ldots, P_n\}$ and they are placed on one side or on the other side of the straight line g in K, that means in K_1 or in K_2.

Using formula (4), we have

$$E(\varepsilon_{ij}) = \frac{\int_{K\times K}\left(\int_{K_1^{n-2}} dP_3\dots dP_n + \int_{K_2^{n-2}} dP_3\dots dP_n\right) dP_1 dP_2}{\int_{K^n} dP_1\dots dP_n}$$

or

$$E(\varepsilon_{ij}) = \frac{\int_{K\times K}[s_k^{n-2} + (S-s_k)^{n-2}]dP_1dP_2}{S^n},$$

and we obtain

$$E(X_n) = \frac{C_n^2}{S^2}\int_{K\times K}\left(1-\frac{s_k}{S}\right)^{n-2} dP_1dP_2 + \frac{C_n^2}{S^2}\int_{K\times K}\left(\frac{s_k}{S}\right)^{n-2} dP_1dP_2.$$

Since $\frac{s_k}{S} \le \frac{1}{2}$, it results that

$$\frac{C_n^2}{S^2}\int_{K\times K}\left(\frac{s_k}{S}\right)^{n-2} dP_1dP_2 \le \frac{C_n^2}{2^{n-2}}$$

and

$$\lim_{n\to\infty}\frac{C_n^2}{2^{n-2}} = 0.$$

Using Landau symbol, we have

$$E(X_n) = \frac{C_n^2}{S^2}\int_{K\times K}\left(1-\frac{s_k}{S}\right)^{n-2} dP_1dP_2 + O(1) \tag{5}$$

Similarly we obtain

$$E(Y_{ij}) = \frac{\int_{K\times K}[(s_k^{n-2} + (S-s_k)^{n-2}]\|P_1P_2\|dP_1dP_2}{S^n} \tag{6}$$

$$E(Z_{ij}) = \frac{1}{2}\cdot\frac{\int_{K\times K}[(s_k^{n-2} + (S-s_k)^{n-2}]H(P_1,P_2)\|P_1P_2\|dP_1dP_2}{S^n} \tag{7}$$

for all $i<j,\ \ i,j=\overline{1,n}$, and from this we get that

$$E(Y_n) = \frac{C_n^2}{S^2}\int_{K\times K}\left(1-\frac{s_k}{S}\right)^{n-2}\|P_1P_2\|dP_1dP_2 + O(1) \tag{8}$$

$$E(Z_n) = \frac{C_n^2}{2S^2}\int_{K\times K}\left(1-\frac{s_k}{S}\right)^{n-2}H(P_1,P_2)\|P_1P_2\|dP_1dP_2 + O(1) \tag{9}$$

We compute the elementary measure of the set of pairs of points P_1, P_2 in M_2.

We denote $OP_1 = x = -Hn + \rho_1 t$ and $OP_2 = y = -Hn + \rho_2 t$.

Let (x_1, x_2) be the coordinates of the point P_1 in the frame $\{O, t, n\}$ and let (y_1, y_2) be the coordinates of the point P_2. It results that

$$dx_1 = d\rho_1 + \lambda^{-1}Hd\varphi,\ \ dx_2 = -dH + \rho_1 d\varphi$$

and

$$dy_1 = d\rho_2 + \lambda^{-1}Hd\varphi,\ \ dy_2 = -dH + \rho_2 d\varphi.$$

Therefore, we have

$$dP_1dP_2 = |dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2| = |\rho_1 - \rho_2||d\rho_1 \wedge d\rho_2 \wedge dH \wedge d\varphi| \quad (10)$$

Let g be the straight line determined by the points P_1, P_2 and we denote by $\|g \cap K\|$ the Minkowski length of the chord determined by the straight line g in K.

If we suppose that the terminal points of the chord correspond to $\rho = 0$, respectively to $\rho = \|g \cap K\|$, then for a fixed position of the straight line g, we have

$$\int_{g\cap K\neq\emptyset} |\rho_1 - \rho_2| d\rho_1 d\rho_2 = \int_0^{\|g\cap K\|} \left[\int_0^{\rho_1} (\rho_1 - \rho_2) d\rho_2 + \int_{\rho_1}^{\|g\cap K\|} (\rho_2 - \rho_1) d\rho_2 \right] d\rho_1 =$$

$$= \frac{\|g \cap K\|^3}{3} \quad (11)$$

Since $\|P_1P_2\| = |\rho_1 - \rho_2|\|t\| = |\rho_1 - \rho_2|$, it results that $\|P_1P_2\| dP_1dP_2 = |\rho_1 - \rho_2|^2 d\rho_1 d\rho_2 dH d\varphi$.

Similar to (10), we obtain

$$\int_{g\cap K\neq\emptyset} |\rho_1 - \rho_2|^2 d\rho_1 d\rho_2 = \frac{1}{6}\|g \cap K\|^4 \quad (12)$$

Using relations (10), (11) and (12), we obtain for the mean values from (5), (8) and (9) the following forms:

$$E(X_n) = \frac{C_n^2}{3S^2} \int_0^{2\pi} \left(\int_0^{H(\varphi)} \left(1 - \frac{s_k}{S}\right)^{n-2} \|g \cap K\|^3 dH \right) d\varphi + O(1) \quad (13)$$

$$E(Y_n) = \frac{C_n^2}{6S^2} \int_0^{2\pi} \left(\int_0^{H(\varphi)} \left(1 - \frac{s_k}{S}\right)^{n-2} \|g \cap K\|^4 dH \right) d\varphi + O(1) \quad (14)$$

$$E(Z_n) = \frac{C_n^2}{12S^2} \int_0^{2\pi} \left(\int_0^{H(\varphi)} \left(1 - \frac{s_k}{S}\right)^{n-2} \|g \cap K\|^4 H dH \right) d\varphi + O(1) \quad (15)$$

where the Minkowski parameters (H, φ) correspond to all intersection positions of the straight line g with the boundary ∂K of the domain K.

If $0 < \varepsilon < \frac{s_k}{S} < 1$, then $C_n^2 \left(1 - \frac{s_k}{S}\right)^2 < C_n^2 (1 - \varepsilon)^2 \longrightarrow 0$ when $n \longrightarrow \infty$.

5. **Conclusions**

The computation of the integrals from the relations (13)-(15) is made for those pairs (H, φ) for which the quotient $\dfrac{s_k}{S}$ can be very small. This is possible when the straight line g gets near the tangent points of the curve ∂K with the anti-osculating circle associated to ∂K in each point. In these cases, the straight line g cuts both the curve ∂K and the anti-osculating circle associated to the curve ∂K in each point, fact that allows afterwards to compute the integrals from relations (13)-(15).

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