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SOME INTEGRAL INEQUALITIES FOR HARMONICALLY h-CONVEX FUNCTIONS

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In this paper, we introduce and investigate a new class of harmonically convex functions, which is called harmonically h-convex function. It is shown that this class unifies several new and known classes of harmonically convex functions. We derive some new Hermite-Hadamard like inequalities for harmonically h-convex functions. Harmonically s-convex function and Harmonically s-Godunova-Levin functions of first kinds are also defined and their properties are investigated. Some special cases are also discussed. Results obtained in this paper continue to hold for the new and known classes of harmonically convex functions.

Keywords: Convex function, harmonically convex function, harmonically h-convex function, hypergeometric, Hermite-Hadamard inequality.

MSC2010: 26D15, 26A51.

1. Introduction

In recent years, theory of convex functions has received special attention by many researchers because of its importance in different fields of pure and applied sciences such as optimization and economics. Consequently the classical concepts of convex functions has been extended and generalized in different directions using novel and innovative ideas, see [1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 19, 22, 26, 27, 28]. A significant generalization of convex functions was the introduction of h-convex functions by Varosanec [24]. For different properties and other aspects of h-convex functions, readers are referred to [1, 3, 4, 5, 10, 18, 20, 21, 25, 26]. Iscan [11] introduced a new class of convex functions, which is called harmonically convex functions. For some recent investigations and extensions of harmonically convex functions interested readers are referred to [11, 12, 17, 18, 27].
It is known that theory of convex functions is closely related to theory of inequalities. Many known inequalities are proved for convex functions. An intensively studied inequality in the literature is Hermite-Hadamard's inequality which was proved independently by Hermite (1883) and Hadamard (1896). This inequality gives a necessary and sufficient condition for a function to be convex. The classical Hermite-Hadamard inequality reads as:

Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

For useful details on Hermite-Hadamard type of integral inequalities, see [3, 4, 5, 6, 7, 8, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 23, 25, 27, 28, 29].

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Motivated by this ongoing research, we introduce a new class of harmonically convex functions which is called harmonically h-convex functions. It is shown that for suitable choices of function h, one can obtain a number of new classes of harmonically convex functions such as harmonically s-convex functions, harmonically P-functions, harmonically Godunova-Levin functions and harmonically s-Godunova-Levin functions. We obtain several new Hermite-Hadamard type inequalities for harmonically h-convex functions. Several special cases are also discussed. The ideas used in this paper may inspire interested readers to find the novel and innovative applications of harmonically h-convex functions in various branches of pure and applied sciences. This is the main motivation of this paper.

2. Preliminaries and Basic Results

In this section, we recall some basic results and define the concept of harmonically h-convex functions.

Definition 2.1. A set $K \subseteq \mathbb{R}$ is said to be *convex set* in the classical sense, if

$$(1-t)x + ty \in K, \quad \forall x, y \in K, \quad t \in [0,1].$$

Definition 2.2. A function $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *convex function*, if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in K, \quad t \in [0,1].$$

Definition 2.3 ([22])**.** A set $\Omega \subset \mathbb{R}_+$ is said to be *harmonically convex set*, if

$$\frac{xy}{tx + (1-t)y} \in \Omega, \quad \forall x, y \in \Omega, \quad t \in [0,1].$$

Definition 2.4 ([11])**.** A function $f : \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically convex function*, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)(f(x)) + t(f(y)), \quad \forall x, y \in \Omega, \quad t \in [0,1].$$

Definition 2.5 ([17])**.** A function $f : \Omega \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *harmonically* log-*convex function*, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq f(x)^{1-t} f(y)^t, \quad \forall x, y \in \Omega, \quad t \in [0,1].$$

Now we define some new concepts.

Definition 2.6. A function $f : \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically s-convex function of second kind*, where $s \in (0,1]$, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)^s f(x) + t^s f(y), \quad \forall x, y \in I, \quad t \in [0,1].$$

Definition 2.7. A function $f : \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically P-function*, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq f(x) + f(y), \quad \forall x, y \in I, \quad t \in [0,1].$$

Definition 2.8. A function $f : \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically Godunova-Levin function*, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{1-t} f(x) + \frac{1}{t} f(y), \quad \forall x, y \in I, \quad t \in (0,1).$$

Definition 2.9. A function $f : \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically s-Godunova-Levin function of second kind*, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{(1-t)^s} f(x) + \frac{1}{t^s} f(y), \quad \forall x, y \in I, \quad t \in (0,1), \quad s \in [0,1].$$

In order to unify the above concepts, let us define the class of harmonically h-convex functions.

Definition 2.10. Let $h\colon [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f\colon \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically h-convex function*, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)f(x)+h(t)f(y), \quad \forall x,y \in I, \quad t \in (0,1).$$

Note that for $t = \frac{1}{2}$, we have Jensen's type harmonically h-convex function or harmonically-arithmetically (HA) h-convex functions

$$f\left(\frac{2xy}{x+y}\right) \leq h(\frac{1}{2})[f(x)+f(y)].$$

Remark 2.1. It is obvious that for $h(t) = t$, $h(t) = t^s$, $h(t) = 1$, $h(t) = \frac{1}{t}$ and $h(t) = \frac{1}{t^s}$ in Definition 2.10, we have the definitions of harmonically convex functions, harmonically s-convex functions of second kind, harmonically P-functions, harmonically Godunova-Levin functions and harmonically s-Godunova-Levin functions of second kind respectively.

Now we define the concepts of harmonically s-convex functions and harmonically s-Godunova-Levin functions of first kinds respectively.

Definition 2.11. A function $f\colon \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically s-convex function of first kind*, where $s \in [0,1]$, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t^s)f(x)+t^s f(y), \quad \forall x,y \in I, \quad t \in [0,1].$$

Definition 2.12. A function $f\colon \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be *harmonically s-Godunova-Levin function of first kind*, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{1}{(1-t^s)}f(x)+\frac{1}{t^s}f(y), \quad \forall x,y \in I, \quad t \in (0,1), \quad s \in (0,1].$$

Remark 2.2. It is worth mentioning here that the concepts of harmonically s-convex function of first kind and harmonically s-Godunova-Levin function of first kind are not contained in the class of harmonically h-convex functions.

Definition 2.13 ([18])**.** Let $h\colon [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f\colon \Omega \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *harmonically* log-*h-convex function*, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq f(x)^{h(1-t)}f(y)^{h(t)}, \quad \forall x,y \in I, \quad t \in (0,1).$$

Iscan [11] proved following Hermite-Hadamard type inequality for harmonically convex functions.

Theorem 2.14. *Let $f\colon \Omega \subset \mathbb{R}_+ \to \mathbb{R}$ be harmonically convex function and $a,b \in \Omega$ with $a < b$. If $f \in L[a,b]$, then*

$$f\Big(\frac{2ab}{a+b}\Big) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

The following result of Iscan [11] plays an important role in the development of some of our man results.

Lemma 2.15 ([11])**.** *Let $f\colon I\to\mathbb{R}$ be a differentiable function on I° (interior of I) and $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx$$
$$=\frac{ab(b-a)}{2}\int\limits_0^1\frac{1-2t}{(tb+(1-t)a)^2}f'\left(\frac{ab}{tb+(1-t)a}\right)dt$$

Now we give the definition of hypergeometric series which will be used in the obtaining some integrals.

Definition 2.16 ([13])**.** For the real or complex numbers a,b,c, other than $0,-1,-2,\ldots$, the *hypergeometric series* is defined by

$${}_2F_1[a,b,c;z]=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}.$$

Here $(\phi)_m$ is the Pochhammer symbol, which is defined by

$$(\phi)_m=\begin{cases}1, & m=0,\\ \phi(\phi+1)\cdots(\phi+m-1), & m>0.\end{cases}$$

Definition 2.17 ([24])**.** Two functions f and g are said to be *similarly ordered*, if

$$(f(x)-f(y))(g(x)-g(y))\geq 0,\quad \forall x,y\in\mathbb{R}.$$

3. Main Results

In this section, we prove our main results. Throughout this section $h(\frac{1}{2})\neq 0$, $I\subset\mathbb{R}_+$ be the interval and I° be the interior of I, unless otherwise specified.

Proposition 3.1. *Let f and g be two harmonically h-convex functions. If f and g are similarly ordered functions and $h(t)+h(1-t)\leq 1$, then the product fg is also harmonically convex function.*

Proof. Let f and g be harmonically convex functions. Then

$$\begin{aligned}
&f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\\
&\leq [h(1-t)f(a)+h(t)f(b)][h(1-t)g(a)+h(t)g(b)]\\
&=[h(1-t)]^2f(a)g(a)+h(t)h(1-t)[f(a)g(b)+f(b)g(a)]+[h(t)]^2f(b)g(b)\\
&\leq [h(1-t)]^2f(a)g(a)+h(t)h(1-t)[f(a)g(a)+f(b)g(b)]+[h(t)]^2f(b)g(b)\\
&=[h(1-t)f(a)g(a)+h(t)f(b)g(b)][h(t)+h(1-t)]\\
&\leq h(1-t)f(a)g(a)+h(t)f(b)g(b). \qquad (1)
\end{aligned}$$

This shows that the product of two harmonically h-convex functions is again harmonically h-convex function. □

Theorem 3.2. *Let $f\colon I\to\mathbb{R}$ be harmonically h-convex function where $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\leq[f(a)+f(b)]\int\limits_0^1 h(t)dt.$$

Proof. Since f is harmonically h-convex function, so, we have

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)f(x)+h(t)f(y).$$

For $t=\frac{1}{2}$, it follows

$$f\left(\frac{2xy}{x+y}\right) \leq h(\frac{1}{2})[f(x)+f(y)].$$

Let $x=\frac{ab}{ta+(1-t)b}$ and $y=\frac{ab}{tb+(1-t)a}$, then, we obtain

$$f\left(\frac{2ab}{a+b}\right) \leq h(\frac{1}{2})\left[f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right)\right].$$

Integrating both sides of above inequality with respect to t on [0,1], we get

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})}f\left(\frac{2ab}{a+b}\right) &\leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\\ &= \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\\ &\leq [f(a)+f(b)]\int_0^1 h(t)dt.\end{aligned}$$

This completes the proof. □

We now discuss some special cases.

I. If $h(t)=t^s$ then Theorem 3.2 reduces to the following result.

Corollary 3.3. *Let $f\colon I\to\mathbb{R}$ be harmonically s-convex function of second kind, where $a,b\in I$ with $a<b$ and $s\in[0,1]$. If $f\in L[a,b]$, then*

$$2^{s-1}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{s+1}.$$

II. If $h(t)=1$ then Theorem 3.2 collapses to the following result.

Corollary 3.4. *Let $f\colon I\to\mathbb{R}$ be harmonically P-function where $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then*

$$\frac{1}{2}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq f(a)+f(b).$$

III. If $h(t)=t^{-s}$ then we have the following result.

Corollary 3.5. *Let $f\colon I\to\mathbb{R}$ be harmonically s-Godunova-Levin function of second kind, where $a,b\in I$ with $a<b$ and $s\in[0,1)$. If $f\in L[a,b]$, then*

$$\frac{1}{2^{s+1}}f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{1-s}.$$

Our coming result is the Hermite-Hadamard inequality for product of two harmonically convex functions.

Theorem 3.6. *Let $f, g\colon I \to \mathbb{R}$ be two harmonically convex functions where $a, b \in I$ with $a < b$. If $fg \in L[a, b]$, then*

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right) dx \le M(a,b)\int_0^1 (h(t))^2 dt + N(a,b)\int_0^1 h(t)h(1-t)dt,$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b), \tag{2}$$

and

$$N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. Let $f, g\colon I \to \mathbb{R}$ be two harmonically h-convex functions, then we have

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right) dx \\
&= \int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right) g\left(\frac{xy}{tx+(1-t)y}\right) dt \\
&\le \int_0^1 (h(1-t)f(a) + h(t)f(b))(h(1-t)g(a) + h(t)w(b))dt \\
&= M(a,b)\int_0^1 (h(t))^2 dt + N(a,b)\int_0^1 h(t)h(1-t)dt.
\end{aligned}$$

This completes the proof. □

Theorem 3.7. *Under the conditions of Theorem 3.6, if f and g are similarly ordered functions, then, we have*

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right) dx \le M(a,b)\int_0^1 h(t)dt,$$

where $M(a,b)$ is given by (2).

Proof. Integrating inequality (1) completes the proof. □

Now using Lemma 2.15, we prove our next results.

Theorem 3.8. *Let $f\colon I \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$, $q \ge 0$ is harmonically h-convex function, then, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\mathcal{C}_1^{1-\frac{1}{q}}\left(\mathcal{C}_2|f'(a)|^q + \mathcal{C}_3|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\mathcal{C}_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right), \tag{3}$$

$$\mathcal{C}_2 = \int_0^1 \frac{|1-2t|h(t)}{(tb+(1-t)a)^2}dt, \tag{4}$$

and

$$\mathcal{C}_3 = \int_0^1 \frac{|1-2t|h(1-t)}{(tb+(1-t)a)^2}dt, \tag{5}$$

respectively.

Proof. Using Lemma 2.15, power mean inequality and the fact that $|f'|^q$ is harmonically h-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\frac{|1-2t|}{(tb+(1-t)a)^2}\left[h(t)|f'(a)|^q + h(1-t)|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{ab(b-a)}{2}\mathcal{C}_1^{1-\frac{1}{q}}\left(\mathcal{C}_2|f'(a)|^q + \mathcal{C}_3|f'(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.9. *Under the conditions of Theorem* 3.8*, if* $q=1$*, then, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\mathcal{C}_2|f'(a)| + \mathcal{C}_3|f'(b)|\right),$$

where $\mathcal{C}_2$*,* $\mathcal{C}_3$ *are given by* (4) *and* (5) *respectively.*

If $h(t) = t^s$ in Theorem 3.8, we have result for harmonically s-convex functions of second kind.

Corollary 3.10. *Let* $f\colon I \to \mathbb{R}$ *be a differentiable function on* I° *where* $a, b \in I$ *with* $a < b$ *and* $f' \in L[a,b]$*, if* $|f'|^q$*,* $q \geq 0$ *is harmonically* s*-convex function of second kind, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\mathcal{C}_1^{1-\frac{1}{q}}\left(\mathcal{K}_1|f'(a)|^q + \mathcal{K}_2|f'(b)|^q\right)^{\frac{1}{q}},$$

where $\mathcal{C}_1$ *is given by* (3)*, and*

$$\mathcal{K}_1 = \int_0^1 \frac{|1-2t|t^s}{(tb+(1-t)a)^2}dt, \tag{6}$$

$$\mathcal{K}_2 = \int_0^1 \frac{|1-2t|(1-t)^s}{(tb+(1-t)a)^2}dt,$$

respectively.

Theorem 3.11. *Let $f\colon I \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$, if $|f'|^q$, $q \geq 0$ is harmonically s-convex function of first kind, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{2}\mathcal{C}_1^{1-\frac{1}{q}}\left(\mathcal{K}_1|f'(a)|^q + (\mathcal{C}_1 - \mathcal{K}_1)|f'(b)|^q\right)^{\frac{1}{q}},$$

where $\mathcal{C}_1$ and $\mathcal{K}_1$ are given by (3) and (6) respectively.

If $h(t) = 1$ in Theorem 3.8, we have result for harmonically P-functions.

Corollary 3.12. *Let $f\colon I \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$, if $|f'|^q$, $q \geq 0$ is harmonically P-function, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{2}\mathcal{C}_1\left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}},$$

where $\mathcal{C}_1$ is given by (3).

If $h(t) = t^{-s}$ in Theorem 3.8, we have result for harmonically s-Godunova-Levin functions of second kind.

Corollary 3.13. *Let $f\colon I \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$, if $|f'|^q$, $q \geq 0$ is harmonically s-Godunova-Levin function of second kind, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{2}\mathcal{C}_1^{1-\frac{1}{q}}\left(\mathcal{L}_1|f'(a)|^q + \mathcal{L}_2|f'(b)|^q\right)^{\frac{1}{q}},$$

where $\mathcal{C}_1$ is given by (3) and

$$\mathcal{L}_1 = \int_0^1 \frac{|1-2t|t^{-s}}{(tb+(1-t)a)^2}dt, \tag{7}$$

$$\mathcal{L}_2 = \int_0^1 \frac{|1-2t|(1-t)^{-s}}{(tb+(1-t)a)^2}dt,$$

respectively.

Theorem 3.14. *Let $f\colon I \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$, if $|f'|^q$, $q \geq 0$ is harmonically s-Godunova-Levin function of first kind, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{2}\mathcal{C}_1^{1-\frac{1}{q}}\left(\mathcal{L}_1|f'(a)|^q + \mathcal{L}_2^\star|f'(b)|^q\right)^{\frac{1}{q}},$$

where $\mathcal{C}_1$ and $\mathcal{L}_1$ are given by (3) and (7) respectively, and

$$\mathcal{L}_2^\star = \int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^2(1-t^s)}dt \approx \mathcal{C}_1 + \mathcal{K}_1.$$

Theorem 3.15. *Let $f\colon I \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$, if $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, is harmonically h-convex. Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathcal{C}_4|f'(a)|^q+\mathcal{C}_5|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\mathcal{C}_4=\int_0^1 \frac{h(t)}{(tb+(1-t)a)^{2q}}dt,$$

and

$$\mathcal{C}_5=\int_0^1 \frac{h(1-t)}{(tb+(1-t)a)^{2q}}dt,$$

respectively.

Proof. Using Lemma 2.15, Holder's inequality and the fact that $|f'|^q$ is harmonically h-convex function, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left\{h(t)|f'(a)|^q+h(1-t)|f'(b)|^q\right\}dt\right)^{\frac{1}{q}} \\
&= \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mathcal{C}_4|f'(a)|^q+\mathcal{C}_5|f'(b)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

If $h(t)=t^s$ in Theorem 3.15, we have result for harmonically s-convex functions of second kind.

Corollary 3.16. *Let $f\colon I \to \mathbb{R}$ be a differentiable function on I° where $a, b \in I$ with $a < b$ and $f' \in L[a,b]$, if $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically s-convex function of second* *kind. Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\vartheta_1|f'(a)|^q+\vartheta_2|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\vartheta_1=\int\limits_0^1\frac{t^s}{(tb+(1-t)a)^{2q}}dt=\frac{a^{-2q}{}_2F_1[2q,1+s,2+s,1-\frac{b}{a}]}{1+s},$$

and

$$\vartheta_2=\int\limits_0^1\frac{(1-t)^s}{(tb+(1-t)a)^{2q}}dt=\frac{a^{-2q}{}_2F_1[1,2q,2+s,1-\frac{b}{a}]}{1+s},$$

respectively.

Theorem 3.17. *Let $f\colon I\to\mathbb{R}$ be a differentiable function on I° where $a,b\in I$ with $a<b$ and $f'\in L[a,b]$, if $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically s-convex function of first kind. Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\vartheta_1^\star|f'(a)|^q+\vartheta_2^\star|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\vartheta_1^\star=\int\limits_0^1\frac{t^s}{(tb+(1-t)a)^{2q}}dt=\frac{a^{-2q}{}_2F_1[2q,1+s,2+s,1-\frac{b}{a}]}{1+s},$$

and

$$\vartheta_2^\star=\int\limits_0^1\frac{(1-t^s)}{(tb+(1-t)a)^{2q}}dt=\frac{bb^{-2q}-aa^{-2q}}{(b-a)(1-2q)}-\vartheta_1,$$

respectively.

If $h(t)=1$ in Theorem 3.15, we have result for harmonically P-functions.

Corollary 3.18. *Let $f\colon I\to\mathbb{R}$ be a differentiable function on I° where $a,b\in I$ with $a<b$ and $f'\in L[a,b]$, if $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically P-function, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\mathcal{C}^{\frac{1}{q}}\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}},$$

where

$$\mathcal{C}=\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}dt=\frac{bb^{-2q}-aa^{-2q}}{(b-a)(1-2q)}.$$

If $h(t)=t^{-s}$ in Theorem 3.15, we have result for harmonically s-Godunova-Levin functions of second kind.

Corollary 3.19. *Let $f\colon I\to\mathbb{R}$ be a differentiable function on I° where $a,b\in I$ with $a<b$ and $f'\in L[a,b]$, if $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically s-Godunova-Levin function of second kind. Then, we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\&\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\varphi_1|f'(a)|^q+\varphi_2|f'(b)|^q\right)^{\frac{1}{q}},\end{aligned}$$

where

$$\varphi_1=\int_0^1 \frac{t^{-s}}{(tb+(1-t)a)^{2q}}dt=\frac{a^{-2q}{}_2F_1[2q,1-s,2-s,1-\frac{b}{a}]}{1-s}, \tag{8}$$

and

$$\varphi_2=\int_0^1 \frac{(1-t)^{-s}}{(tb+(1-t)a)^{2q}}dt=\frac{a^{-2q}{}_2F_1[1,2q,2-s,1-\frac{b}{a}]}{1-s},$$

respectively.

Theorem 3.20. *Let $f\colon I\to\mathbb{R}$ be a differentiable function on I° where $a,b\in I$ with $a<b$ and $f'\in L[a,b]$, if $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$ is harmonically s-Godunova-Levin function of first kind. Then, we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\&\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\varphi_1|f'(a)|^q+\varphi_2^\star|f'(b)|^q\right)^{\frac{1}{q}},\end{aligned}$$

where φ_1 is given by (8) *and*

$$\varphi_2^\star=\int_0^1 \frac{(1-t)^{-s}}{(tb+(1-t)a)^{2q}}dt=\vartheta_1^\star+\frac{bb^{-2q}-aa^{-2q}}{(b-a)(1-2q)},$$

respectively.

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