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SOME INVARIANTS CONNECTED WITH EULER-LAGRANGE EQUATIONS

Irena ČOMIĆ*

Lucrarea descrie mai multe tipuri de omogenitate definite în spaţiul Osc^kM. Sunt prezentate ecuaţiile Euler-Lagrange pentru diverse tipuri de variabile, iar legătura dintre Lagrangienii spaţiilor $L^{(k)n}$ şi Hamiltonienii spaţiilor $H^{(k)n}$ este precizată. Ecuaţiile diferă semnificativ, funcţie de paritatea ordinului k. Se arată că în Osc^3M, doar E_i^0 şi E_i^3 sunt covectori. Se demonstrează că ecuaţiile Euler-Lagrange au caracter invariant, şi se determină un nou câmp scalar.

In Osc^kM, different kinds of homogeneity are defined, and properties of homogeneous functions are pointed out. The Euler-Lagrange equations for different types of variables in Osc^kM are given; they are coordinate invariant, and have different forms for $k = 2l$ and for $k = 2l+1$. The connection between Lagrangians in the space $L^{(k)n}$ and Hamiltonians in $H^{(k)n}$, is outlined. It is shown that in Osc^3M, only E_i^0 and E_i^3 are covectors, and one new scalar field is determined.

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1. Different kinds of homogeneity. Euler's equations

Let us denote by Osc^kM the k-th order tangent space of M. In some local chart the point $u \in Osc^kM$ has coordinates $(x^i = y^{0i}, y^{1i}, \dots, y^{ki})$, $i = \overline{1,n}$, $y^{Ai} = \frac{d^Ay^{0i}}{dt^A}$, $A = \overline{0,k}$. The transformation group is well known ([1], [2], [3], [4], etc). Some properties of Osc^3M are given by (41)-(43).

Definition 1. The scalar function $F(y^{0i}, y^{1i}, \dots, y^{ki})$ is homogeneous of first kind and order r ($r \in \mathbf{Z}$) if

$$F(y^{0i}, \lambda y^{1i}, \lambda^2 y^{2i}, \dots, \lambda^k y^{ki}) = \lambda^r F(y^{0i}, y^{1i}, \dots, y^{ki}), \quad \lambda > 0. \tag{1}$$

*Professor, Faculty of Technical Sciences, University of Novi Sad, Serbia

It is homogeneous of second kind and order r if

$$F(y^{0i}, \lambda y^{1i}, \lambda y^{2i}, \dots, \lambda y^{ki}) = \lambda^r F(y^{0i}, y^{1i}, \dots, y^{ki}). \tag{2}$$

It is homogeneous of third kind and order r with respect to variable y^{Ai}, if

$$F(y^{0i}, y^{1i}, \dots, y^{(A-1)i}, \lambda y^{Ai}, y^{(A+1)i}, \dots, y^{ki}) = \lambda^r F(y^{0i}, y^{1i}, \dots, y^{ki}). \tag{3}$$

It is homogeneous of fourth kind and order r with respect to variable y^{Ai} if

$$F(y^{0i}, y^{1i}, \dots, y^{(A-1)i}, \lambda^A y^{Ai}, y^{(A+1)i}, \dots, y^{ki}) = (\lambda^A)^r F(y^{0i}, y^{1i}, \dots, y^{ki}). \tag{4}$$

Theorem 1. *For the homogeneous function $F(y^{0i}, y^{1i}, \dots, y^{ki})$ of first kind and order r the following Euler type equations are valid:*

$$\frac{\partial F}{\partial y^{1i}} y^{1i} + \frac{\partial F}{\partial y^{2i}} 2y^{2i} + \cdots + \frac{\partial F}{\partial y^{ki}} k y^{ki} = rF, \quad r \geq 0 \tag{5}$$

$$\frac{\partial^2 F}{\partial y^{1i} \partial y^{1j}} y^{1i} y^{1j} + \frac{\partial^2 F}{\partial y^{2i} \partial y^{2j}} 2 \cdot 2 y^{2i} y^{2j} + \cdots + \frac{\partial^2 F}{\partial y^{ki} \partial y^{kj}} k \cdot k y^{ki} y^{kj} + \tag{6}$$

$$2 \left[\frac{\partial^2 F}{\partial y^{1i} \partial y^{2j}} 1 \cdot 2 y^{1i} y^{2j} + \frac{\partial^2 F}{\partial y^{1i} \partial y^{3j}} 1 \cdot 3 y^{1i} y^{3j} + \cdots \right.$$

$$\left. + \frac{\partial^2 F}{\partial y^{1i} \partial y^{kj}} 1 \cdot k y^{1i} y^{kj} + \cdots + \frac{\partial^2 F}{\partial y^{(k-1)i} \partial y^{kj}} (k-1) k y^{(k-1)i} y^{kj} \right] +$$

$$\frac{\partial F}{\partial y^{2i}} 2 \cdot 1 y^{2i} + \frac{\partial F}{\partial y^{3i}} 3 \cdot 2 y^{3i} + \cdot + \frac{\partial F}{\partial y^{ki}} k(k-1) y^{ki} = r(r-1)F, \ r \geq 1.$$

Proof. If we take the first derivative of (1) with respect to λ, we get

$$\sum_{A=1}^{k} \frac{\partial F}{\partial (\lambda^A y^{Ai})} A \lambda^{A-1} y^{Ai} = r \lambda^{r-1} F. \tag{7}$$

If we put $\lambda = 1$ in (7), we get (5). If we take the derivative of (7) with respect to λ, we get

$$\sum_{B=1}^{k} \sum_{A=1}^{k} \frac{\partial^2 F}{\partial (\lambda^A y^{Ai}) \partial (\lambda^B y^{Bi})} A \lambda^{A-1} y^{Ai} B \lambda^{B-1} y^{Bi} + \tag{8}$$

$$\sum_{A=2}^{k} \frac{\partial F}{\partial (\lambda^A y^{Ai})} \cdot A(A-1) \lambda^{A-2} y^{Ai} = r(r-1) \lambda^{r-2} F.$$

If we put $\lambda = 1$ in (8), we obtain (6). $\square$

Theorem 2. *For the homogeneous function $F(y^{0i}, y^{1i}, \ldots, y^{ki})$ of second kind and order r the following equations are valid:*

$$\frac{\partial F}{\partial y^{1i}} y^{1i} + \frac{\partial F}{\partial y^{2i}} y^{2i} + \cdots + \frac{\partial F}{\partial y^{ki}} y^{ki} = rF, \quad r \geq 0 \tag{9}$$

$$\frac{\partial^2 F}{\partial y^{1i} \partial y^{1j}} y^{1i} y^{1j} + \frac{\partial^2 F}{\partial y^{2i} \partial y^{2j}} y^{2i} y^{2j} + \cdots + \frac{\partial^2 F}{\partial y^{ki} \partial y^{kj}} y^{ki} y^{kj} + \tag{10}$$

$$2\left[\frac{\partial^2 F}{\partial y^{1i} \partial y^{2j}} y^{1i} y^{2j} + \cdots + \frac{\partial^2 F}{\partial y^{1i} \partial y^{kj}} y^{1i} y^{kj} + \cdots + \frac{\partial^2 F}{\partial y^{(k-1)i} \partial y^{kj}} y^{(k-1)i} y^{kj}\right] =$$

$$r(r-1)F, \qquad r \geq 1.$$

Proof. The first and second derivative of (2) with respect to λ give:

$$\sum_{A=1}^{k} \frac{\partial F}{\partial (\lambda y^{Ai})} y^{Ai} = r\lambda^{r-1} F \tag{11}$$

$$\sum_{B=1}^{k} \sum_{A=1}^{k} \frac{\partial^2 F}{\partial (\lambda y^{Ai}) \partial (\lambda y^{Bj})} y^{Ai} y^{Bj} = r(r-1)F. \tag{12}$$

For $\lambda = 1$ the explicit form of (11) and (12) are (9) and (10), respectively. The higher order derivatives give more complicated formulae. □

Theorem 3. *If the function $F(y^{0i}, y^{1i}, \ldots, y^{Ai}, \ldots, y^{ki})$ is homogeneous of third kind and order r with respect to the variable y^{Ai}, then*

$$\frac{\partial F}{\partial y^{Ai}} y^{Ai} = rF, \frac{\partial^2 F}{\partial y^{Ai} \partial y^{Aj}} y^{Ai} y^{Aj} = r(r-1)F, \ldots, \tag{13}$$

$$\frac{\partial^r F}{\partial y^{Ai_1} \partial y^{Ai_2} \ldots y^{Ai_r}} y^{Ai_1} y^{Ai_2} \ldots y^{Ai_r} = r!F.$$

Proof. The proof is similar to the proof of previous Theorems. □

Theorem 4. *If the function $F(y^{0i}, y^{1i}, \ldots, y^{Ai}, \ldots, y^{ki})$ is homogeneous of fourth kind and order r with respect to variable y^{Ai}, then*

$$\frac{\partial F}{\partial y^{Ai}} A y^{Ai} = rAF, \quad A = \overline{1, k} \tag{14}$$

$$\frac{\partial^2 F}{\partial y^{Ai} \partial y^{Aj}} A^2 y^{Ai} y^{Aj} + \frac{\partial F}{\partial y^{Ai}} A(A-1) y^{Ai} = rA(rA-1)F. \tag{15}$$

Proof. The first and second derivatives of (4) with respect to λ are

$$\frac{\partial F}{\partial(\lambda^A y^{Ai})} A\lambda^{A-1} y^{Ai} = rA\lambda^{rA-1} F \tag{16}$$

$$\frac{\partial^2 F}{\partial(\lambda^A y^{Ai})\partial(\lambda^A y^{Aj})} (A\lambda^{A-1})^2 y^{Ai} y^{Aj} + \tag{17}$$

$$\frac{\partial F}{\partial(\lambda^A y^{Ai})} A(A-1)\lambda^{A-2} y^{Ai} = rA(rA-1)\lambda^{rA-2} F.$$

For $\lambda = 1$, (16) and (17) give (14) and (15) respectively. □

Remark 1. For the scalar function on $Osc^1 M$, i.e. $F = F(x, y) = F(y^{0i}, y^{1i})$ the definitions of first, second and third kinds of homogeneity coincide.

Example 1. The function

$$F(y^{0i}, y^{1i}, y^{2i}) = \frac{a_i(x) y^{2i}}{b_j(x) y^{1j}} \quad (y^{0i} = x^i)$$

is homogeneous of first kind and order 1, or of second kind and order 0, namely

$$F(y^{0i}, \lambda y^{1i}, \lambda^2 y^{2i}) = \frac{a_i(x)\lambda^2 y^{2i}}{b_j(x)\lambda y^{1j}} = \lambda F(y^{0i}, y^{1i}, y^{2i})$$

$$F(y^{0i}, \lambda^{1i}, \lambda y^{2i}) = F(y^{0i}, y^{1i}, y^{2i}).$$

For this function, the relations (5) and (9) give

$$-\frac{a_i(x) y^{2i}}{b_j(x)(y^{1j})^2} \cdot y^{1j} + \frac{a_i(x)}{b_j(x) y^{1j}} \cdot 2y^{2i} = \frac{a_i(x) y^{2i}}{b_j(x) y^{1j}}$$

$$-\frac{a_i(x) y^{2i}}{b_j(x)(y^{1j})^2} \cdot y^{1j} + \frac{a_i(x)}{b_j(x) y^{1j}} \cdot y^{2i} = 0.$$

Example 2. The function

$$F(y^{0i}, y^{1i}, y^{2i}) = a_i(x) y^{2i} + b_{ij}(x) y^{1i} y^{1j} + \frac{c_i(x) y^{3i}}{d_j(x) y^{1j}}$$

is homogeneous of first kind and order 2. For this example (5) and (6) reduce to:

$$\frac{\partial F}{\partial y^{1i}} \cdot y^{1i} + \frac{\partial F}{\partial y^{2i}} 2y^{2i} + \frac{\partial F}{\partial y^{3i}} \cdot 3y^{3i} = 2F$$

$$\frac{\partial^2 F}{\partial y^{1i} \partial y^{1j}} \cdot y^{1i} y^{1j} + 2\frac{\partial^2 F}{\partial y^{1i} \partial y^{3j}} \cdot 3y^{1i} y^{3j} + \frac{\partial F}{\partial y^{2i}} \cdot 2y^{2i} + \frac{\partial F}{\partial y^{3i}} \cdot 6y^{3i} = 2F.$$

Straightforward calculations lead to these equations,

Example 3. The function

$$F(x, y^{(1)}, y^{(2)}, y^{(3)}) = a_{ijk}(x) y^{1i} (y^{2j})^3 (y^{3k})^2$$

is homogeneous of third kind and order 3 with respect to $y^{(2)}$ and homogeneous of third kind and order 2 with respect to $y^{(3)}$, namely

$$F(x, y^{(1)}, \lambda y^{(2)}, y^{(3)}) = \lambda^3 F(x, y^{(1)}, y^{(2)}, y^{(3)})$$

$$F(x, y^{(1)}, y^{(2)}, \lambda y^{(3)}) = \lambda^2 F(x, y^{(1)}, y^{(2)}, y^{(3)}).$$

For this example, (13) has the form

$$\frac{\partial F}{\partial y^{2i}} y^{2i} = 3F, \quad \frac{\partial^2 F}{\partial y^{2i} \partial y^{2j}} y^{2i} y^{2j} = 3 \cdot 2F$$

$$\frac{\partial F}{\partial y^{3i}} y^{3i} = 2F, \quad \frac{\partial^2 F}{\partial y^{3i} \partial y^{3j}} y^{3i} y^{3j} = 2 \cdot 1F.$$

Example 4. The vector field [5]

$$z^{2i} = y^{2i} + \frac{1}{2} \gamma_j{}^i{}_k y^{1j} y^{1k}$$

in $L^{(2)n}$ is homogeneous of first kind and order 2. The scalar functions:

$$\alpha(x, y^1, y^2) = [a_{ij}(x) z^{2i} z^{2j}]^{1/2}, \quad \beta(x, y^1, y^2) = b_i(x) z^{2i}$$

are homogeneous of first kind and order 2. The fundamental functions

$$F(x, y^{(1)}, y^{(2)}) = \alpha + \beta \qquad \text{(Randers metric)}$$

$$F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2}{\beta} \qquad \text{(Kropina metric)}$$

$$F(x, y^{(1)}, y^{(2)}) = \frac{\alpha^2}{\alpha - \beta} \qquad \text{(Matsumoto metric)}$$

are homogeneous of first kind and order 2.

In the Lagrange space $L^{(2)n} = (M, L(x, y^{(1)}, y^{(2)})$ with

$$L(x, y^{(1)}, y^{(2)}) = F^2(x, y^{(1)}, y^{(2)}),$$

the metric tensor:

$$c_{ij}(x, y^{(1)}, y^{(2)}) = \frac{1}{2}\frac{\partial^2 L(x, y^{(1)}, y^{(2)})}{\partial y^{2i}\partial y^{2j}}$$

is homogeneous of first kind and order 0, i.e.

$$c_{ij}(x, \lambda y^{(1)}, \lambda^2 y^{(2)}) = \lambda^0 c_{ij}(x, y^{(1)}, y^{(2)}).$$

2. The Euler-Lagrange equations for functions of several higher order variables

To obtain the general Euler-Lagrange equations for functions of several higher order variables we shall restrict our attention to one special case. Let us consider

$$I = \int_0^1 F(t, x, \dot{x}, \ddot{x}, x^{(3)}, p, \dot{p}, \ddot{p}, p^{(3)})dt, \tag{18}$$

where $x = x(t), p = p(t), \dot{x} = \dot{x}(t), \dot{p} = \dot{p}(t), \ldots$ and $\delta x(t), \delta\dot{x}(t)$, $\delta\ddot{x}(t)$, $\delta p(t)$, $\delta\dot{p}(t)$, $\delta\ddot{p}(t)$ at $t = 0$ and $t = 1$ have value zero. Using the partial integration and the property: $\frac{d}{dt}\delta x = \delta\dot{x}, \ldots, \frac{d}{dt}\delta\ddot{x} = \delta x^{(3)}$ (similarly for p), from (18) we get the first variation of I [6]:

$$\begin{aligned}
\delta I &= \int_0^1 \left(\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial \dot{x}}\delta\dot{x} + \frac{\partial F}{\partial \ddot{x}}\delta\ddot{x} + \frac{\partial F}{\partial x^{(3)}}\delta x^{(3)} + \right. \\
&\quad \left. + \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial \dot{p}}\delta\dot{p} + \frac{\partial F}{\partial \ddot{p}}\delta\ddot{p} + \frac{\partial F}{\partial p^{(3)}}\delta p^{(3)}\right) dt \\
&= \int_0^1 \left(\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial p}\delta p\right) dt + \int_0^1 \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\delta x + \frac{\partial F}{\partial \ddot{x}}\delta\dot{x} + \frac{\partial F}{\partial x^{(3)}}\delta\ddot{x}\right) dt + \\
&\quad + \int_0^1 \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{p}}\delta p + \frac{\partial F}{\partial \ddot{p}}\delta\dot{p} + \frac{\partial F}{\partial p^{(3)}}\delta\ddot{p}\right) dt
\end{aligned}$$

$$-\int_0^1 \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\delta x + \frac{d}{dt}\frac{\partial F}{\partial \ddot{x}}\delta \dot{x} + \frac{d}{dt}\frac{\partial F}{\partial x^{(3)}}\delta \ddot{x} \right) dt$$

$$-\int_0^1 \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{p}}\delta p + \frac{d}{dt}\frac{\partial F}{\partial \ddot{p}}\delta \dot{p} + \frac{d}{dt}\frac{\partial F}{\partial p^{(3)}}\delta \ddot{p} \right) dt.$$

We have

$$\int_0^1 \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\delta x + \frac{\partial F}{\partial \ddot{x}}\delta \dot{x} + \frac{\partial F}{\partial x^{(3)}}\delta \ddot{x} \right) dt = \frac{\partial F}{\partial \dot{x}}\delta x\Big|_0^1 + \frac{\partial F}{\partial \ddot{x}}\delta \dot{x}\Big|_0^1 + \frac{\partial F}{\partial x^{(3)}}\delta \ddot{x}\Big|_0^1 = 0$$

because

$$\delta x(1) = \delta \dot{x}(1) = \delta \ddot{x}(1) = 0, \quad \delta x(0) = \delta \dot{x}(0) = \delta \ddot{x}(0) = 0.$$

The same is true if in the above equation x is changed with p. Now we have

$$\begin{aligned}
\delta I \;=\;& \int_0^1 \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} \right) \delta x dt + \int_0^1 \left(\frac{\partial F}{\partial p} - \frac{d}{dt}\frac{\partial F}{\partial \dot{p}} \right) \delta p dt \\
& - \int_0^1 \frac{d}{dt}\left(\frac{d}{dt}\frac{\partial F}{\partial \ddot{x}}\delta x + \frac{d}{dt}\frac{\partial F}{\partial x^{(3)}}\delta \dot{x} \right) dt + \int_0^1 \left(\frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}}\delta x + \frac{d^2}{dt^2}\frac{\partial F}{\partial x^{(3)}}\delta \dot{x} \right) dt \\
& - \int_0^1 \frac{d}{dt}\left(\frac{d}{dt}\frac{\partial F}{\partial \ddot{p}}\delta p + \frac{d}{dt}\frac{\partial F}{\partial p^{(3)}}\delta \dot{p} \right) dt + \int_0^1 \left(\frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{p}}\delta p + \frac{d^2}{dt^2}\frac{\partial F}{\partial p^{(3)}}\delta \dot{p} \right) dt = \\
& \int_0^1 \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}} \right) \delta x dt + \int_0^1 \left(\frac{\partial F}{\partial p} - \frac{d}{dt}\frac{\partial F}{\partial p} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{p}} \right) \delta p dt + A,
\end{aligned}$$

where

$$\begin{aligned}
A \;=\;& \int_0^1 \left(\frac{d^2}{dt^2}\frac{\partial F}{\partial x^{(3)}}\delta \dot{x} + \frac{d^2}{dt^2}\frac{\partial F}{\partial p^{(3)}}\delta \dot{p} \right) dt = \\
=\;& \int_0^1 \frac{d}{dt}\left(\frac{d^2}{dt^2}\frac{\partial F}{\partial x^{(3)}}\delta x + \frac{d^2}{dt^2}\frac{\partial F}{\partial p^{(3)}}\delta p \right) dt - \int_0^1 \left(\frac{d^3}{dt^3}\frac{\partial F}{\partial x^{(3)}}\delta x + \frac{d^3}{dt^3}\frac{\partial F}{\partial p^{(3)}}\delta p \right) dt.
\end{aligned}$$

Using the same procedure as before we have

$$\delta I = \int_0^1 \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}} - \frac{d^3}{dt^3}\frac{\partial F}{\partial x^{(3)}} \right) \delta x dt + \tag{19}$$

$$\int_0^1 \left(\frac{\partial F}{\partial p} - \frac{d}{dt}\frac{\partial F}{\partial \dot{p}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{p}} - \frac{d^3}{dt^3}\frac{\partial F}{\partial p^{(3)}} \right) \delta p dt.$$

We shall use the notations $x = x^{(0)}, \dot{x} = x^{(1)}, \ddot{x} = x^{(2)}, \ldots, \frac{d^0}{dt^0}G = G$. From (19), using similar calculations, we obtain

Theorem 5. *The extreme value of functional*

$$I = \int_0^1 F(t, x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(k)}, p^{(0)}, p^{(1)}, p^{(2)}, \ldots, p^{(k)}) dt,$$

where $x^{(l)} = x^{(l)}(t), p^{(l)} = p^{(l)}(t),\ l = \overline{0,k}$ *and* $\delta x^{(l)}(1) = 0,\ \delta x^{(l)}(0) = 0,$ $\delta p^{(l)}(1) = 0, \delta p^{(l)}(0) = 0, \quad l = \overline{0,k}$, *can be obtained, when*

$$\sum_{l=0}^{k} (-1)^l \frac{d^l}{dt^l}\frac{\partial F}{\partial x^{(l)}} = 0, \quad \sum_{l=0}^{k} (-1)^l \frac{d^l}{dt^l}\frac{\partial F}{\partial p^{(l)}} = 0,$$

i.e.

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x^{(1)}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial x^{(2)}} - \cdots + (-1)^k \frac{d^k}{dt^k}\frac{F}{\partial x^{(k)}} = 0 \tag{20}$$

$$\frac{\partial F}{\partial p} - \frac{d}{dt}\frac{\partial F}{\partial p^{(1)}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial p^{(2)}} - \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial F}{\partial p^{(k)}} = 0. \tag{21}$$

The above equations are Euler-Lagrange equations. In the case when x denotes $(x^1, x^2, \ldots, x^n) = (x^i)$ and $p = (p_1, p_2, \ldots, p_n) = (p_i)$, i.e. when x and p are points in n-dimensional spaces, then the Euler-Lagrange equations have the form:

$$\frac{\partial F}{\partial x^{(0)i}} - \frac{d}{dt}\frac{\partial F}{\partial x^{(1)i}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial x^{(2)i}} - \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial F}{\partial x^{(k)i}} = 0 \tag{22}$$

$$\frac{\partial F}{\partial p_{(0)i}} - \frac{d}{dt}\frac{\partial F}{\partial p_{(1)i}} + \frac{d^2}{dt^2}\frac{\partial F}{\partial p_{(2)i}} - \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial F}{\partial p_{(k)i}} = 0.$$

In the recent literature it is more usual to use the notations:

$$x^{(l)i} = y^{(l)i} = y^{li},\ p_{(l)i} = p_{li},\ x^{0i} = x^i,\ p_{0i} = p_i,\ l = \overline{0,k},\ i = \overline{1,n}. \tag{23}$$

3. The connection between Lagrangians and Hamiltonians of higher order

Let $L(t, y^{(0)}(t), y^{(1)}(t), \dots, y^{(k)}(t))$ be a regular Lagrangian in $L^{(k)n}$ and $H(t, p_{(0)}(t), p_{(1)}(t), \dots, p_{(k)}(t))$ the Hamiltonian in the $H^{(k)n}$ space. We shall examine the variation of

$$\begin{aligned} &F(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}, p_{(0)}, p_{(1)}, \dots, p_{(k)}) = \\ &H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)}) + L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}) \\ &-y^{0i}p_{ki} - y^{1i}p_{(k-1)i} - \dots - y^{(k-1)i}p_{1i} - y^{ki}p_{0i}. \end{aligned} \tag{24}$$

To obtain the general formula we shall similar to (18) first consider the special form of F:

$$\begin{aligned} &F = F(t, x, \dot{x}, \ddot{x}, x^{(3)}, p, \dot{p}, \ddot{p}, p^{(3)}) = \\ &H(t, p_i, \dot{p}_i, \ddot{p}_i, p_i^{(3)}) + L(t, x^i, \dot{x}^i, \ddot{x}^i, x^{(3)i}) - \\ &-x^i p_i^{(3)} - \dot{x}^i \ddot{p}_i - \ddot{x}^i \dot{p}_i - x^{(3)i} p_i. \end{aligned} \tag{25}$$

From (18) and (22) it follows that $\delta I = \int_0^1 F dt = 0$ if

$$\frac{\partial F}{\partial x^i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{x}^i} - \frac{d^3}{dt^3}\frac{\partial F}{\partial x^{(3)i}} = 0, \quad i = \overline{1,n},$$

$$\frac{\partial F}{\partial p_i} - \frac{d}{dt}\frac{\partial F}{\partial \dot{p}_i} + \frac{d^2}{dt^2}\frac{\partial F}{\partial \ddot{p}_i} - \frac{d^3}{dt^3}\frac{\partial F}{\partial p_i^{(3)}} = 0, \quad i = \overline{1,n}.$$

From the above we get

$$\frac{\partial L}{\partial x^i} - p_i^{(3)} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} - \ddot{p}_i\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}^i} - \dot{p}_i\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial x^{(3)i}} - p_i\right) = 0$$

$$\frac{\partial H}{\partial p_i} - x^{(3)i} - \frac{d}{dt}\left(\frac{\partial H}{\partial \dot{p}_i} - \ddot{x}^i\right) + \frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \ddot{p}_i} - \dot{x}^i\right) - \frac{d^3}{dt^3}\left(\frac{\partial H}{\partial p_i^{(3)}} - x^i\right) = 0.$$

As $\dot{p}_i = \frac{d}{dt}p_i, \dot{x}^i = \frac{d}{dt}x^i, \ldots$, we have

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^i} - \frac{d^3}{dt^3}\frac{\partial L}{\partial x^{(3)i}} = 0 \tag{26}$$

$$\frac{\partial H}{\partial p_i} - \frac{d}{dt}\frac{\partial H}{\partial \dot{p}_i} + \frac{d^2}{dt^2}\frac{\partial H}{\partial \ddot{p}_i} - \frac{d^3}{dt^3}\frac{\partial H}{\partial p_i^{(3)}} = 0. \tag{27}$$

The Euler-Lagrange equation of F given by (25) are (26) and (27). From (25) we get

$$\begin{aligned} dF = & \left(\frac{\partial H}{\partial t} + \frac{\partial L}{\partial t}\right) dt + \left(\frac{\partial L}{\partial x^i} - p_i^{(3)}\right) dx^i + \left(\frac{\partial L}{\partial \dot{x}^i} - \ddot{p}_i\right) d\dot{x}^i + \\ & \left(\frac{\partial L}{\partial \ddot{x}^i} - \dot{p}_i\right) d\ddot{x}^i + \left(\frac{\partial L}{\partial x^{(3)i}} - p_i\right) dx^{(3)i} + \\ & \left(\frac{\partial H}{\partial p_i} - x^{(3)i}\right) dp_i + \left(\frac{\partial H}{\partial \dot{p}_i} - \ddot{x}^i\right) d\dot{p}_i + \\ & \left(\frac{\partial H}{\partial \ddot{p}_i} - \dot{x}^i\right) d\ddot{p}_i + \left(\frac{\partial H}{\partial p_i^{(3)}} - x\right) dp_i^{(3)} = 0. \end{aligned} \tag{28}$$

As the variables are independent, from (28) it follows, that $dF = 0$ is equivalent to the following relations

$$\begin{aligned} \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \quad & \frac{\partial L}{\partial x^i} = p_i^{(3)} \quad \frac{\partial L}{\partial \dot{x}^i} = \ddot{p}_i \quad \frac{\partial L}{\partial \ddot{x}^i} = \dot{p}_i \quad \frac{\partial L}{\partial x^{(3)i}} = p_i \\ & \frac{\partial H}{\partial p_i} = x^{(3)i} \quad \frac{\partial H}{\partial \dot{p}_i} = \ddot{x}^i \quad \frac{\partial H}{\partial \ddot{p}_i} = \dot{x}^i \quad \frac{\partial H}{\partial p_i^{(3)}} = x^i. \end{aligned} \tag{29}$$

If we replace equations (29) into (26) and (27) we obtain $0 = 0$. For the function

$$\begin{aligned} & F(t, x, \dot{x}, \ddot{x}, p, \dot{p}, \ddot{p}) = H(t, p, \dot{p}, \ddot{p}) + \\ & L(t, x, \dot{x}, \ddot{x}) - x^i\ddot{p}_i - \dot{x}^i\dot{p}_i - \ddot{x}^i p_i, \end{aligned}$$

the Euler-Lagrange equations (26) and (27) have the form:

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^i} - \ddot{p}_i = 0 \tag{30}$$

$$\frac{\partial H}{\partial p_i} - \frac{d}{dt}\frac{\partial H}{\partial \dot{p}_i} + \frac{d^2}{dt^2}\frac{\partial H}{\partial \ddot{p}_i} - \ddot{x}^i = 0 \tag{31}$$

and equation (29) have the form

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \qquad \frac{\partial L}{\partial x^i} = \ddot{p}_i \quad \frac{\partial L}{\partial \dot{x}^i} = \dot{p}_i \quad \frac{\partial L}{\partial \ddot{x}^i} = p_i \tag{32}$$

$$\frac{\partial H}{\partial p_i} = \ddot{x}^i \quad \frac{\partial H}{\partial \dot{p}_i} = \dot{x}^i \quad \frac{\partial H}{\partial \ddot{p}_i} = x^i.$$

If we substitute (32) into (30) and (31) we obtain $0 = 0$, i.e. (32) satisfy (30) and (31). From the above it follows:

Theorem 6. *The Euler-Lagrange equations for the function F determined by (24) are:*

(a) for $k = 2l + 1$

$$A = \frac{\partial L}{\partial y^{0i}} - \frac{d}{dt}\frac{\partial L}{\partial y^{1i}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial y^{2i}} - \frac{d^3}{dt^3}\frac{\partial L}{\partial y^{3i}} + \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial y^{ki}} = 0 \tag{33}$$

$$B = \frac{\partial H}{\partial p_{0i}} - \frac{d}{dt}\frac{\partial H}{\partial p_{1i}} + \frac{d^2}{dt^2}\frac{\partial H}{\partial p_{2i}} - \frac{d^3}{dt^3}\frac{\partial H}{\partial p_{3i}} + \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial H}{\partial p_{ki}} = 0 \tag{34}$$

(b) for $k = 2l$ the Euler-Lagrange equations have the form:

$$A - p_{ki} = 0, \quad B - y^{ki} = 0 \tag{35}$$

where $p_{ki} = \frac{d^k p_i}{dt^k}, y^{ki} = \frac{d^k x^i}{dt^k}$. The equations which give relations between L and H for both cases (a) and (b) have the form

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}, \ \frac{\partial L}{\partial y^{0i}} = p_{ki}, \ \frac{\partial L}{\partial y^{1i}} = p_{(k-1)i}, \tag{36}$$

$$\frac{\partial L}{\partial y^{2i}} = p_{(k-2)i}, \dots, \frac{\partial L}{\partial y^{ki}} = p_{0i} = p_i$$

$$\frac{\partial H}{\partial p_{0i}} = y^{ki}, \ \frac{\partial H}{\partial p_{1i}} = y^{(k-1)i}, \dots, \frac{\partial H}{\partial p_{(k-1)i}} = y^{1i}, \ \frac{\partial H}{\partial p_{ki}} = y^{0i}.$$

Theorem 7. *If $L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)})$ and $H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)})$ are homogeneous functions of first kind and order k, i.e.*

$$L(t, y^{(0)}, \lambda y^{(1)}, \dots, \lambda^k y^k) = \lambda^k L(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}), \tag{37}$$

$$H(t, p_{(0)}, \lambda p_{(1)}, \dots, \lambda^k p_{(k)}) = \lambda^k H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)}), \qquad (38)$$

then the function $F(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}, p_{(0)}, p_{(1)}, \dots, p_{(k)})$ *defined by (24) is also homogeneous of first kind and order* k, *i.e.*

$$F(t, y^{(0)}, \lambda y^{(1)}, \dots, \lambda^k y^{(k)}, p_{(0)}, \lambda p_{(1)}, \dots, \lambda^k p_{(k)}) =$$

$$\lambda^k F(t, y^{(0)}, y^{(1)}, \dots, y^{(k)}, p_{(0)}, p_{(1)}, \dots, p_{(k)}).$$

If we suppose that $L(t, y^{(0)}, \dots, y^{(k)})$ with the property (37) is the fundamental function in the Finsler space of order $k(F^{(k)n})$, then the metric tensor in this space can be defined by

$$g_{ij}(t, y^{(0)}, \dots, y^{(k)}) = \frac{\partial^2 L^2(t, y^{(0)}, \dots, y^{(k)})}{\partial y^{ki} \partial y^{kj}}. \qquad (39)$$

If $H(t, p_{(0)}, p_{(1)}, \dots, p_{(k)})$ with the property (38) is the fundamental function in the Cartan space of order k $(C^{(k)n})$, then the metric tensor in this space can be determined by

$$c^{ij}(t, p_{(0)}, p_{(1)}, \dots, p_{(k)}) = \frac{\partial^2 H^2(t, p_{(0)}, p_{(1)}, \dots, p_{(k)})}{\partial p_{ki} \partial p_{kj}}. \qquad (40)$$

Both metric tensors g_{ij} and c^{ij} given by (39) and (40) respectively are homogeneous of first kind and order zero. The former statements can be very useful in the investigation of $F^{(k)n}$ and $C^{(k)n}$ spaces recently introduced. The connection between the function F in (24) and energies of higher order can be investigated.

4. Invariant form of Euler-Lagrange equations

We want to show that Euler-Lagrange equations of type (20) are coordinate invariant. For that reason we restrict k to $k = 3$, and let $F = L(t, y^{0i}, y^{1i}, y^{2i}, y^{3i})$ be a regular Lagrangian in $Osc^3 M$.

We shall use the notations

$$\partial_{li} = \frac{\partial}{\partial y^{li}}, \; d_t^l = \frac{d^l}{dt^l}, \quad l = \overline{0, 3}.$$

In $Osc^3 M$ the allowable transformations are: ([1], [7], [2], [3], [4], etc)

$$y^{0i'} = y^{0i'}(y^{0i}), \quad y^{0i} = x^i \qquad (41)$$

$$y^{1i'} = (\partial_{0i} y^{0i'}) y^{1i} = B_i^{i'} y^{1i}$$

$$y^{2i'} = (d_t^1 B_i^{i'}) y^{1i} + B_i^{i'} y^{2i}$$

$$y^{3i'} = (d_t^2 B_i^{i'}) y^{1i} + 2(d_t^1 B_i^{i'}) y^{2i} + B_i^{i'} y^{3i}.$$

The transformations of the form (41) form a group. The natural basis of $T(Osc^3 M)$ is $\{\partial_{0i}, \partial_{1i}, \partial_{2i}, \partial_{3i}\}$ and it is transforming as follows:

$$\begin{array}{lclclclcl}
\partial_{0i} & = & (\partial_{0i} y^{0i'})\partial_{0i'} & + & (\partial_{0i} y^{1i'})\partial_{1i'} & + & (\partial_{0i} y^{2i'})\partial_{2i'} & + & (\partial_{0i} y^{3i'})\partial_{3i'} \\
\partial_{1i} & = & & & (\partial_{1i} y^{1i'})\partial_{1i'} & + & (\partial_{1i} y^{2i'})\partial_{2i'} & + & (\partial_{1i} y^{3i'})\partial_{3i'} \\
\partial_{2i} & = & & & & & (\partial_{2i} y^{2i'})\partial_{2i'} & + & (\partial_{2i} y^{3i'})\partial_{3i'} \\
\partial_{3i} & = & & & & & & & (\partial_{3i} y^{3i'})\partial_{3i'}.
\end{array} \tag{42}$$

It is known that ([8], [1], [9], [7], [2], [3], [4])

$$\partial_{0i} y^{0i'} = \partial_{1i} y^{1i'} = \partial_{2i} y^{2i'} = \partial_{3i} y^{3i'}, \tag{43}$$

$$\partial_{0i} y^{1i'} = \frac{1}{2}\partial_{1i} y^{2i'} = \frac{1}{3}\partial_{2i} y^{3i'}, \partial_{0i} y^{2i'} = \frac{1}{3}\partial_{1i} y^{3i'}, \ldots (\partial_{0i} y^{si'} = \frac{1}{\binom{r+s}{r}}\partial_{ri} y^{(r+s)i'}),$$

$$d_t^1 \partial_{0i} y^{0i'} = \partial_{0i} y^{1i'},\ d_t^2 \partial_{0i} y^{0i'} = \partial_{0i} y^{2i'},\ d_t^1 \partial_{0i} y^{1i'} = \partial_{0i} y^{2i'},\ d_t^1 \partial_{0i} y^{2i'} = \partial_{0i} y^{3i'}.$$

Let us denote

$$\delta x^i = \varepsilon v^i(t) = \varepsilon v^{0i},\ \delta \dot{x}^i = \varepsilon d_t^1 v^i = \varepsilon v^{1i},\ \delta \ddot{x} = \varepsilon d_t^2 v^i = \varepsilon v^{2i},\ \delta x^{(3)} = \varepsilon d_t^3 v^i = \varepsilon v^{3i}.$$

Using the similar procedure as in (19) (only the initial conditions are not substituted), we obtain for $I = \int\limits_0^1 L dt$:

$$\frac{\delta I}{\varepsilon} = \int\limits_0^1 [v^{0i}(\partial_{0i} L) + v^{1i}(\partial_{1i} L) + v^{2i}(\partial_{2i} L) + v^{3i}(\partial_{3i} L)] dt. \tag{44}$$

Using the notations

$$E_i^0 = \partial_{0i} - d_t^1 \partial_{1i} + d_t^2 \partial_{2i} - d_t^3 \partial_{3i} \tag{45}$$

$$E_i^1 = \partial_{1i} - d_t^1 \partial_{2i} + d_t^2 \partial_{3i}$$

$$E_i^2 = \partial_{2i} - d_t^1 \partial_{3i}$$

$$E_i^3 = \partial_{3i}$$

(44) can be written in the form

$$\frac{\delta I}{\varepsilon} = \int_0^1 \{v^{0i}(E_i^0 L) + d_t^1[v^{0i}(E_i^1 L) + v^{1i}(E_i^2 L) + v^{2i}(E_i^3 L)]\} dt. \tag{46}$$

The comparison of (44) and (46) gives

$$(v^{0i}\partial_{0i} + v^{1i}\partial_{1i} + v^{2i}\partial_{2i} + v^{3i}\partial_{3i})L = \tag{47}$$
$$v^{0i}(E_i^0 L) + d_t^1[v^{0i}(E_i^1 L) + v^{1i}(E_i^2 L) + v^{2i}(E_i^3 L).$$

From (45), it follows that

$$E_i^0 + d_t^1 E_i^1 = \partial_{0i},\ E_i^1 + d_t^1 E_i^2 = \partial_{1i}, \tag{48}$$
$$E_i^2 + d_t^1 E_i^3 = \partial_{2i},\ E_i^3 = \partial_{3i}.$$

If we use the condition that $v^{0i}(t)$, $v^{1i}(t)$ and $v^{2i}(t)$ vanish for $t = 0$ and $t = 1$, then from (46), we get

$$\frac{\delta I}{\varepsilon} = \int_0^1 v^{0i}(E_i^0 L) dt. \tag{49}$$

Lemma 8. *Under the transformation (41), E_1^0 is covariant vector field, i.e. ([8], [3], [4])*

$$E_i^0 = (\partial_{0i} y^{0i'}) E_{i'}^0 = B_i^{i'} E_{i'}^0. \tag{50}$$

Proof. Using (42) and (43), we have

$$\begin{aligned} E_i^0 \ = \ & (\partial_{0i} y^{0i'})\partial_{0i'} + (\partial_{0i} y^{1i'})\partial_{1i'} + (\partial_{0i} y^{2i'})\partial_{2i'} + (\partial_{0i} y^{3i'})\partial_{3i'} \\ & -d_t^1[(\partial_{0i} y^{0i'})\partial_{1i'} + 2(\partial_{0i} y^{1i'})\partial_{2i'} + 3(\partial_{0i} y^{2i'})\partial_{3i'}] \\ & +d_t^2[(\partial_{0i} y^{0i'})\partial_{2i'} + 3(\partial_{0i} y^{1i'})\partial_{3i'}] - d_t^3[(\partial_{0i} y^{0i'})\partial_{3i'}] = \\ & (\partial_{0i} y^{0i'})\partial_{0i'} + [(\partial_{0i} y^{1i'}) - (\partial_{0i} y^{1i'})]\partial_{1i'} \\ & +[(\partial_{0i} y^{2i'}) - 2(\partial_{0i} y^{2i'}) + (\partial_{0i} y^{2i'})]\partial_{2i'} \\ & +[(\partial_{0i} y^{3i'}) - 3(\partial_{0i} y^{3i'}) + 3(\partial_{0i} y^{3i'}) - (\partial_{0i} y^{3i'})]\partial_{3i'} \\ & -(\partial_{0i} y^{0i'}) d_t^1 \partial_{1i'} + (\partial_{0i} y^{0i'}) d_t^2 \partial_{2i'} - (\partial_{0i} y^{0i'}) d_t^3 \partial_{3i'} \end{aligned}$$

$$+(\partial_{0i}y^{1i'})[-2d_t^1\partial_{2i'}+2d_t^1\partial_{2i'}]$$

$$+(\partial_{0i}y^{2i'})[-3d_t^1\partial_{3i'}+6d_t^1\partial_{3i'}-3d_t^1\partial_{3i'}]$$

$$+(\partial_{0i}y^{1i'})[-3d_t^2\partial_{3i}+3d_t^2\partial_{3i'}]=$$

$$(\partial_{0i}y^{0i'})[\partial_{0i'}-d_t^1\partial_{1i'}+d_t^2\partial_{2i'}-d_t^3\partial_{3i'}]=B_i^{i'}E_i^0.$$

In the above the Leibniz rule of differentiation was used. □

Theorem 9. *The Euler-Lagrange equation* $(v^{0i}E_i^0)L = 0$ *is coordinate invariant if* $v^{0i}=B_{i'}^iv^{0i'}$, *i.e. if* $v^i=v^{0i}$ *is a contravariant vector field.*

Proof. From (50), we get

$$v^{0i}E_i^0=v^{0i}B_i^{j'}E_{j'}^0=v^{0j'}E_{j'}^0 \tag{51}$$

i.e., $v^{0j'}=v^{0i}B_i^{j'}$. The fields E_i^1, E_i^2, E_i^3 defined by (45) are transforming in the following way:

$$E_i^1=(\partial_{0i}y^{0i'})E_{i'}^1+(\partial_{0i}y^{1i'})E_{i'}^2+(\partial_{0i}y^{2i'})E_{i'}^3 \tag{52}$$

$$E_i^2=(\partial_{0i}y^{0i'})E_{i'}^2+2(\partial_{0i}y^{1i'})E_{i'}^3,\quad E_i^3=(\partial_{0i}y^{0i'})E_{i'}^3.$$

From (50) and (52) it is clear, that only E_i^0 and E_i^3 are covector fields. □

Theorem 10. *The expression* $v^{0i}E_i^0+d_t^1B$, *which appears in (46) is coordinate invariant, where*

$$B=v^{0i}E_i^1+v^{1i}E_i^2+v^{2i}E_i^3. \tag{53}$$

Proof. From (51), we have $v^{0i}E_i^0=v^{0i'}E_{i'}^0$. Further:

$$\begin{aligned} B \;=\; & (\partial_{0j'}y^{0i})v^{0j'}[(\partial_{0i}y^{0i'})E_{i'}^1+(\partial_{0i}y^{1i'})E_{i'}^2+(\partial_{0i}y^{2i'})E_{i'}^3]+ \\ & [(\partial_{0j'}y^{1i})v^{0j'}+(\partial_{0j'}y^{0i})v^{1j'}][(\partial_{0i}y^{0i'})E_{i'}^2+2(\partial_{0i}y^{1i'})E_{i'}^3]+ \\ & [(\partial_{0j'}y^{2i'})v^{0j'}+2(\partial_{0j'}y^{1i})v^{1j'}+(\partial_{0j'}y^{0i})v^{2j'}](\partial_{0i}y^{0i'})E_{i'}^3. \end{aligned} \tag{54}$$

Using the relations

$$y^{1i'}=B_h^{i'}y^{1h},\ y^{1i}=B_{h'}^iy^{1h'},\ y^{2i'}=B_h{}^{i'}{}_ky^{1h}y^{1k}+B_h^{i'}y^{2h},$$

$$y^{2i}=B_{h'}{}^i{}_{k'}y^{1h'}y^{1k'}+B_{h'}^iy^{2h'},$$

$$\partial_{0i}y^{1h}=0,\quad \partial_{0i}y^{2h}=0,\quad \partial_{0i'}y^{1h'}=0,\quad \partial_{0i'}y^{2h'}=0$$

obtained from (41) and (42), and

$$(\partial_{0j'}y^{0i})(\partial_{0i}y^{0i'}) = B^i_{j'}B^{i'}_i = \delta^{i'}_{j'},\ B_{j'}{}^i{}_{k'}B^{i'}_i + B^i_{j'}B_i{}^{i'}{}_kB^k_{k'} = 0,$$

$$B^i_{j'k'h'}B^{i'}_i + B_{j'}{}^i{}_{k'}B_i{}^{i'}{}_hB^h_{h'} + B_{j'}{}^i{}_{h'}B_i{}^{i'}{}_kB^k_{k'} + B^i_{j'}B^{i'}_{ihk}B^h_{h'}B^k_{k'} + B^i_{j'}B_i{}^{i'}{}_kB_{k'}{}^k{}_{h'} = 0$$

and similar type relations obtained from $B^i_{j'}B^{j'}_j = \delta^i_j$, after longer calculations, we get

$$B = v^{0i'}E^1_{i'} + v^{1i'}E^2_{i'} + v^{2i'}E^3_{i'}. \tag{55}$$

The other coefficients besides $v^{0j'}E^2_{i'}$, $v^{0j'}E^3_{i'}$, $v^{1j'}E^3_{i'}$ in (54) are equal to zero. Using (55), (51) and (53), we get

$$[v^{0i}E^0_i + d^1_t(v^{0i}E^1_i + v^{1i}E^2_i + v^{2i}E^3_i)]L = \tag{56}$$

$$[v^{0i'}E^0_{i'} + d^1_t(v^{0i'}E^1_{i'} + v^{1i'}E^2_{i'} + v^{3i'}E^3_{i'})]L.$$

From (53) and (55), we obtain that B defined by (53) is a scalar field. □

R E F E R E N C E S

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