DATA ENVELOPMENT SCENARIO ANALYSIS IN A FORM OF MULTIPLICATIVE MODEL

Najmeh Malekmohammadi[1], Mahboubeh Farid[2]

In this paper a new target model which is in the framework of multiplicative data envelopment analysis (DEA) models is presented. This model is proposed especially for limited resources in total input consumption and total output production. Considering the application of multiplicative models in DEA especially when we encounter the ratio variables in the data set, a new target model is established in a multiplicative types. With the numerical results, the advantages of the new model over our previous one is discussed completely.

Keywords: Data Envelopment Analysis, Target Setting, Limited Resources, Ratio Analysis.

1. Introduction

Evaluating the relative efficiency of decision making units has been proposed under the name of data envelopment analysis (DEA) by Charnes et al. [1]. DEA model originally is a non-linear fractional mathematical programming model, known as the Charnes, Cooper and Rhodes (CCR) model. The objective function in this model is considered to reach the best set of weights for the single ratio of the weighted outputs to the weighted inputs for a particular decision making unit (DMU) denoted by DMUo.

Evaluating the relative efficiency is not the only usage of DEA; target setting and resource allocation are the another attitude of DEA. For instance, there are some situations in which all the units belong to the same organization and there is a centralized decision maker who supervises these units and he/she also desire to set a target to future planning of his/her organization. Many models and methods have been proposed for target setting and resource allocation which include the decision maker's preference in target setting process, for example, Athanassopoulos ([2], [3], [4]), Golany [5], Korhonen and Syrjnen [6], Lozano and Villa [7], Thanassoulis and Dyson [8] and Malekmohammadi et al. [9].
Malekmohammadi et al. [10] suggested a target model for limited resources in total input consumption or total output production. By using the proposed model, we are able to increase total output production and avoid total output production being unchanged. Considering the importance of imprecise data in organizations, the

[1]Department of Mathematics, Faculty of Science, South Tehran Branch, Islamic Azad University, Tehran, Iran, e-mail: n_malekmohammadi@azad.ac.ir; n.malekmohammadi@gmail.com
[2] Department of Biological and Agricultural Engineering, Faculty of Engineering, University Putra Malaysia, 43400 Serdang, Selangor, Malaysia

target model was defined for imprecise data types such as interval and ordinal data.

Cooper et al. [11] introduced a model which the data are stated in natural logarithmic units. By using anti-logs, the another DEA model was derived which has been called multiplicative model. This approach resulted in Cobb-Douglas form of production frontiers which has the wide application in econometric and statistical studies. On the other hand, Hollingsworth and Smith [12] explained that the CCR formulation of DEA can not be used in the case of ratio variable, also Emrouznejad and Amin [13] have illustrated that in the standard DEA model, the incorrect results can be occurred if ratio variables are considered in the data set. Therefore, considering the ratio data in DEA models has received a great deal of attention for researchers. Fernandez-Castro and Smith [14] proposed a model named the general non-parametric corporate performance (GNCP) for measuring the efficiency of DMUs with ratio data. Emrouznejad and Cabanda [15] (see also Emrouznejad and Amin, [13] and Emrouznejad et al. [16]) have mentioned that the GNCP model has two shortcomings: the convexity and the proportionality properties, so, they proposed a new model (in the framework of multiplicative models) known as the multiplicative non-parametric corporate performance model (MNCP) to rectify such a situations. It is worthy to mention that Emrouznejad et al. [17] have extended (GNCP) model and (MNCP) model to the interval ratios.

In this paper a new model is proposed for limited resources in total input consumption and total output production in Cobb-Douglas form of production frontiers which is inspired from the conventional multiplicative DEA models. Since the production possibility set is included the geometric combination form of the data, the model become a kind of non-linear mathematical programming models. By the existing method, the model will be converted to the linear one which results in the logarithmic form of the data set.

This paper proceeds as follows. In section 2, data envelopment scenario analysis is presented. In section 3, we introduce the new target in the form of multiplicative model. A numerical example and conclusions are provided in sections 4 and 5, respectively.

2. **Data Envelopment Analysis**

In this section, the CCR envelopment model (the first DEA model) proposed by Charnes et al. in 1978 is introduced.
Suppose we have a set of $j = 1, ..., n$ decision making units (DMUs) and each unit uses input $X \in R^m_+$ quantities to produce output quantities $Y \in R^s_+$. We consider the index sets of inputs,$I = 1, ..., m$ and outputs, $O = 1, ...s$. Also $o(o \in 1, ..., n)$ is the DMU under assessment (usually denoted by DMU_o). For evaluating the relative efficiency of DMU_o, we need to solve the following linear mathematical

programming problem:

$$\begin{aligned} Min \quad & \theta_o \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta_o x_{io}, \qquad i \in I, \\ & \sum_{j=1}^{n} \lambda_j y_{kj} - s_k^+ = y_{ko}, \qquad k \in O, \end{aligned}$$

$$\lambda_j \geq 0, \forall j, \theta_o \quad free. \tag{1}$$

DMU_o is CCR efficient if $\theta^* = 1$ and the slacks s_k^{+*}, s_i^{-*}in every optimal solution are zero. The target input and output for DMU_ocan be obtained as follows:

$$\begin{aligned} \bar{x}_{io} &= \theta_o^* x_{io} - s_i^{*-}, \forall i \in I \\ \bar{y}_{ko} &= y_{ko} + s_k^{*+}, \forall k \in O \end{aligned}$$

Data envelopment analysis can also be used for target setting and resource allocation. Thereafter along with evaluating the efficiency, the acceptance of the projected DMU into the efficient frontier is also considerable for researchers. Various type of models and methods for target setting can be found in Athanassopoulos ([2], [3], [4]), Golany [5], Korhonen and Syrjnen [6], Lozano and Villa [7], Thanassoulis and Dyson [8]and Malekmohammadi et al. [9], [10] .

3. The New Target Model in Multiplicative Form

In this section firstly, by introducing Cooper et al. model [11], we will explain the multiplicative data envelopment analysis model. Secondly, the new target model for limited resources will be proposed. Consider the following model:

$$\begin{aligned} Min \quad & \theta_o \\ s.t. \quad & \prod_{j=1}^{n} x_{ij}^{\lambda_j} . e^{s_i^-} = x_{io}^{\theta_o}, \qquad i \in I, \\ & \prod_{j=1}^{n} y_{kj}^{\lambda_j} . e^{-s_k^+} = y_{ko}, \qquad k \in O, \end{aligned}$$

$$\lambda_j \geq 0, \ j = 1, ..., n, \theta_o \quad free, \ s_i^-, \ s_k^+ \geq 0, \ i \in I, \ k \in O. \tag{2}$$

The version of an Additive model which is initiated from model (2) is also introduced by Cooper et al. [11]. Model (2) has four shortcomings, firstly, it is needed to solve n mathematical programming for evaluating the relative efficiency of each DMU. Secondly, in the existence of the centralized decision maker, model (2) is not useful. Thirdly, the model is *radial* which causes the same reduction and expansion for input and output respectively. Fourthly, since the decision maker 's preferences is not involved in target setting process, the limited resources will be occurred for total input or total output respectively.

Now, a new target model for limited resources will be proposed which rectify the problem mentioned above. The model computes one rather than n mathematical

programming problems along with reducing total input and increasing total output simultaneously, as follows:
Let $j, r = 1, ..., n$ be the indices for decision making units (DMUs) while each unit uses input quantities $X \in R_+^m$ to deliver output quantities $Y \in R_+^s$. We can also consider the indices sets of inputs, $I = 1, ..., m$ and outputs ,$O = 1, ...s$, and their subsets $I \equiv I_f \cup \bar{I}_f$ and $O \equiv O_f \cup \bar{O}_f$ where I_f and $0_f$are used to indicate inputs and outputs which have limited resources.

$$
\begin{aligned}
Max \quad & \sum_{k \in 0} P_k^+ Z_k - \sum_{i \in I} P_i^- \theta_i \\
s.t. & \prod_{r=1}^{n} \prod_{j=1}^{n} x_{ij}^{\lambda_{jr}} = \theta_i \prod_{j=1}^{n} x_{ij}, \qquad i \in I, \\
& \prod_{r=1}^{n} \prod_{j=1}^{n} y_{kj}^{\lambda_{jr}} = Z_k \prod_{r=1}^{n} y_{kr}, \qquad k \in O, \\
& \prod_{j=1}^{n} \theta_i x_{ij} \geq G_i^L, \quad i \in I_f, \qquad (3.a) \\
& \prod_{r=1}^{n} Z_k y_{kr} \leq G_k^{'U}, \quad k \in O_f, \qquad (3.b)
\end{aligned}
$$

$$
\lambda_{jr} \geq 0, \;\; \forall j, r, \;\; \theta_i \;\; free, \; i \in I, \; Z_k \;\; free, \; k \in O, \qquad (3)
$$

Where x_{ij} is the quantity of input i of unit j; y_{kr}, y_{kj} the quantity of output k of units r and j, respectively; $[G_i^L, G_i^U]$ and $[G_k^{'L}, G_k^{'U}]$ indicate the intervals of existing resources and bounds for total input i consumption and total output k production. P_i, P_k the user-specified constants reflecting the decision makers' preferences over the improvement of input/output components; θ_i, Z_k the contraction rate of input i and expansion rate of output k(characterization of *non-radial* models).
In the constraint sets (3.a) and (3.b), $G_i^L, G_k^{'U}$are the bounds that can be chosen by centralized DM when the lower and upper bounds of existing resources for total input i and total output k for all DMUs together are G_i^L and $G_k^{'U}$, respectively.

By using log transformation, model(3) will be converted to the following model:

$$Max \quad \sum_{k\in 0} P_k^+ Z_k - \sum_{i\in I} P_i^- \theta_i$$

$$s.t. \sum_{r=1}^{n} \ln(\prod_{j=1}^{n} x_{ij}^{\lambda_{jr}}) = \ln\theta_i + \sum_{j=1}^{n} \ln(x_{ij}), \qquad i \in I,$$

$$\sum_{r=1}^{n} \ln(\prod_{j=1}^{n} y_{kj}^{\lambda_{jr}}) = \ln Z_k + \sum_{r=1}^{n} \ln(y_{kr}), \qquad k \in O,$$

$$\sum_{j=1}^{n} \ln(\theta_i x_{ij}) \geq \ln(G_i^L), \quad i \in I_f,$$

$$\sum_{r=1}^{n} \ln(Z_k y_{kr}) \leq \ln(G_k^{'U}), \quad k \in O_f,$$

$$\lambda_{jr} \geq 0, \quad \forall j,r, \quad \theta_i \quad free, \; i \in I, \; Z_k \quad free, \; k \in O, \tag{4}$$

With the following transformation, model (5) will be established:

$$\hat{y_{kj}} = \ln(y_{kj}); \; \hat{y_{kr}} = \ln(y_{kr}); \; \hat{x_{ij}} = \ln(x_{ij}); \; \ln(\theta_i) = \tau_i; \; \ln(Z_k) = \gamma_k$$

$$\ln(G_i^L) = L_i; \; \ln(G_k^{'U}) = U_k;$$

$$\ln(\theta_i x_{ij}) = \ln\theta_i + \ln x_{ij} = \tau_i + \hat{x_{ij}}; \; \ln(Z_k y_{kr}) = \ln Z_k + \ln y_{kr} = \gamma_k + \hat{y_{kr}}.$$

Model (5) is the following model;

$$Max \quad \sum_{k\in 0} P_k^+ Z_k - \sum_{i\in I} P_i^- \theta_i$$

$$s.t. \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr}\hat{x_{ij}} = \tau_i + \sum_{j=1}^{n} \hat{x_{ij}}, \qquad i \in I,$$

$$\sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr}\hat{y_{kj}} = \gamma_k + \sum_{r=1}^{n} \hat{y_{kr}}, \qquad k \in O,$$

$$\sum_{j=1}^{n} \tau_i + \hat{x_{ij}} \geq L_i, \quad i \in I_f,$$

$$\sum_{r=1}^{n} \gamma_k + \hat{y_{kr}} \leq U_k, \quad k \in O_f,$$

$$\sum_{j=1}^{n} \lambda_{jr} = 1, \quad \forall r$$

$$\lambda_{jr} \geq 0, \quad \forall j,r, \quad \theta_i \quad free, \; i \in I, \; Z_k \quad free, \; k \in O, \tag{5}$$

The vector $(\lambda_{1r}, \lambda_{2r}, ..., \lambda_{nr})$ such that $\sum \lambda_{jr} = 1$, is imposed for convex combination between inputs or outputs for n DMUs.

Proposition1. For any DMU_j and DMU_r, the points $x_{ij}^* = \prod_{j=1}^{n} \theta_i^* x_{ij}, \; \forall i =$

$1,...,n$ and $y^*_{kr} = \prod_{j=1}^{n} Z^*_k y_{kr},\ \forall k = 1,...,s$ from model (3) indicate the lowest total input consumption for input i and the highest total output production for output k , respectively.
Proof: Let us assume that the proposition is false and we will always arrive at a contradiction. If $(x^*_{11}, x^*_{22}..., x^*_{mn})$ is not is smallest total input consumption then, there exist $\hat{x}_{ij}$ and $\hat{\theta}_i (i = 1,...,m)$ and $(j = 1,...,n)$ such that $\hat{x}_{ij} = \prod_{j=1}^{n} \hat{\theta}_i x_{ij} \leq \prod_{j=1}^{n} \theta^*_i x_{ij} = x^*_{ij}$ and at least for one input $i^{'}$the inequality is strict. Let us assume that it is for input $i^{'}$ for which $\prod_{j=1}^{n} \hat{\theta}_{i'} x_{i'j} < \prod_{j=1}^{n} \theta^*_{i'} x_{i'j} \Rightarrow \hat{\theta}_{i'} < \theta^*_{i'}$ which is a contradiction, because by model (3) we will get another minimum feasible solution for $\theta_i,\ i = 1,...,m$. The same proof can be done for y^*_{kr}.

4. Numerical Results

The application of model (3) will be illustrated in this section. Table 1 presents the data set which can be found in (Wang et al, [18]), but the data have been changed a little to be more suitable for the proposed model.
There are eight dependent manufacturing enterprises (DMUs) with two inputs and one output. Each manufacturing enterprise manufactures the same type of product. All of the manufacturing enterprises are under the control of a centralized decision maker (DM) who prefers to reduce total input consumption and increase total output production. It is mentionable that the purchase cost (PC) is changed to capital (C) in Wang et al.'s [18] data set. This was done since it is not logical to reallocate these two data item (PC) among the units.
Suppose the centralized DM finds that the permissible reduction for total input 1

Table 1. The input-output data for the eight DMUs

DMU	Input	Input	Output
	C	NOE	GOV
1	2166	1875	14950
2	1455	1342	13584
3	2562	2359	18452
4	2346	2018	15900
5	1517	1548	14638
6	2034	1760	14582
7	2256	1982	17169
8	2465	2254	18256

(capital) is 15962. Also, input 2 (NOE) should not be less than 14533 since; each DMU needs at least a reasonable number of employees to handle its organization. Also for output (GOV), the DM cannot increase total output so as to lie outside interval [127531,136000]. The DM has to consider the permissibility of increasing the total output. To reach the optimum total input consumption and output production, model (3) can be used. In constrains (3a) and (3b) in model (3), we can have $G'^{L}_1 =$

$127531; G_1'^U = 136000; G_1^L = 15962; G_2^L = 14533$. $P_k^+ = P_i^- = 1$ for $i = 1, ..., m$ and $k = 1, ..., s$ have been chosen, also.
Since model (3) is non-linear, model (5) has been used instead. Table 2 shows the results obtained from model (5) (solved by LINGO, a powerful software package). By considering Proposition 1 the optimum targets can be defined. In Table 2, the optimum total input consumption and output production have been shown. It is mentionable that the new result from model (3) and (5) is also compared with our previous research (Malekmohammadi et. al, [10]) and also with the existing total inputs and outputs. According to the results in the new approach we have less

TABLE 2. The total input-output data for the eight DMUs

Total	Total C	Total NOE	Total GOV
Existing	16801	15138	127531
New approach	16090	14534	134342
Previous approach	15962	14533	129722

reduction in inputs than the previous approach but the expansion of output is much more considerable. In the new approach the proposed model has been initiated from multiplicative DEA model but in the previous approach the framework of the suggested model was the conventional DEA models. This research can be the main step for more and more extended multiplicative models in DEA.

5. Conclusion

In this paper, a new target model is suggested in the form of multiplicative DEA model which is suitable for limited resources in total inputs and total outputs. The main aim of this paper is the consideration of the more comprehensive research in multiplicative DEA models which is applicable in some cases, for example the existence of the ratio data in our data set. By the numerical result the proposed model has been applied to manufacturing enterprises which has shown that in some cases the multiplicative model can be more considerable.

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