ABELIAN FINITE GROUP GRADINGS ON THE SKEW POLYNOMIAL RING $\Bbbk[X][Y, \varphi]$

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In acest articol dăm o descriere completă a graduărilor după grupuri abeliene finite ale inelului de polinoame încrucişate $\Bbbk[X][Y, \varphi]$, unde φ este un $\Bbbk$-automorfism al lui $\Bbbk[X]$.

In this paper we give a precise description of the gradings on the skew polynomial ring $\Bbbk[X][Y, \varphi]$ over finite abelian groups, where φ is a $\Bbbk$-automorphism of $\Bbbk[X]$.

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1. Introduction and preliminaries

Let $\Bbbk$ a field, A a $\Bbbk$-algebra and G a group A G- grading of A is a vector space decomposition, $A = \bigoplus_{g \in G} A_g$, such that $A_g A_h \subset A_{gh}$, for all $g, h \in G$. The set $h(A) = \bigcup_{g \in G} A_g$ is the set of *homogeneous elements* of A. A nonzero element $a \in A_g$ is said to be *homogeneous of degree g*. An element x of A has a unique decomposition as $x = \sum_{g \in G} x_g$ with $x_g \in A_g$, for all $g \in G$, but the sum being a finite sum i.e almost all x_g zero.

If R is a ring and $\varphi : R \longrightarrow R$ is an injective ring homomorphism, then the ring $R[X, \varphi]$ consisting of all finite sums $\sum_{i=0}^{n} r_i X^i$, with $r_i \in R$, where addition is component-wise, as in $R[X]$, and multiplication is defined by $X \cdot r = \varphi(r) X$, for all $r \in R$, is called *the skew polynomial ring*.

Skew polynomial rings form an interesting class of noncommutative rings, having recent applications to coding theory ([1]).

Many authors have studied the gradings of different algebras over certain groups. For instance, M. Kochetov has studied the gradings on finite-dimensional simple Lie algebras; several authors, as Yu Bahturin, C. Boboc, S. Caenepeel, J. Chun, S. Dăscălescu, J. Lee, C. Năstăsescu, S. Segal,

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M. V. Zaicev have studied the gradings on matrix algebra, C. Buruiană has studied in [2] the gradings over finite abelian groups on the polynomial algebra in one variable, but a systematic study of the gradings over finite abelian groups on the skew polynomial ring has not been done so far and this paper aims to begin this.

This paper treats the case of the skew polynomial ring in two variables, $\Bbbk[X][Y, \varphi]$ where φ is a $\Bbbk$-automorphism of $\Bbbk[X]$. We start by studying first the gradings of $\Bbbk[X][Y, \varphi]$ over cyclic groups. An issue here is the computation of all $\Bbbk$-automorphisms of $\Bbbk[X][Y, \varphi]$, a problem far from being trivial. Using an elementary technique, we solve this problem. Further, we study under which conditions two C_m- and C_n- gradings of $\Bbbk[X][Y, \varphi]$ are compatible and when they are, we can construct a $C_m \times C_n$- grading of $\Bbbk[X][Y, \varphi]$. In case of compatibility, we give a precise description of homogeneous components of $\Bbbk[X][Y, \varphi]$. Moreover, we can extend the results obtained for all finite abelian groups. In fact, this paper is a generalization of [2].

2. Main results

Let n be a positive integer and $C_n = <g>$ be the cyclic group of order n. Throughout this section $\Bbbk$ will be a field containing a primitive n-th root ξ of unity, which implies that the characteristic of $\Bbbk$ does not divide n.

We will start by studying gradings of the skew polynomial ring $\Bbbk[X][Y, \varphi]$ over cyclic groups.

Let $A = \Bbbk[X][Y, \varphi]$, where φ is a $\Bbbk$-automorphism of $\Bbbk[X]$. It is well known that the $\Bbbk$-automorphisms of $\Bbbk[X]$ are of the following form: $\varphi(f) = f(aX + b)$, $a, b \in \Bbbk$, $a \neq 0$, for any $f \in \Bbbk[X]$. The multiplication rule is $YX = \varphi(X)Y = aXY + bY$. One can easily prove by induction that $Y^n X^m = (\varphi^n(X))^m Y^n$, for any $m, n \in \mathbb{N}$.

Because $\varphi(X) = aX + b$, we obtain $\varphi^n(X) = a^n X + a^{n-1}b + ... + ab + b$, for any $n \in \mathbb{N}^*$. Hence $Y^n X^m = a^{mn} X^m Y^n + \ldots$ the lower degree terms.

Moreover, if we have two monoms $X^m Y^n, X^p Y^q$, then we obtain $X^m Y^n X^p Y^q = X^m (\varphi^n(X))^p Y^{n+q} = a^{np} X^{m+p} Y^{n+q} + \ldots$ the lower degree terms.

Using the above notations, let us give first the following useful lemma.

Lemma 2.1. *Let $A = \Bbbk[X][Y, \varphi]$, where φ is a $\Bbbk$-automorphism of $\Bbbk[X]$, $\varphi(f) = f(aX + b)$, $a, b \in \Bbbk$, $a \neq 0$, φ different from the identity map. Then the $\Bbbk$- automorphisms of A are of the following form:*

(i) *If $char\Bbbk = 2$ or $b \neq 0$ or $a \neq -1$, then the $\Bbbk$-automorphism of A are of the form $\Psi : A \longrightarrow A$, where $\Psi(X) = a_1 X + b_1$, $\Psi(Y) = c_1 Y$, $a_1, b_1, c_1 \in \Bbbk$, $a_1, c_1 \neq 0$ such that $ba_1 + b_1 = ab_1 + b$;*

(ii) *If $\mathrm{char}\Bbbk \neq 2$, $b = 0$ and $a = -1$, then A has two types of $\Bbbk$-automorphisms: $\Psi_1, \Psi_2 : A \longrightarrow A$, where $\Psi_1(X) = a_1 X$, $\Psi_1(Y) = c_1 Y$ and $\Psi_2(X) = a_2 Y$, $\Psi_2(Y) = c_2 X$, $a_1, a_2, c_1, c_2 \in \Bbbk^*$.*

Proof. Let $\Psi : A \longrightarrow A$ a $\Bbbk$-automorphism of A. Denote $\Psi(X) = \sum_{i,j} a_{ij} X^i Y^j$ and $\Psi(Y) = \sum_{i,j} b_{ij} X^i Y^j$; let $X^{m_1} Y^{m_2}$ be the leading monom of $\Psi(X)$ and $X^{n_1} Y^{n_2}$ the leading monom of $\Psi(Y)$. Because Ψ is an automorphism of A, there exists an unique $f \in \Bbbk[X][Y,\varphi]$ such that $\Psi(f) = X$. Let $f = \sum_{i,j} c_{ij} X^i Y^j$ and $X^{p_1} Y^{p_2}$ be the leading monom of f. Then the leading monom of $\Psi(f)$ is $X^{p_1 m_1 + p_2 n_1} Y^{p_1 m_2 + p_2 n_2}$. From $\Psi(f) = X$ we will obtain

$$X^{p_1 m_1 + p_2 n_1} Y^{p_1 m_2 + p_2 n_2} = X$$

Hence

$$
\begin{aligned}
p_1 m_1 + p_2 n_1 &= 1 \\
p_1 m_2 + p_2 n_2 &= 0
\end{aligned}
$$

where $m_1, m_2, p_1, p_2 \in \mathbb{N}$.

Solving the above system we will obtain two types of $\Bbbk$- automorphisms: $\Psi_1, \Psi_2 : A \longrightarrow A$, where $\Psi_1(X) = a_1 X + b_1$, $\Psi_1(Y) = c_1 Y + d_1$ and $\Psi_2(X) = a_2 Y + b_2$, $\Psi_2(Y) = c_2 X + d_2$, $a_i, b_i, c_i, d_i \in \Bbbk$, for any $i = 1, 2$.

Now, because Ψ_1, Ψ_2 are $\Bbbk$- automorphisms we have $\Psi_i(Y^n X^m) = \Psi_i(Y^n)\Psi_i(X^m)$ for any $m, n \in \mathbb{N}^*$ and $i = 1, 2$.

Let us study first what happens for Ψ_1. We have

$$
\begin{aligned}
\Psi_1(Y)\Psi_1(X) &= (c_1 Y + d_1)(a_1 X + b_1) \\
&= a_1 c_1 YX + a_1 d_1 X + b_1 c_1 Y + b_1 d \\
&= a_1 c_1 (aXY + bY) + a_1 d_1 X + b_1 c_1 Y + b_1 d_1 \\
&= a a_1 c_1 XY + (b a_1 c_1 + b_1 c_1)Y + a_1 d_1 X + b_1 d_1
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_1(YX) &= \Psi_1(aXY + bY) \\
&= a\Psi_1(XY) + b\Psi_1(Y) \\
&= a(a_1 X + b_1)(c_1 Y + d_1) + b(c_1 Y + d_1) \\
&= a a_1 c_1 XY + (ab_1 c_1 + bc_1)Y + a a_1 d_1 X + ab_1 d_1 + bd_1
\end{aligned}
$$

By identifying the coefficients in $\Psi_1(YX) = \Psi_1(Y)\Psi_1(X)$, we obtain:

$$
\begin{aligned}
ba_1c_1 + b_1c_1 &= ab_1c_1 + bc_1 \\
a_1d_1 &= aa_1d_1 \\
b_1d_1 &= ab_1d_1 + bd_1.
\end{aligned}
$$

Because $a_1 \neq 0$, from the second equation we obtain $d_1 = ad_1$ and we will have $d_1 = 0$ or $a = 1$. If $d_1 \neq 0$, we obtain $a = 1$ and from the third equation we have $b_1 = b_1 + b$, so $b = 0$ and in this case φ is the identity map, contradiction. Hence $d_1 = 0$. Also, we can remark that from the first equation we have that $ba_1 + b_1 = ab_1 + b$.

Now let us study what happens for Ψ_2. We have:

$$
\begin{aligned}
\Psi_2(YX) &= \Psi_2(aXY + bY) = a\Psi_2(XY) + b\Psi_2(Y) \\
&= a(a_2Y + b_2)(c_2X + d_2) + b(c_2X + d_2) \\
&= a^2a_2c_2XY + (aa_2d_2 + aba_2c_2)Y + (ab_2c_2 + bc_2)X + ab_2d_2 + bd_2.
\end{aligned}
$$

and

$$
\Psi_2(Y)\Psi_2(X) = (c_2X + d_2)(a_2Y + b_2) = a_2c_2XY + c_2b_2X + a_2d_2Y + b_2d_2
$$

From the equality $\Psi_2(YX) = \Psi_2(Y)\Psi_2(X)$ we will obtain:

$$
\begin{aligned}
a^2 &= 1 \\
aa_2d_2 + aba_2c_2 &= a_2d_2 \\
ab_2c_2 + bc_2 &= c_2b_2 \\
ab_2d_2 + bd_2 &= b_2d_2
\end{aligned}
$$

If $a = 1$, from the third equation we obtain $b = 0$ and in this case φ is the identity map, contradiction. Hence $a = -1$. From the second equation we obtain $bc_2 = -2d_2$ and from the third $b = 2b_2$. If $char\Bbbk = 2$, we have $a = 1$, $b = 0$ and φ is the identity map, contradiction. Therefore, when $char\Bbbk = 2$, Ψ_2 can not be an automorphism of A. Let $char\Bbbk \neq 2$. Using $a = -1$, we have

$$
\begin{aligned}
\Psi_2(Y^2)\Psi_2(X) &= (c_2X + d_2)^2(a_2Y + b_2) \\
&= (c_2^2X^2 + 2c_2d_2X + d_2^2)(a_2Y + b_2) \\
&= a_2c_2^2X^2Y + b_2c_2^2X^2 + 2a_2c_2d_2XY + 2c_2b_2d_2X \\
&+ a_2d_2^2Y + b_2d_2^2
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_2(Y^2X) &= \Psi_2(XY^2) = (a_2Y + b_2)(c_2X + d_2)^2 \\
&= a_2c_2^2X^2Y - 2a_2c_2d_2XY + (2a_2c_2d_2b + a_2d_2^2)Y \\
&+ b_2c_2^2X^2 + 2c_2b_2d_2X + b_2d_2^2
\end{aligned}
$$

By identifying the coefficients and using $char\Bbbk \neq 2$ we will obtain $d_2 = 0$. But we have $bc_2 = -2d_2$ and $b = 2b_2$,therefore $b = b_2 = 0$. When $a = -1$ and $b = 0$, obviously $b_1 = 0$ and in this case $\Psi_1(X) = a_1X$. $\qquad\square$

Now, using the same techniques as in [3], we will show how to obtain the C_n- gradings of $\Bbbk[X][Y, \varphi]$.

Theorem 2.2. *Let* $A = \Bbbk[X][Y, \varphi]$ *be a skew polynomial ring, where* $\Bbbk$ *is a field containing a primitive n-th root* ξ *of unity and* $\varphi(f) = f(aX + b)$, $a, b \in \Bbbk$, $a \neq 0$, φ *different from identity map. Then:*

(i) *If* $b \neq 0$ *or* $a \neq -1$, *a* C_n- *grading of* A *is of the following form:*
$A_{g^i} = \{\frac{1}{n}[f(X,Y) + \xi^{-i}f(a_1X + b_1, c_1Y) + \xi^{-2i}f(a_1^2X + a_1b_1 + b_1, c_1^2Y^2) + \ldots + \xi^{-(n-1)i}f(a_1^{n-1}X + a_1^{n-2}b_1 + \ldots + a_1b_1 + b_1, c_1^{n-1}Y^{n-1})] \mid f \in \Bbbk[X]\}$,
where $a_1, b_1, c_1 \in \Bbbk$, *such that* $ba_1 + b_1 = ab_1 + b$, $a_1^n = c_1^n = 1$.

(ii) *If* $b = 0$ *and* $a = -1$, *then:*

(a) *If* n *is even, a* C_n- *grading of* A *is of the one of the following types:*

(I) $A_{g^i} = \{\frac{1}{n}[f(X,Y) + \xi^{-i}f(a_1X, c_1Y) + \xi^{-2i}f(a_1^2X, c_1^2Y) + \ldots + \xi^{-(n-1)i}f(a_1^{n-1}X, c_1^{n-1}Y)] \mid f \in \Bbbk[X]\}$, *where* $a_1, c_1 \in \Bbbk$, $a_1^n = c_1^n = 1$.

(II) $A_{g^i} = \{\frac{1}{n}[f(X,Y) + \xi^{-i}f(a_2Y, c_2X) + \xi^{-2i}f(a_2c_2X, a_2c_2Y) + \ldots + \xi^{-(n-1)i}f(a_2^{\frac{n+1}{2}}c_2^{\frac{n-1}{2}}Y, a_2^{\frac{n-1}{2}}c_2^{\frac{n+1}{2}}X)] \mid f \in \Bbbk[X]\}$, *where* $a_2, c_2 \in \Bbbk$, $(a_2c_2)^{\frac{n}{2}} = 1$.

(b) *If* n *is odd,* A *has only* C_n- *gradings of the first type.*

Proof. Let $A = \bigoplus_{i \in Z_n} A_{g^i}$ be a C_n- grading of A and let us define the map $\Psi : A \longrightarrow A$ by $\Psi(f) = \sum_{i \in Z_n} \xi^i f_{g^i}$ for any $f \in A$. Then Ψ is a linear map and for any $f, h \in A$ we have

$$\begin{aligned}
\Psi(f)\Psi(h) &= \left(\sum_{i \in Z_n} \xi^i f_{g^i}\right)\left(\sum_{j \in Z_n} \xi^j h_{g^j}\right) \\
&= \sum_{i,j \in Z_n} \xi^{i+j} f_{g^i} h_{g^j} \\
&= \sum_{s \in Z_m} \sum_{i+j=s} \xi^s f_{g^i} h_{g^j} \\
&= \sum_{s \in Z_m} \xi^s (fh)_{g^s} = \Psi(fh)
\end{aligned}$$

Hence Ψ is an algebra morphism. Also, for any j, we can easily prove that $\Psi^j(f) = \sum_{i \in Z_n} \xi^{ji} f_{g^i}$, for any $f \in A$, where $\Psi^k = \underbrace{\Psi \circ \Psi \circ \ldots \circ \Psi}_{k \text{ times}}$. In particular, $\Psi^n = Id$ and Ψ is an algebra automorphism of A. If we multiply the following equations

$$\begin{aligned}
f &= \sum_{i \in Z_n} f_{g^i} \\
\Psi(f) &= \sum_{i \in Z_n} \xi^i f_{g^i} \\
&\ldots \\
\Psi^{n-1}(f) &= \sum_{i \in Z_n} \xi^{(n-1)i} f_{g^i}
\end{aligned}$$

by 1, ξ^{-i}, ξ^{-2i}, ..., $\xi^{-(n-1)i}$, respectively, and add them, we get:

$$nf_{g^i} = f + \xi^{-i}\Psi(f) + ... + \xi^{-(n-1)i}\Psi^{n-1}(f) \qquad (2.1)$$

From Lemma 2.1 we know the $\Bbbk$-automorphisms of A. In our case we need only those that have the property $\Psi^n = Id$. For the automorphisms of the form $\Psi(f) = f(a_1 X + b_1, c_1 Y)$, where $a_1, b_1, c_1 \in \Bbbk$, $ba_1 + b_1 = ab_1 + b$, we have

$$\Psi^{\Bbbk}(f) = f(a_1^{\Bbbk-1}X + a_1^{\Bbbk-2}b_1 + ... + a_1 b_1 + + b_1, c_1^{\Bbbk-1}Y)$$

for any $\Bbbk \in \mathbb{N}$ and $f \in A$. From $\Psi^n(X) = X$, $\Psi^n(Y) = Y$ we obtain $a_1^n = 1 = c_1^n$; if $a_1 = 1$, we must have $b_1 = 0$.

Let now $b = 0$ and $a = -1$. We remark that for an automorphism of the form $\Psi(f) = f(a_1 Y, c_1 X)$, $a_1, c_1 \in \Bbbk$, we have $\Psi^r(X) = a_1^{\frac{r}{2}} c_1^{\frac{r}{2}} X$ and $\Psi^r(Y) = a_1^{\frac{r}{2}} c_1^{\frac{r}{2}} Y$ when r is even, and $\Psi^r(X) = a_1^{\frac{r+1}{2}} c_1^{\frac{r-1}{2}} Y$ and $\Psi^r(Y) = a_1^{\frac{r-1}{2}} c_1^{\frac{r+1}{2}} X$ when r is odd; from the condition $\Psi^n = Id$, we obtain $(a_2 c_2)^{\frac{n}{2}} = 1$, when n is even. Obviously, we can not have an automorphism of second type, when n is odd. If n is even, from hypothesis $char\Bbbk \neq 2$. Using Lemma 2.1 and the above descriptions of $\Psi^k(f)$, for any k, where Ψ is an $\Bbbk$-automorphism of A, the proof ends. $\qquad \square$

We will denote by $A^I(a_1, b_1, c_1)$ a C_n - grading of A from (i) and by $A^{II}(a_2, c_2)$ a C_n - grading of A of second type. Obviously, for a C_n - grading of A of the first type from (ii), we have $A^I(a_1, 0, c_1)$.

Remark 2.3. From the proof of the above theorem we can see that $f \in A_{g^i}$ if and only if $\Psi(f) = \xi^i f$.

Definition 2.4. If G is a group, a G- grading of $\Bbbk[X][Y, \varphi]$ is a *good grading* if the variables X, Y are homogeneous elements.

Proposition 2.5. *Let $A = \Bbbk[X][Y, \varphi]$, with $\Bbbk$ a field containing a primitive n-th root ξ of unity. Then:*

(i) *If $b \neq 0$ or $a \neq -1$, a C_n- grading of A of type $A^I(a_1, b_1, c_1) = \bigoplus_{i=0}^{n-1} A_{g^i}^I(a_1, b_1, c_1)$ is a good grading if and only if $b_1 = 0$.*

(ii) *If $b = 0$ and $a = -1$, then only the C_n- gradings of A of the first type are good gradings.*

Proof. (i) Let us compute the homogeneous components of X and Y.

Let $b \neq 0$ or $a \neq -1$. Using (2.1) we have:

$$\begin{aligned}
X_{g^i} &= \frac{1}{n}[X + \xi^{-i}(a_1 X + b_1) + \xi^{-2i}(a_1^2 X + a_1 b_1 + b_1) + \ldots \\
&\quad + \xi^{-(n-1)i}(a_1^{n-1}X + +a_1^{n-2}b_1 + ... + a_1 b_1 + b_1)]
\end{aligned}$$

for any $i = 0, 1, ..., n-1$, $a_1^n = 1$.

If $a_1 = 1$, from Theorem 2.2 we obtain $b_1 = 0$. In this case we have $X_e = X$ and $X_{g^i} = \frac{1}{n}X(1 + \xi^{-i} + \ldots + \xi^{-(n-1)i}) = 0$, for any $i = 1, \ldots, n - 1$. Now let $a_1 \neq 1$. Then $a_1 = \xi^t$, for some $t \in \{1, \ldots, n - 1\}$ and we have

$$
\begin{aligned}
X_e &= \frac{1}{n}[X + (\xi^t X + b_1) + \ldots + (\xi^{(n-1)t}X + \xi^{(n-2)t}b_1 + \ldots + \xi b_1 + b_1)] \\
&= \frac{1}{n}X(1 + \xi^t + \ldots + \xi^{(n-1)t}) + \frac{1}{n}b_1[1 + (\xi^t + 1) + \ldots + (\xi^{(n-2)t} + \ldots \\
&\quad + \xi + 1)] \\
&= \frac{1}{n}b_1(1 + \frac{\xi^{2t} - 1}{\xi^t - 1} + \ldots + \frac{\xi^{(n-1)t} - 1}{\xi^t - 1}) = -\frac{b_1}{\xi^t - 1}
\end{aligned}
$$

It is straightforward to compute the other homogeneous components of X and we will obtain: $X_{g^t} = X + \frac{b_1}{\xi^t - 1}$ and $X_{g^s} = 0$, for any $s \in \{1, \ldots, n - 1\}$, $s \neq t$. Now, using again (2.1) we have

$$
Y_{g^i} = \frac{1}{n}(Y + \xi^{-i}c_1 Y + \xi^{-2i}c_1^2 Y + \ldots + \xi^{-(n-1)i}c_1^{n-1}Y)
$$

for any $i = 0, 1, \ldots, n - 1$, $c_1^n = 1$. If $c_1 = 1$, we have $Y_e = Y$ and $Y_{g^i} = 0$, for any $i = 1, \ldots, n - 1$.

If $c_1 = \xi^l$, for some $l \in \{1, \ldots, n - 1\}$, it is easy to prove that $Y_{g^l} = Y$ and $Y_{g^s} = 0$, for any $s = 0, 1, \ldots, n - 1$, $s \neq l$.

From the above we can see that Y is a homogeneous element. Also, we remark that X is homogeneous if and only if $b_1 = 0$.

(ii) Let $b = 0$ and $a = -1$. Obviously, the C_n- gradings of A of first type are good gradings. We will prove that the C_n- gradings of A of second type are not good gradings. For this, let us compute the homogeneous components of X and Y. Let n be an even number and let $A^{II}(a_2, c_2) = \bigoplus_{i=0}^{n-1} A_{g^i}^{II}(a_2, c_2)$ be a C_n- grading of A of second type. Using Theorem 2.2 we have that

$$
\begin{aligned}
X_{g^i} &= \frac{1}{n}(X + \xi^{-i}a_2 Y + \xi^{-2i}a_2 c_2 X + \xi^{-3i}a_2^2 c_2 Y + \ldots + \xi^{-(n-1)i}a_2^{\frac{n+1}{2}}c_2^{\frac{n-1}{2}}Y) \\
&= \frac{1}{n}(X + a_2\xi^{-i}Y)[1 + \xi^{-2i}a_2 c_2 + (\xi^{-2i}a_2 c_2)^2 + \ldots + (\xi^{-2i}a_2 c_2)^{\frac{n-1}{2}}]
\end{aligned}
$$

for any $i = 0, 1, \ldots, n - 1$, with $(a_2 c_2)^{\frac{n}{2}} = 1$. If $a_2 c_2 = 1$, we obtain

$$
\begin{aligned}
X_e &= \frac{X + aY}{2} \\
X_{g^{\frac{n}{2}}} &= \frac{X - aY}{2} \\
X_{g^i} &= 0
\end{aligned}
$$

for any $i = 1, \ldots, n - 1$, $i \neq \frac{n}{2}$.

If $a_2 c_2 \neq 1$, then $a_2 c_2 = \xi^{2t}$, for some $t \in \{1, ..., \frac{n}{2} - 1\}$ (we have used that if ξ is a primitive n-th root of unity, then ξ^2 is a primitive $\frac{n}{2}$-th root of unity). In this case we obtain

$$
\begin{aligned}
X_{g^t} &= \frac{X + a\xi^{-t}Y}{2} \\
X_{g^{(\frac{n}{2}+1)t}} &= \frac{X - a\xi^{-t}Y}{2} \\
X_{g^i} &= 0
\end{aligned}
$$

for any $i = 0, 1, ..., n-1$, $i \neq t, (\frac{n}{2} + 1)t$.

In both cases we observe that X is not homogeneous. In the same way we can prove that Y is not homogeneous.

Hence a C_n- grading of A of second type is not a good grading. $\qquad\square$

Example 2.6. *Using Th. 2.2 and Prop. 2.5, the good C_2- gradings of $\Bbbk[X][Y, \varphi]$ are:*

  (i) *The trivial grading;*
  (ii) $A_e = < \{X^i Y^{2s} \mid i, s \in \mathbb{N}\} >$ *and* $A_g = < \{X^i Y^{2s+1} \mid i, s \in \mathbb{N}\} >$;
  (iii) $A_e = < \{X^{2t} Y^j / t, j \in \mathbb{N}\} >$ *and* $A_g = < \{X^{2t+1} Y^j / t, j \in \mathbb{N}\} >$;
  (iv) $A_e = < \{X^i Y^j \mid i, j \in \mathbb{N}, \quad i + j \quad is \; even\} >$ *and*
     $A_g = < \{X^i Y^j / i, j \in \mathbb{N}, i + j \; is \; odd\} >$.

In the following, we continue the study of the gradings of $\Bbbk[X][Y, \varphi]$ over abelian groups, using the same techniques as in [2]. First, we will introduce some remarks and definitions.

Let G and H two finite groups and let $A = \bigoplus\limits_{(g,h) \in G \times H} A_{(g,h)}$ be a $G \times H$- grading of $\Bbbk[X][Y, \varphi]$. Let $S_g = \bigoplus\limits_{h \in H} A_{(g,h)}$ and $T_h = \bigoplus\limits_{g \in G} A_{(g,h)}$, for each $g \in G$ and $h \in H$. Then $S = \bigoplus\limits_{g \in G} S_g$ and $T = \bigoplus\limits_{h \in H} T_h$ are G- grading and H- grading of $\Bbbk[X][Y, \varphi]$, respectively. We call them *gradings associated* with A.

We have $A_{(g,h)} = S_g \cap T_h$, for any $g \in G$ and $h \in H$. Hence every $G \times H$- grading of $\Bbbk[X][Y, \varphi]$ induces a natural G- grading and H- grading of $\Bbbk[X][Y, \varphi]$.

Now we would like to know if the converse is true. In order to answer this question we will give the following definition.

Definition 2.7. Let G and H be two finite groups. A G- grading $S = \bigoplus\limits_{g \in G} S_g$ and an H- grading $T = \bigoplus\limits_{h \in H} T_h$ of $\Bbbk[X][Y, \varphi]$ are *compatible* if $S_g = \bigoplus\limits_{h \in H} (S_g \cap T_h)$ for all $g \in G$ and $T_h = \bigoplus\limits_{g \in G} (S_g \cap T_h)$ for all $h \in H$.

The following proposition is proved in [2] for the matrix algebra, so we will give it without proof, because the main idea of the proof is the same.

Proposition 2.8. *Let G and H be two finite groups and let $S = \bigoplus\limits_{g \in G} S_g$ and $T = \bigoplus\limits_{h \in H} T_h$ be a G- grading and an H- grading of $\Bbbk[X][Y,\varphi]$, respectively. Then S and T are compatible if and only if there exists a $G \times H$- grading of $\Bbbk[X][Y,\varphi]$, whose associated G and H- gradings are S and T, respectively.*

Proposition 2.9. *Let $\Bbbk$ be a field containing a primitive m-th root ξ and a primitive n-th root η of unity and $A = \Bbbk[X][Y,\varphi]$. Then:*

(i) *If $b \neq 0$ or $a \neq -1$, any two C_m and C_n- gradings of $\Bbbk[X][Y,\varphi]$ are compatible.*

(ii) *If $b = 0$ and $a = -1$, then*

    (a) *Any two C_m and C_n- gradings of the first type of $\Bbbk[X][Y,\varphi]$ are compatible.*

    (b) *If $S = \bigoplus\limits_{i=0}^{m-1} S_{g^i} = \bigoplus\limits_{i=0}^{m-1} A_{g^i}^{II}(a_1,c_1)$ and $T = \bigoplus\limits_{j=0}^{n-1} T_{h^j} = \bigoplus\limits_{j=0}^{n-1} A_{h^j}^{II}(a_2,c_2)$ are C_m and C_n- gradings of second type, respectively, then S and T are compatible if and only if $a_1 c_2 = a_2 c$.*

    (c) *A C_m- grading $S = \bigoplus\limits_{i=0}^{m-1} S_{g^i} = \bigoplus\limits_{i=0}^{m-1} A_{g^i}^{I}(a_1,0,c_1)$ of the first type is compatible with a C_n- grading of the second type if and only if $a_1 = c_1$.*

Proof. (i) Let $S = \bigoplus\limits_{i=0}^{m-1} S_{g^i} = \bigoplus\limits_{i=0}^{m-1} A_{g^i}^{I}(a_1,b_1,c_1)$ and $T = \bigoplus\limits_{j=0}^{n-1} T_{h^j}$

$= \bigoplus\limits_{j=0}^{n-1} A_{h^j}^{I}(a_2,b_2,c_2)$ be   two C_m and

C_n- gradings of first type, respectively and $\Psi_1, \Psi_2 : \Bbbk[X][Y,\varphi] \longrightarrow \Bbbk[X][Y,\varphi]$, $\Psi_1(f) = f(a_1 X + b_1, c_1 Y)$ and $\Psi_2(f) = f(a_2 X + b_2, c_2 Y)$ be two automorphisms of $\Bbbk[X][Y,\varphi]$ such that $\Psi_1^m = \Psi_2^n = Id$ and $ba_i + b_i = ab_i + b$, for each $i = 1,2$. First, we will prove that $\Psi_1(\Psi_2(f)) = \Psi_2(\Psi_1(f))$, for all $f \in \Bbbk[X]$. We have

$$\Psi_1(\Psi_2(f)) = \Psi_1(f(a_2 X + b_2, c_2 Y)) = f(a_1 a_2 X + a_2 b_1 + b_2, c_1 c_2 Y)$$
$$\Psi_2(\Psi_1(f)) = \Psi_2(f(a_1 X + b_1, c_1 Y)) = f(a_1 a_2 X + a_1 b_2 + b_1, c_1 c_2 Y)$$

If $a_1 = 1$ or $a_2 = 1$, then from $\Psi_1^m(X) = \Psi_2^n(X) = X$ we obtain $b_1 = 0$ or $b_2 = 0$ and in this case we have $a_2 b_1 + b_2 = a_1 b_2 + b_1$. So, let $a_1, a_2 \neq 1$. In this case from $ba_i + b_i = ab_i + b$, for each $i = 1,2$, we obtain $\frac{b}{a-1} = \frac{b_1}{a_1 - 1} = \frac{b_2}{a_2 - 1}$ and obviously: $a_2 b_1 + b_2 = a_1 b_2 + b_1$. Hence $\Psi_1(\Psi_2(f)) = \Psi_2(\Psi_1(f))$, for all $f \in \Bbbk[X]$. Now we can easily prove that $\Psi_1^r(\Psi_2^p(f)) = \Psi_2^p(\Psi_1^r(f))$, for all

$f \in \Bbbk[X]$ and $r, p \in \mathbb{N}^*$. Let $f \in S_{g^i}$ and $f = \sum\limits_{j=0}^{n-1} f_{h^j}^i$, where $f_{h^j}^i \in T_{h^j}$, for any $j = 0, ..., n-1$.

From (2.1) we have $f_{h^j} = \frac{1}{n} \sum\limits_{\Bbbk=0}^{n-1} \eta^{-kj} \Psi_2^{\Bbbk}(f)$, with $\Psi_2^0(f) = f$ and we obtain

$$f_{h^j}(a_1 X + b_1, c_1 Y) = \Psi_1(f_{h^j}^i) = \frac{1}{n}(\sum_{\Bbbk=0}^{n-1} \eta^{-kj} \Psi_1(\Psi_2^{\Bbbk}(f))) = \frac{1}{n}(\sum_{\Bbbk=0}^{n-1} \eta^{-kj} \Psi_2^{\Bbbk}(\Psi_1(f)))$$

From $f \in A_{g^i}(a_1, b_1)$ we have $\xi^i f(X) = f(a_1 X + b_1, c_1 Y)$. Hence $\xi^i f_{h^j}(X) = f_{h^j}(a_1 X + b_1, c_1 Y)$.

From above we obtain that $f_{h^j} \in S_{g^i}$, therefore $S_{g^i} \subset \bigoplus\limits_{j=0}^{n-1}(S_{g^i} \cap T_{h^j})$.

Obviously $S_{g^i} \supset \bigoplus\limits_{j=0}^{n-1}(S_{g^i} \cap T_{h^j})$ and it implies that $S_{g^i} = \bigoplus\limits_{j=0}^{n-1}(S_{g^i} \cap T_{h^j})$.

Similarly we can prove $T_{h^j} = \bigoplus\limits_{i=0}^{m-1}(S_{g^i} \cap T_{h^j})$.

(ii) From (i), we obtain that any two C_m and C_n- gradings of the first type of $\Bbbk[X][Y, \varphi]$ are compatible.

Let $S = \bigoplus\limits_{i=0}^{m-1} S_{g^i} = \bigoplus\limits_{i=0}^{m-1} A_{g^i}^{II}(a_1, c_1)$ and $T = \bigoplus\limits_{j=0}^{n-1} T_{h^j} = \bigoplus\limits_{j=0}^{n-1} A_{h^j}^{II}(a_2, c_2)$ a C_m and C_n- gradings of second type, respectively. Suppose that S and T are compatible. Let $f \in S_{g^i}$. Then $f = \sum\limits_{j=0}^{n-1} f_{h^j}^i$, where $f_{h^j}^i \in S_{g^i} \cap T_{h^j}$ for any $j = 0, ..., n-1$. From Remark 2.3, we know that $f \in S_{g^i}$ if and only $\Psi_1(f) = \xi^i f$ and $f \in T_{h^j}$ if and only if $\Psi_2(f) = \eta^j f$, where $\Psi_1, \Psi_2 : \Bbbk[X][Y, \varphi] \longrightarrow \Bbbk[X][Y, \varphi], \Psi_i(f) = f(a_i Y, c_i X)$ for each $i = 1, 2$. We have that

$$\Psi_1(\Psi_2(f)) = \Psi_1(\eta^j f) = \eta^j(\sum_{j=0}^{n-1} \Psi_1(f_{h^j}^i)) = \sum_{j=0}^{n-1} \xi^i \eta^j f_{h^j}^i$$

$$\Psi_2(\Psi_1(f)) = \Psi_2(\xi^i f) = \xi^i(\sum_{j=0}^{n-1} \Psi_2(f_{h^j}^i)) = \sum_{j=0}^{n-1} \xi^i \eta^j f_{h^j}^i$$

Hence $\Psi_1(\Psi_2(f)) = \Psi_2(\Psi_1(f))$, for all $f \in S_{g^i}$. Therefore $\Psi_1(\Psi_2(f)) = \Psi_2(\Psi_1(f))$, for all $f \in \Bbbk[X][Y, \varphi]$ and we obtain $a_1 c_2 = a_2 c_1$. Conversely, when $a_1 c_2 = a_2 c_1$ we can prove that S and T are compatible, in a similar way to (i).

We can prove (c) in the same way as above. $\square$

Remark 2.10. The previous proposition works for any m and n, in particular for $m = n$. Therefore we obtain a characterization of $(C_m)^2$- gradings on skew polynomial rings.

The following two results are a generalization of the results obtained by the author in [2], for the polynomial algebra $\Bbbk[X]$. By using the same idea, the proof is straightforward.

Theorem 2.11. *Let $A = \Bbbk[X][Y,\varphi]$, where $\Bbbk$ is a field containing a primitive m-th root ξ and a primitive n-th root η of unity and $A = \bigoplus_{\substack{0 \le i \le m-1 \\ 0 \le j \le n-1}} A_{(g^i,h^j)}$ be a*
$C_m \times C_n$- grading of $\Bbbk[X][Y,\varphi]$, where $C_m = \langle g \rangle$ and $C_n = \langle h \rangle$. Then:

(i) *If $b \ne 0$ or $a \ne -1$ and $S = \bigoplus_{i=0}^{m-1} S_{g^i} = \bigoplus_{i=0}^{m-1} A_{g^i}^I(a_1, b_1, c_1)$ and*

$T = \bigoplus_{j=0}^{n-1} T_{h^j} = \bigoplus_{j=0}^{n-1} A_{h^j}^I(a_2, b_2, c_2)$ *are two gradings associated with A,*
such that $a_1 b_2 + b_1 = a_2 b_1 + b_2$, then

$$
\begin{aligned}
A_{(g^i,h^j)} &= \{\frac{1}{mn} \sum_{\substack{0 \le k \le m-1 \\ 0 \le t \le n-1}} \xi^{-ki}\eta^{-tj} f(a_1^k a_2^t X + a_2^t(a_1^{k-1} b_1 + \dots + a_1 b_1 + b_1) \\
&\quad + a_2^{t-1} b_2 + \dots + a_2 b_2 + b_2, c_1^k c_2^t Y) \mid f \in \Bbbk[X][Y,\varphi]\}
\end{aligned}
$$

for any $i = 0, 1, \dots, m-1$ and $j = 0, 1, \dots, n-1$.
(ii) *If $b = 0$ and $a = -1$, then*

    (a) *If $S = \bigoplus_{i=0}^{m-1} A_{g^i}^I(a_1, 0, c_1)$ and $T = \bigoplus_{j=0}^{n-1} A_{h^j}^I(a_2, 0, c_2)$ are two gradings*
    of the first type associated with A, we have that

$$
A_{(g^i,h^j)} = \{\frac{1}{mn} \sum_{\substack{0 \le k \le m-1 \\ 0 \le t \le n-1}} \xi^{-ki}\eta^{-tj} f(a_1^k a_2^t X, c_1^k c_2^t Y)/f \in \Bbbk[X][Y,\varphi]\}
$$

    for any $i = 0, 1, \dots, m-1$ and $j = 0, 1, \dots, n-1$.
    (b) *If m and n are even and $S = \bigoplus_{i=0}^{m-1} A_{g^i}^{II}(a_1, c_1)$ and $T = \bigoplus_{j=0}^{n-1} A_{h^j}^{II}(a_2, c_2)$*
    are two gradings of the second type associated with A, such that

$a_1 c_2 = a_2 c_1$, we obtain

$$
\begin{aligned}
A_{(g^i, h^j)} \;=\; & \Big\{ \frac{1}{mn} \Big[\sum_{\substack{0 \le \Bbbk \le m-1 \\ 0 \le t \le n-1 \\ \Bbbk, t \ even}} \xi^{-ki} \eta^{-tj} f\big(a_1^{\frac{\Bbbk}{2}} c_1^{\frac{\Bbbk}{2}} a_2^{\frac{t}{2}} c_2^{\frac{t}{2}} X,\; a_1^{\frac{\Bbbk}{2}} c_1^{\frac{\Bbbk}{2}} a_2^{\frac{t}{2}} c_2^{\frac{t}{2}} Y\big) \\[2mm]
& + \sum_{\substack{0 \le \Bbbk \le m-1 \\ 0 \le t \le n-1 \\ \Bbbk \ even, t \ odd}} \xi^{-ki} \eta^{-tj} f\big(a_1^{\frac{\Bbbk}{2}} c_1^{\frac{\Bbbk}{2}} a_2^{\frac{t+1}{2}} c_2^{\frac{t-1}{2}} Y,\; a_1^{\frac{\Bbbk}{2}} c_1^{\frac{\Bbbk}{2}} a_2^{\frac{t-1}{2}} c_2^{\frac{t+1}{2}} X\big) \\[2mm]
& + \sum_{\substack{0 \le \Bbbk \le m-1 \\ 0 \le t \le n-1 \\ \Bbbk \ odd, t \ even}} \xi^{-ki} \eta^{-tj} f\big(a_1^{\frac{\Bbbk+1}{2}} c_1^{\frac{\Bbbk-1}{2}} a_2^{\frac{t}{2}} c_2^{\frac{t}{2}} Y,\; a_1^{\frac{\Bbbk-1}{2}} c_1^{\frac{\Bbbk+1}{2}} a_2^{\frac{t}{2}} c_2^{\frac{t}{2}} X\big) \\[2mm]
& + \sum_{\substack{0 \le \Bbbk \le m-1 \\ 0 \le t \le n-1 \\ \Bbbk, t \ odd}} \xi^{-ki} \eta^{-tj} f\big(a_1^{\frac{\Bbbk-1}{2}} c_1^{\frac{\Bbbk+1}{2}} a_2^{\frac{t+1}{2}} c_2^{\frac{t-1}{2}} X,\; a_1^{\frac{\Bbbk+1}{2}} c_1^{\frac{\Bbbk-1}{2}} a_2^{\frac{t-1}{2}} c_2^{\frac{t+1}{2}} Y\big) \Big] \Big\}
\end{aligned}
$$

(c) If $S = \bigoplus\limits_{i=0}^{m-1} A_{g^i}^{I}(a_1, 0, a_1)$ and $T = \bigoplus\limits_{j=0}^{n-1} A_{h^j}^{II}(a_2, c_2)$ are two gradings associated with A, of the first and of the second type, respectively, then we have that

$$
\begin{aligned}
A_{(g^i, h^j)} \;=\; & \Big\{ \frac{1}{mn} \Big[\sum_{\substack{0 \le \Bbbk \le m-1 \\ 0 \le t \le n-1 \\ t \ even}} \xi^{-ki} \eta^{-tj} f\big(a_1^{\Bbbk} a_2^{\frac{t}{2}} c_2^{\frac{t}{2}} X,\; a_1^{\Bbbk} a_2^{\frac{t}{2}} c_2^{\frac{t}{2}} Y\big) \\[2mm]
& + \sum_{\substack{0 \le \Bbbk \le m-1 \\ 0 \le t \le n-1 \\ t \ odd}} \xi^{-ki} \eta^{-tj} f\big(a_1^{\Bbbk} a_2^{\frac{t+1}{2}} c_2^{\frac{t-1}{2}} Y,\; a_1^{\Bbbk} a_2^{\frac{t-1}{2}} c_2^{\frac{t+1}{2}} X\big) \Big] \Big\}
\end{aligned}
$$

Proposition 2.12. *Let G and H two finite groups and let $R = \bigoplus\limits_{(g,h) \in G \times H} R_{(g,h)}$ be a $G \times H-$ grading of $\Bbbk[X][Y, \varphi]$. Then R is a good grading if and only if the two gradings S and T associated with R are good gradings.*

Example 2.13. *Let $\Bbbk$ be a field of characteristic different from two and $A = \Bbbk[X][Y, \varphi]$. Using Prop 2.5, Th 2.11 and Prop 2.12 we obtain that the good $C_2 \times C_2$- gradings of A are:*

(i) *The trivial grading;*

(ii) $A_{(e,e)} = < \{X^i Y^{2j}/i, j \in \mathbb{N}\} >$, $A_{(g,g)} = < \{X^i Y^{2j+1}/i, j \in \mathbb{N}\} >$, $A_{(e,g)} = A_{(g,e)} = 0$;

(iii) $A_{(e,e)} = < \{X^i Y^{2j}/i, j \in \mathbb{N}\} >$, $A_{(e,g)} = < \{X^i Y^{2j+1}/i, j \in \mathbb{N}\} >$, $A_{(g,g)} = A_{(g,e)} = 0$;

(iv) $A_{(e,e)} = < \{X^i Y^{2j}/i, j \in \mathbb{N}\}$, $A_{(g,e)} = < \{X^i Y^{2j+1}/i, j \in \mathbb{N}\} >$, $A_{(g,g)} = A_{(e,g)} = 0$;

(v) $A_{(e,e)} =< \{X^{2i}Y^j/i,j \in \mathbb{N}\} >$, $A_{(e,g)} =< \{X^{2i+1}Y^j/i,j \in \mathbb{N}\} >$, $A_{(g,g)} = A_{(g,e)} = 0$;

(vi) $A_{(e,e)} =< \{X^iY^j/i,j \in \mathbb{N}, i+j \text{ is even}\} >$, $A_{(e,g)} =< \{X^iY^j/i,j \in \mathbb{N}, i+j \text{ is odd}\} >$, $A_{(g,g)} = A_{(g,e)} = 0$;

(vii) $A_{(e,e)} =< \{X^{2i}Y^{2j}/i,j \in \mathbb{N}\} >$, $A_{(e,g)} =< \{X^{2i+1}/i \in \mathbb{N}\} >$, $A_{(g,e)} =< \{Y^{2j+1}/j \in \mathbb{N}\} >$, $A_{(g,g)} =< \{X^{2i+1}Y^{2j+1}/i,j \in \mathbb{N}\} >$;

(viii) $A_{(e,e)} =< \{X^{2i}Y^{2j}/i,j \in \mathbb{N}\} >$, $A_{(g,e)} =< \{X^{2i+1}Y^{2j+1}/i,j \in \mathbb{N}\} >$, $A_{(e,g)} =< \{X^{2i+1}/i \in \mathbb{N}\} >$, $A_{(g,g)} =< \{Y^{2j+1}/j \in \mathbb{N}\} >$;

(ix) $A_{(e,e)} =< \{X^{2i}Y^j/i,j \in \mathbb{N}\} >$, $A_{(g,e)} =< \{X^{2i+1}Y^j/i,j \in \mathbb{N}\} >$, $A_{(g,g)} = A_{(e,g)} = 0$;

(x) $A_{(e,e)} =< \{X^iY^j/i,j \in \mathbb{N}, i+j \text{ is even}\} >$, $A_{(g,e)} =< \{X^iY^j/i,j \in \mathbb{N}, i+j \text{ is odd}\} >$, $A_{(g,g)} = A_{(e,g)} = 0$;

(xi) $A_{(e,e)} =< \{X^{2i}Y^{2j}/i,j \in \mathbb{N}\} >$, $A_{(e,g)} =< \{Y^{2j+1}/j \in \mathbb{N}\} >$, $A_{(g,e)} =< \{X^{2i+1}/i \in \mathbb{N}\} >$, $A_{(g,g)} =< \{X^{2i+1}Y^{2j+1}/i,j \in \mathbb{N}\} >$;

(xii) $A_{(e,e)} =< \{X^{2i}Y^{2j}/i,j \in \mathbb{N}\} >$, $A_{(e,g)} =< \{X^{2i+1}Y^{2j+1}/i,j \in \mathbb{N}\} >$, $A_{(g,e)} =< \{X^{2i+1}/i \in \mathbb{N}\} >$, $A_{(g,g)} =< \{Y^{2j+1}/j \in \mathbb{N}\} >$;

(xiii) $A_{(e,e)} =< \{X^{2i}Y^j/i,j \in \mathbb{N}\} >$, $A_{(g,g)} =< \{X^{2i+1}Y^j/i,j \in \mathbb{N}\} >$, $A_{(g,e)} = A_{(e,g)} = 0$;

(xiv) $A_{(e,e)} =< \{X^{2i}Y^{2j}/i,j \in \mathbb{N}\} >$, $A_{(g,g)} =< \{X^{2i+1}/i \in \mathbb{N}\} >$, $A_{(e,g)} =< \{Y^{2j+1}/j \in \mathbb{N}\} >$, $A_{(g,e)} =< \{X^{2i+1}Y^{2j+1}/i,j \in \mathbb{N}\} >$;

(xv) $A_{(e,e)} =< \{X^{2i}Y^{2j}/i,j \in \mathbb{N}\} >$, $A_{(g,g)} =< \{X^{2i+1}/i \in \mathbb{N}\} >$, $A_{(e,g)} =< \{X^{2i+1}Y^{2j+1}/i,j \in \mathbb{N}\} >$, $A_{(g,e)} =< \{Y^{2j+1}/j \in \mathbb{N}\} >$;

(xvi) $A_{(e,e)} =< \{X^iY^j/i,j \in \mathbb{N}, i+j \text{ is even}\} >$, $A_{(g,g)} =< \{X^iY^j/i,j \in \mathbb{N}, i+j \text{ is odd}\} >$, $A_{(g,e)} = A_{(e,g)} = 0$.

Let us now introduce the following definition.

Definition 2.14. Let $A = \bigoplus\limits_{\substack{0\leq i_1 \leq m_1-1 \\ \vdots \\ 0\leq i_n \leq m_n-1}} A_{(g_1^{i_1},...,g_n^{i_n})}$ be a $C_{m_1} \times .. \times C_{m_n}$- grading of $\Bbbk[X][Y,\varphi]$, where $m_1,\ldots,m_n \in \mathbb{N}^*$ and $C_{m_k} =< g_k >$. For each $1 \leq t \leq n$, we denote $S(t) = \bigoplus\limits_{i=0}^{m_t-1} S(t)_{g_t^i} = \bigoplus\limits_{i=0}^{m_t-1} \bigoplus\limits_{(i_1,..i_{t-1},i,i_{t+1},...,i_n)} A_{(g_1^{i_1},...,g_{t-1}^{i_{t-1}},g_t^i,g_{t+1}^{i_{t+1}},...,g_n^{i_n})}$. Then $S(t)$ is a C_{m_t}- grading of $\Bbbk[X][Y,\varphi]$, which is called the C_{m_t}-*grading associated with A.*

Now, remember that any finite abelian group is a direct product of finite cyclic groups. Using this and Prop 2.9, Th 2.11 and Prop 2.12, we can provide a complete description (although lengthy) of all G- gradings of $\Bbbk[X][Y,\varphi]$, for any G finite abelian group.

Acknowledgements

R E F E R E N C E S

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