A STUDY ON INTERIOR Γ-HYPERFILTERS IN ORDERED Γ-SEMIHYPERGROUPS

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In this paper, interior Γ-hyperfilters of ordered Γ-semihypergroups are defined and various types of them are studied. Results related to productional ordered Γ-semihypergroups were investigated. Moreover, we investigate some properties of the inverse images of strong interior Γ-hyperfilters in ordered Γ-semihypergroups. Finally, we discuss the relationship between two fundamental notions of ordered Γ-semihypergroup, the several types of interior Γ-hyperfilters and the (completely) prime interior Γ-hyperideals.

Keywords: ordered Γ-semihypergroup; **I**-Γ-hyperfilter; strong **I**-Γ-hyperfilter; weak **I**-Γ-hyperfilter; (completely) prime.

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1. Introduction and prerequisites

The notion of ordered semihypergroups was proposed by Heidari and Davvaz [7] in 2011. In [4], Davvaz et al. initiated the study of pseudoorders in ordered semihypergroups. Connections between ordered semigroups and ordered semihypergroups are considered in [4]. Gu and Tang [6] attempted to study the ordered regular equivalence relations of the ordered semihypergroups. They answered to an open problem on ordered semihypergroups which appeared in [4].

In 2010, Anvariyeh et al. [1] introduced the notion of a Γ-semihypergroup which is a generalization of semihypergroup. In 2015, Yaqoob and Aslam [23] introduced the idea of rough quasi-Γ-hyperideals in Γ-semihypergroups. Tang et al. [22] inspected useful results on fuzzy **I**-Γ-hyperideals in ordered Γ-semihypergroups. In [5], Gan and Jiang defined ordered semiring and investigated some useful results. In 2016, Omidi and Davvaz [16] made a first step in extending the theory of ordered rings to ordered (semi)hyperrings. In 2017, Omidi and Davvaz [14] studied the prime (m, n)-bi-hyperideals of ordered semihyperrings. There have been approaches to the constructions of ordered hyperstructyures as can be seen in [6, 13, 18, 19].

Hypergroups were originally proposed in 1934 by Marty [11] at the 8^{th} Congress of Scandinavian Mathematicians. The notion of hyperrings was proposed by Krasner [10] in

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1983. Jun [8] attempted to study the geometric aspects of the Krasner hyperrings. In [3], Corsini and Leoreanu provided many applications of hyperstructures.

Some researchers worked on hyperfilters by applying them to different types of ordered hyperstructures. In [20], Tang et al. applied fuzzy set theory to the hyperfilters of ordered semihypergroups. As generalizations of filters in ordered semigroups, the concept of Γ-hyperfilters of an ordered Γ-semihypergroup was first introduced by Omidi et al. [17] in 2018. After that so many authors, for example [12, 18, 21], conducted research on this and developed it. In [21], Tang et al. defined and analyzed the weak hyperfilters of ordered semihypergroups. (m,n)-Hyperfilters of ordered semihypergroups were investigated by Mahboob and Khan [12]. Later on, Rao et al. [18] introduced and studied the concept of (m,n)-Γ-hyperfilters in ordered Γ-semihypergroups. In [24], Yaqoob and Tang applied rough set theory to different types of hyperfilters in ordered LA-semihypergroups and explored some results. Roughness has also studied in hyperfilters of ordered LA-semihypergroups [2].

Previous studies on the hyperfilters of ordered hyperstructures motivated us to study the interior Γ-hyperfilter (briefly, **I**-Γ-hyperfilter) of an ordered Γ-semihypergroup. In Section 1, some notions on ordered Γ-semihypergroups are explained to facilitate the terminology (see [17] and [18] for more details and basic definitions). In Section 2, we define the concepts of strong and weak interior Γ-hyperfilters of the ordered Γ-semihypergroup S which are two new classes of **I**-Γ-hyperfilters. In Section 3, several properties of strong and weak interior Γ-hyperfilters are provided. Furthermore, we discuss the relationship between two fundamental notions of ordered Γ-semihypergroup, the several types of **I**-Γ-hyperfilters and the (completely) prime **I**-Γ-hyperideals. The study ends with some conclusions and ideas for future works.

Let $P^*(S)$ be the family of all non-empty subsets of $S \neq \emptyset$. A mapping $\circ : S \times S \to P^*(S)$ is called a *hyperoperation* on S. If $\emptyset \neq U, V \subseteq S$ and $x \in S$, then

$$U \circ V = \bigcup_{\substack{u \in U \\ v \in V}} u \circ v,\ x \circ U = \{x\} \circ U \text{ and } V \circ x = V \circ \{x\}.$$

A non-empty set equipped with a (binary) hyperoperation is called hypergroupoid. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every $a, b, c \in S$,

$$a \circ (b \circ c) = (a \circ b) \circ c.$$

Let $S \neq \emptyset$ be a set equipped with the hyperoperations $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$. If

(1) $a\gamma b \subseteq S$ for all $a, b \in S$ and all $\gamma \in \Gamma$,
(2) If $a, b, x, y \in S$ such that $a = x$ and $b = y$, then $a\gamma b = x\gamma y$,
(3) $x\alpha(a\beta b) = (x\alpha a)\beta b$,

hold, then S is said to be a Γ-*semihypergroup.* The reader may see [1, 9] for detailed discussion.

Definition 1.1. [15] *An ordered* Γ*-semihypergroup* $(T, \Gamma, \leq)$ *is a* Γ*-semihypergroup* (T, Γ) *endowed with a suitable (partial) order relation* $\leq$ *such that: for all* $a, b, x \in T$ *and* $\gamma \in \Gamma$, $a \leq b$ *implies* $a\gamma x \preceq b\gamma x$ *and* $x\gamma a \preceq x\gamma b$*, where for every* $\emptyset \neq U, V \subseteq S$, $U \preceq V$ *if and only if for each* $u \in U$*, there exists* $v \in V$ *such that* $u \leq v$.

$(T, \Gamma, \leq)$ is called *regular* if for every $a \in T$ there exist $x \in T$, $\gamma, \delta \in \Gamma$ such that $a \preceq a\gamma x\delta a$. A subset $F \neq \emptyset$ of an ordered Γ-semihypergroup T is said to be a *sub* Γ*-semihypergroup* if and only if $a\gamma b \subseteq F$ for all $a, b \in F$ and $\gamma \in \Gamma$. $(F]$ is defined as follows:

$$(F] := \{x \in T \mid x \leq f \text{ for some } f \in F\}.$$

Example 1.1. [15] *Let* $S = [0, 1]$ *and* $\Gamma = \mathbb{N}$*. For every* $x, y \in S$ *and* $\gamma \in \Gamma$*, we define* $\gamma : S \times \Gamma \times S \to \mathcal{P}^*(S)$ *by* $x\gamma y = [0, \frac{xy}{\gamma}]$*. For every* $x, y, z \in S$ *and* $\gamma, \beta \in \Gamma$*, we have*

$$(x\gamma y)\beta z = [0, \tfrac{xyz}{\gamma\beta}] = x\gamma(y\beta z).$$

We set

$$x \leq y \text{ if and only if } [0, \tfrac{xz}{\gamma}] \subseteq [0, \tfrac{yz}{\gamma}] \text{ for all } x, y, z \in S \text{ and } \gamma \in \Gamma.$$

Then $(S, \Gamma, \leq)$ *is an ordered* Γ*-semihypergroup.*

A Γ-hyperideal F of an ordered Γ-semihypergroup T is said to be *completely prime* if for each $x, y \in T$ and $\gamma \in \Gamma$ such that $x\gamma y \cap F \neq \emptyset$, then $x \in F$ or $y \in F$. Recall that a non-empty subset F of T is a Γ-*hyperideal* of T if (1) $T\Gamma F \subseteq F$ and $F\Gamma T \subseteq F$; (2) $(F] \subseteq F$.

Definition 1.2. [22] *An interior* Γ*-hyperideal (in short* ***I****-*Γ*-hyperideal)* F *of an ordered* Γ*-semihypergroup* $(T, \Gamma, \leq)$ *is a sub* Γ*-semihypergroup* F *of* T *such that*

(1) $T\Gamma F\Gamma T \subseteq F$;
(2) $(F] \subseteq F$.

Theorem 1.1. *Let* $(T, \Gamma, \leq)$ *be a regular ordered* Γ*-semihypergroup. Then every* ***I****-*Γ*-hyperideal of* T *is a* Γ*-hyperideal of* T*.*

Proof. For the proof see Theorem 3.6 in [22]. □

Definition 1.3. [17] *A sub* Γ*-semihypergroup* F *of an ordered* Γ*-semihypergroup* $(T, \Gamma, \leq)$ *is called a* Γ*-hyperfilter of* T *if*

(1) *for all* $a, b \in T$ *and* $\gamma \in \Gamma, a\gamma b \cap F \neq \emptyset \Rightarrow a \in F$ *and* $b \in F$;
(2) *for all* $a \in F$ *and* $c \in T, a \leq c \Rightarrow c \in F$, *i.e.,* $[F) \subseteq F$.

Indeed, for $F \subseteq S$ *we put*

$$[F) := \{x \in T \mid f \leq x \text{ for some } f \in F\}.$$

2. Definitions and examples

Throughout the rest of this paper: S will be an ordered Γ-semihypergroup. We begin this section with the definition of an **I**-Γ-hyperfilter on an ordered Γ-semihypergroup S.

Definition 2.1. *Let* $(S, \Gamma, \leq)$ *be an ordered* Γ*-semihypergroup and* $\emptyset \neq F \subseteq S$*. Then,* F *is said to be an interior* Γ*-hyperfilter (briefly,* ***I****-*Γ*-hyperfilter) of* S *if*

(1) F *is a sub* Γ*-semihypergroup of* S;
(1) *for all* $a, b, x \in S$, $(a\Gamma x)\Gamma b \subseteq F \Rightarrow x \in F$;
(2) *for all* $c \in S$ *and* $a \in F, a \leq c \Rightarrow c \in F$.

Example 2.1. *Consider an ordered* Γ*-semihypergroup* $S = \{a, b, c, d\}$ *with the following hyperoperations* $\Gamma = \{\gamma, \beta\}$ *and (partial) order relation* $\leq$:

γ	a	b	c	d
a	a	$\{b,d\}$	c	d
b	$\{b,d\}$	b	$\{b,d\}$	d
c	c	$\{b,d\}$	a	d
d	d	d	d	d

β	a	b	c	d
a	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	d
b	$\{b,d\}$	b	$\{b,d\}$	d
c	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	d
d	d	d	d	d

$$\leq \ := \{(a,a), (b,b), (c,c), (d,a), (d,b), (d,c), (d,d)\}.$$

We give the covering relation $\prec = \{(d,a), (d,b), (d,c)\}$*, and the figure of* S *in Figure 1.*

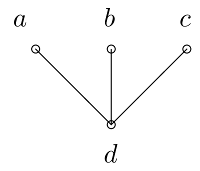

Figure 1: Figure of $(S, \Gamma, \leq)$ for Example 1.

Note that for every $\gamma, \beta \in \Gamma$ and $x, y, z \in S$, we have $x\gamma(y\beta z) = (x\gamma y)\beta z$. Clearly, $F_1 = \{a, c\}$ is a sub Γ-semihypergroup of S, i.e., $x\gamma y \subseteq F_1$ for all $x, y \in F_1$ and $\gamma \in \Gamma$. We hava

$\forall x, y \in S, (x\Gamma b)\Gamma y \nsubseteq F_1$.

$\forall x, y \in S, (x\Gamma d)\Gamma y \nsubseteq F_1$.

*On the other hand, $[F_1) = F_1$. Therefore, F_1 ia an interior Γ-hyperfilter of S. All the **I**-Γ-hyperfilters of S are $F_1 = \{a, c\}$, $F_2 = \{b\}$ and $F_3 = S$.*

Strong **I**-Γ-hyperfilters are sub Γ-semihypergroups in which $\subseteq$ is replaced with non-empty intersection. In the following, we provide the basic definition and results concerning strong **I**-Γ-hyperfilters.

Definition 2.2. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup and $\emptyset \neq F \subseteq S$. Then, F is said to be a strong **I**-Γ-hyperfilter of S if*

(1) *F is a sub Γ-semihypergroup of S;*
(1) *for all $a, b, x \in S$, $((a\Gamma x)\Gamma b) \cap F \neq \emptyset \Rightarrow x \in F$;*
(2) *for all $c \in S$ and $a \in F, a \leq c \Rightarrow c \in F$.*

Clearly, every strong **I**-Γ-hyperfilter of an ordered Γ-semihypergroup is an **I**-Γ-hyperfilter. The converse is not generally true as shown by the following example

Example 2.2. *In Example **2**.1, $F_1 = \{b\}$ is not a strong **I**-Γ-hyperfilter of S. Indeed:*

$$b\Gamma a\Gamma c = \{b, d\} \cap F_1 \neq \emptyset \text{ but } a \notin F_1.$$

Definition 2.3. *Let $\emptyset \neq F$ be a subset of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then F is called a weak **I**-Γ-hyperfilter of S if*

(1) *$(a\gamma b) \cap F \neq \emptyset$ for all $a, b \in F$ and all $\gamma \in \Gamma$;*
(2) *for all $a, b, x \in S$, $((a\Gamma x)\Gamma b) \cap F \neq \emptyset \Rightarrow x \in F$;*
(3) *for all $x \in F$ and $z \in S, x \leq z \Rightarrow z \in F$, i.e., $[F) \subseteq F$.*

Clearly, every strong **I**-Γ-hyperfilter of an ordered Γ-semihypergroup S is a weak **I**-Γ-hyperfilter of S. The converse is not true, in general, that is, a weak **I**-Γ-hyperfilter may not be a strong **I**-Γ-hyperfilter of S.

Example 2.3. *Let us follow the tables used in Example **2**.1. By defining the (partial) order relation*

$$\leq \ := \{(a, a), (b, b), (c, c), (d, a), (d, c), (d, d)\}$$

on S, we get that $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup. Covering relation of S as given below

$$\prec = \{(d, a), (d, c)\}.$$

The Hasse diagram of S is shown in Figure 2.

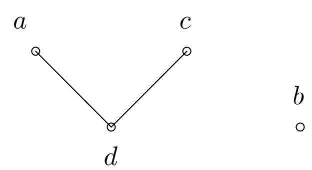

FIGURE 2: Figure of $(S, \Gamma, \leq)$ for Example 4.

*Here, $F = \{a, b, c\}$ is a weak **I**-Γ-hyperfilter of S. Clearly, F is not a strong **I**-Γ-hyperfilter of S. Since $F\Gamma F = S \nsubseteq F$, i.e., $a\Gamma b = \{b, d\} \nsubseteq F$, it follows that F is not a sub Γ-semihypergroup of S.*

3. On two classes of interior Γ-hyperfilters

Lemma 3.1. *Let T be a sub Γ-semihypergroup of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. Then for an **I**-Γ-hyperfilter F of S, either $\emptyset = F \cap T$ or $F \cap T$ is an **I**-Γ-hyperfilter of T.*

Proof. Let $\emptyset \neq F_1 = F \cap T$. We show that F_1 is an **I**-Γ-hyperfilter of T. Clearly, F_1 is a sub Γ-semihypergroup of T. Indeed: $F_1\Gamma F_1 \subseteq F\Gamma F \subseteq F$ and $F_1\Gamma F_1 \subseteq T\Gamma T \subseteq T$. So, $F_1\Gamma F_1 \subseteq F \cap T = F_1$. Therefore, F_1 is a sub Γ-semihypergroup of T. Now, let $a, b, x \in T$ and $(a\Gamma x)\Gamma b \subseteq F_1$. Then $(a\Gamma x)\Gamma b \subseteq F$. Since F is an **I**-Γ-hyperfilter of S, it follows that $x \in F$. So, $x \in F \cap T = F_1$. Now take any $a \in F_1$ and $c \in T$ such that $a \leq c$. Since F is an **I**-Γ-hyperfilter of S and $a \in F$, we get $c \in F$. Hence $c \in F \cap T = F_1$. Therefore, F_1 is an **I**-Γ-hyperfilter of T. □

Lemma 3.2. *Intersection of a non-empty collection of **I**-Γ-hyperfilters of an ordered Γ-semihypergroup S is also an **I**-Γ-hyperfilter of S.*

Proof. Let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a non-empty family of **I**-Γ-hyperfilters of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. We show that $\bigcap\limits_{\lambda\in\Lambda} F_\lambda$ is an **I**-Γ-hyperfilter of S, if $\bigcap\limits_{\lambda\in\Lambda} F_\lambda \neq \emptyset$. Let F_λ be an **I**-Γ-hyperfilter of S for all $\lambda \in \Lambda$ and $a, b \in \bigcap\limits_{\lambda\in\Lambda} F_\lambda$. Then $a, b \in F_\lambda$ for all $\lambda \in \Lambda$. Since F_λ is a sub Γ-semihypergroup of S, we get $a\gamma b \subseteq F_\lambda$ for all $\lambda \in \Lambda$ and $\gamma \in \Gamma$. So, $a\gamma b \subseteq \bigcap\limits_{\lambda\in\Lambda} F_\lambda$. This shows that $\bigcap\limits_{\lambda\in\Lambda} F_\lambda$ is a sub Γ-semihypergroup of S. Let $a, b, x \in R$ and $(a\Gamma x)\Gamma b \subseteq \bigcap\limits_{\lambda\in\Lambda} F_\lambda$. Then, $(a\Gamma x)\Gamma b \subseteq F_\lambda$ for all $\lambda \in \Lambda$. As F_λ is an **I**-Γ-hyperfilter of S, then $x \in F_\lambda$ for all $\lambda \in \Lambda$. It implies that $x \in \bigcap\limits_{\lambda\in\Lambda} F_\lambda$. Now, let $a \in \bigcap\limits_{\lambda\in\Lambda} F_\lambda$, $c \in S$ and $a \leq c$. Then, $a \in F_\lambda$ for all $\lambda \in \Lambda$. Since F_λ is an **I**-Γ-hyperfilter of S for all $\lambda \in \Lambda$, we get $c \in F_\lambda$ for all $\lambda \in \Lambda$. Hence, $c \in \bigcap\limits_{\lambda\in\Lambda} F_\lambda$. Therefore, $\bigcap\limits_{\lambda\in\Lambda} F_\lambda$ is an **I**-Γ-hyperfilter of S. □

Lemma 3.3. *Let us follow the notations used in Lemma **3**.2. Then, $\bigcap\limits_{\lambda\in\Lambda} F_\lambda$ is a strong **I**-Γ-hyperfilter of S, if $\bigcap\limits_{\lambda\in\Lambda} F_\lambda \neq \emptyset$.*

Proof. By the proof of Lemma **3**.2, $\bigcap\limits_{\lambda\in\Lambda} F_\lambda$ is a sub Γ-semihypergroup of S. Let $a, b, x \in S$ and $((a\Gamma x)\Gamma b) \cap (\bigcap\limits_{\lambda\in\Lambda} F_\lambda) \neq \emptyset$. Then there exists $u \in \bigcap\limits_{\lambda\in\Lambda} F_\lambda$ for some $u \in (a\Gamma x)\Gamma b$. Then, $u \in F_\lambda$ for all $\lambda \in \Lambda$. So, $u \in ((a\Gamma x)\Gamma b) \cap F_\lambda$ for all $\lambda \in \Lambda$. It means that $((a\Gamma x)\Gamma b) \cap F_\lambda \neq \emptyset$. As F_λ is a strong **I**-Γ-hyperfilter of S, $x \in F_\lambda$ for all $\lambda \in \Lambda$. Thus, $x \in \bigcap\limits_{\lambda\in\Lambda} F_\lambda$. Clearly, $[\bigcap\limits_{\lambda\in\Lambda} F_\lambda) \subseteq \bigcap\limits_{\lambda\in\Lambda} F_\lambda$. Hence, $\bigcap\limits_{\lambda\in\Lambda} F_\lambda$ is a strong **I**-Γ-hyperfilter of S. □

Let $(S_\lambda, \Gamma_\lambda, \leq_\lambda)$ be an ordered Γ_λ-semihypergroup for all $\lambda \in \Lambda$. Define

$$\odot : (\prod_{\lambda\in\Lambda} S_\lambda) \times (\prod_{\lambda\in\Lambda} \Gamma_\lambda) \times (\prod_{\lambda\in\Lambda} S_\lambda) \to \mathcal{P}^*(\prod_{\lambda\in\Lambda} S_\lambda)$$

by

$$(x_\lambda)_{\lambda\in\Lambda} \odot (\alpha_\lambda)_{\lambda\in\Lambda} \odot (y_\lambda)_{\lambda\in\Lambda} = \{(z_\lambda)_{\lambda\in\Lambda} \mid z_\lambda \in x_\lambda\alpha_\lambda y_\lambda\},$$

for all $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda} \in \prod\limits_{\lambda\in\Lambda} S_\lambda$ and $(\alpha_\lambda)_{\lambda\in\Lambda} \in \prod\limits_{\lambda\in\Lambda} \Gamma_\lambda$. Also,

$$(x_\lambda)_{\lambda\in\Lambda} \leq (y_\lambda)_{\lambda\in\Lambda} \Longleftrightarrow x_\lambda \leq_\lambda y_\lambda \text{ for all } \lambda \in \Lambda.$$

One can easily see that $(\prod_{\lambda\in\Lambda} S_\lambda, \prod_{\lambda\in\Lambda} \Gamma_\lambda, \leq)$ is an ordered $\prod_{\lambda\in\Lambda} \Gamma_\lambda$-semihypergroup [15].

Theorem 3.1. *Let F_λ be a strong **I**-Γ-hyperfilter on the ordered Γ-semihypergroup $(S_\lambda, \Gamma_\lambda, \leq_\lambda)$ for all $\lambda \in \Lambda$. Then, $F = \prod_{\lambda\in\Lambda} F_\lambda$ is a strong **I**-Γ-hyperfilter on $\prod_{\lambda\in\Lambda} S_\lambda$.*

Proof. Let F_λ be a strong **I**-Γ-hyperfilter on the ordered Γ-semihypergroup $(S_\lambda, \Gamma_\lambda, \leq_\lambda)$ for all $\lambda \in \Lambda$. First of all, we show $F = \prod_{\lambda\in\Lambda} F_\lambda$ is a sub Γ-semihypergroup of $\prod_{\lambda\in\Lambda} S_\lambda$. Let $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda} \in F = \prod_{\lambda\in\Lambda} F_\lambda$. Then, $x_\lambda, y_\lambda \in F_\lambda$ for each $\lambda \in \Lambda$. As F_λ's is a sub Γ-semihypergroup of S_λ, $x_\lambda \gamma_\lambda y_\lambda \subseteq F_\lambda$ for all $\gamma_\lambda \in \Gamma_\lambda$. So,

$$(x_\lambda)_{\lambda\in\Lambda} \odot (\gamma_\lambda)_{\lambda\in\Lambda} \odot (y_\lambda)_{\lambda\in\Lambda} = (x_\lambda \gamma_\lambda y_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} F_\lambda = F.$$

Therefore, F is a sub Γ-semihypergroup of $\prod_{\lambda\in\Lambda} S_\lambda$.

Now, let $(a_\lambda)_{\lambda\in\Lambda}, (x_\lambda)_{\lambda\in\Lambda}, (b_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} S_\lambda$ and

$$((a_\lambda)_{\lambda\in\Lambda} \odot (\gamma_\lambda)_{\lambda\in\Lambda} \odot (x_\lambda)_{\lambda\in\Lambda} \odot (\delta_\lambda)_{\lambda\in\Lambda} \odot (b_\lambda)_{\lambda\in\Lambda}) \cap F \neq \emptyset.$$

Then,

$$\begin{aligned}
&((a_\lambda)_{\lambda\in\Lambda} \odot (\gamma_\lambda)_{\lambda\in\Lambda} \odot (x_\lambda)_{\lambda\in\Lambda} \odot (\delta_\lambda)_{\lambda\in\Lambda} \odot (b_\lambda)_{\lambda\in\Lambda}) \cap F \neq \emptyset \\
&\Rightarrow (a_\lambda \gamma_\lambda x_\lambda \delta_\lambda b_\lambda)_{\lambda\in\Lambda} \cap F \neq \emptyset \\
&\Rightarrow a_\lambda \gamma_\lambda x_\lambda \delta_\lambda b_\lambda \cap F_\lambda \neq \emptyset, \forall \lambda \in \Lambda \\
&\Rightarrow x_\lambda \in F_\lambda, \forall \lambda \in \Lambda \\
&\Rightarrow (x_\lambda)_{\lambda\in\Lambda} \in F.
\end{aligned}$$

Let $(a_\lambda)_{\lambda\in\Lambda} \in F$, $(c_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} S_\lambda$ and $((a_\lambda)_{\lambda\in\Lambda} \preceq (c_\lambda)_{\lambda\in\Lambda})$. Then, $a_\lambda \leq_\lambda c_\lambda$ for all $\lambda \in \Lambda$. Since F_λ is a strong **I**-Γ-hyperfilter of S_λ for each $\lambda \in \Lambda$, it follows that $c_\lambda \in F_\lambda$ for each $\lambda \in \Lambda$. So, $(c_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} F_\lambda = F$. Therefore, F is a strong **I**-Γ-hyperfilter of $\prod_{\lambda\in\Lambda} S_\lambda$. □

A mapping $\varphi : S \to T$ of an ordered Γ-semihypergroup $(S, \Gamma, \leq_S)$ into an ordered Γ-semihypergroup $(T, \Gamma', \leq_T)$ is said to be a normal Γ-homomorphism if (1) $\varphi(x\gamma y) = \varphi(x)\gamma'\varphi(y)$ for all $x, y \in S$, $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$; (2) φ is isotone, i.e., for any $a, b \in S$, $a \leq_S b$ implies $\varphi(a) \leq_T \varphi(b)$.

Theorem 3.2. *Let $\varphi : S \to T$ be a normal Γ-homomorphism of ordered Γ-semihypergroups $(S, \Gamma, \leq_S)$ and $(T, \Gamma', \leq_T)$. If F is a strong **I**-Γ-hyperfilter of T, then*

$$\varphi^{-1}(F) = \{a \in S \mid \varphi(a) \in F\}$$

*is a strong **I**-Γ-hyperfilter of S.*

Proof. Let $a, b \in \varphi^{-1}(F)$, $\alpha \in \Gamma$ and $\alpha' \in \Gamma'$. Then $\varphi(a), \varphi(b) \in F$. Since F is a sub Γ-semihypergroup of T and φ a normal Γ-homomorphism, we get

$$\begin{aligned}
\varphi(a\alpha b) &= \varphi(a)\alpha'\varphi(b) \\
&\subseteq F\Gamma' F \\
&\subseteq F.
\end{aligned}$$

So, $a\alpha b \subseteq \varphi^{-1}(F)$. Hence, $\varphi^{-1}(F)$ is sub Γ-semihypergroup of S.

Now, let $a, b, x \in S$, $\alpha, \beta \in \Gamma$ and $a\alpha x\beta b \cap \varphi^{-1}(F) \neq \emptyset$. Then,

$$a\alpha x\beta b \cap \varphi^{-1}(F) \neq \emptyset$$

$$\Rightarrow \varphi(a\alpha x\beta b) \cap F \neq \emptyset$$

$$\Rightarrow (\bigcup_{u \in x\beta b} \varphi(a)\alpha'\varphi(u)) \cap F \neq \emptyset$$

$$\Rightarrow (\varphi(a)\alpha'\varphi(x)\beta'\varphi(b)) \cap F \neq \emptyset$$

$$\Rightarrow \varphi(x) \in F$$

$$\Rightarrow x \in \varphi^{-1}(F).$$

If $a \in \varphi^{-1}(F)$, $c \in S$ and $a \leq_S c$, then $\varphi(a) \in F$. Since φ is a Γ-homomorphism, we get $\varphi(a) \leq_T \varphi(c)$. As F is a strong **I**-Γ-hyperfilter of T, we have $\varphi(c) \in F$. It implies that $c \in \varphi^{-1}(F)$. Therefore, $\varphi^{-1}(F)$ is a strong **I**-Γ-hyperfilter of S. □

In the following, it reveals the relationship between two fundamental notions of ordered Γ-semihypergroup, the several types of **I**-Γ-hyperfilters and the (completely) prime **I**-Γ-hyperideals.

Theorem 3.3. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup and $\emptyset \neq F \subsetneqq S$. If $S \setminus F$ is a sub Γ-semihypergroup of S, then F is a strong **I**-Γ-hyperfilter of S if and only if $S \setminus F$ is a completely prime **I**-Γ-hyperideal of S.*

Proof. Necessity. First, we prove that $S \setminus F$ is an **I**-Γ-hyperideal of S. Let $a, b \in S$, $x \in S \setminus F$, $\gamma, \delta \in \Gamma$ and $(a\gamma x\delta b) \cap F \neq \emptyset$. Since F is a strong **I**-Γ-hyperfilter of S, we get $x \in F$, a contradiction. So, $a\gamma x\delta b \subseteq S \setminus F$, i.e., $S\Gamma(S \setminus F)\Gamma S \subseteq S \setminus F$. Now, let $a \in S \setminus F$, $x \in S$ and $x \leq a$, i.e., $x \in (S \setminus F]$. We show that $x \in S \setminus F$. If $x \in F$, then, since F is a strong **I**-Γ-hyperfilter of S, it follows that $a \in F$, a contradiction. Thus $x \in S \setminus F$, and so $(S \setminus F] \subseteq S \setminus F$. Therefore, $S \setminus F$ is an **I**-Γ-hyperideal of S. Next, let $u, v \in S$, $\alpha \in \Gamma$ and $u\alpha v \cap (S \setminus F) \neq \emptyset$. Then, there exists $t \in u\alpha v$ such that $t \in S \setminus F$. If $u \in F$ and $v \in F$, then, since F is a sub Γ-semihypergroup of S, we get $t \in F$, a contradiction. So, $a \in S \setminus F$ or $b \in S \setminus F$. Therefore, $S \setminus F$ is a completely prime **I**-Γ-hyperideal of S.

Sufficiency. Let $S \setminus F$ be a completely prime **I**-Γ-hyperideal of S. Now, let $m, n \in F$ and $\alpha \in \Gamma$. If $m\alpha n \nsubseteq F$, then $m\alpha n \cap (S \setminus F) \neq \emptyset$. Since $S \setminus F$ is completely prime, we get $m \in S \setminus F$ or $n \in S \setminus F$, which is a contradiction. So, $m\alpha n \subseteq F$. Thus, F is a sub Γ-semihypergroup of S. Let $a, b, x \in S$ and $((a\Gamma x)\Gamma b) \cap F \neq \emptyset$. If $x \in S \setminus F$, then, since $S \setminus F$ is an **I**-Γ-hyperideal, we have $a\Gamma x\Gamma b \subseteq S\Gamma(S \setminus F)\Gamma S \subseteq S \setminus F$, which is a contradiction. It implies that $x \in F$. Now, let $a \in F$, $c \in S$ and $a \leq c$, i.e., $c \in [F)$. We show that $c \in F$. If $c \in S \setminus F$, then, since $S \setminus F$ is an **I**-Γ-hyperideal of S, we get $a \in S \setminus F$, a contradiction. So $c \in F$, and thus $[F) \subseteq F$. Therefore, F is a strong **I**-Γ-hyperfilter of S. □

Example 3.1. *In Example **2**.1, $F_2 = \{a, c\}$ is a strong **I**-Γ-hyperfilter of S. Thus, by Theorem **3**.3, $S \setminus F_2 = \{b, d\}$ is a completely prime **I**-Γ-hyperideal of S.*

Combining Theorem **1**.1 with Theorem **3**.3 we draw the following conclusion.

Corollary 3.1. *Let $(S, \Gamma, \leq)$ be a regular ordered Γ-semihypergroup and $\emptyset \neq F \subsetneqq S$. If $S \setminus F$ is a sub Γ-semihypergroup of S, then F is a strong **I**-Γ-hyperfilter of S if and only if $S \setminus F$ is a completely prime Γ-hyperideal of S.*

In the following, we focus our study on weak **I**-Γ-hyperfilters of ordered Γ-semihypergroups.

Proposition 3.1. *Let F be a weak **I**-Γ-hyperfilter of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If $F \ll a\gamma x\delta b$, then $x \in F$ for all $a, b, x \in S$ and $\gamma, \delta \in \Gamma$. Here, $U \ll V$ means that there exist $u \in U$ and $v \in V$ such that $u \leq v$, for all $\emptyset \neq U, V \subseteq S$.*

Proof. Let F be a weak **I**-Γ-hyperfilter of S and $F \ll a\gamma x\delta b$, where $a, b, x \in S$ and $\gamma, \delta \in \Gamma$. As $F \ll a\gamma x\delta b$, there exists $u \in F$ and $v \in a\gamma x\delta b$ such that $u \leq v$. Since $[F) \subseteq F$, we get $v \in F$. It implies that $(a\gamma x\delta b) \cap F \neq \emptyset$. By condition (2) of Definition **2**.3, we obtain $x \in F$ □

Proposition 3.2. *Let F be a weak **I**-Γ-hyperfilter of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$. If $U \cap F \neq \emptyset$ and $U \preceq V$, then $V \cap F \neq \emptyset$, where $\emptyset \neq U, V \subseteq S$.*

Proof. Since $U \cap F \neq \emptyset$, then there exists $u \in S$ such that $u \in F$ and $u \in U$. As $U \preceq V$ and $u \in U$, there exists $v \in V$ such that $u \leq v$. Since F is a weak **I**-Γ-hyperfilter of S and $u \in F$, we have $v \in F$, by condition (3) of Definition **2**.3. So, $V \cap F \neq \emptyset$. □

Theorem 3.4. *Let $(S, \Gamma, \leq)$ be an ordered Γ-semihypergroup and $\emptyset \neq F \subsetneqq S$. If $S \setminus F$ is a sub Γ-semihypergroup of S, then F is a weak **I**-Γ-hyperfilter of S if and only if $S \setminus F$ is a prime **I**-Γ-hyperideal of S.*

Proof. Necessity. We first show that $S \setminus F$ is an **I**-Γ-hyperideal of S. Let $u, v \in S$, $x \in S \setminus F$ and $\gamma, \delta \in \Gamma$. If $u\gamma x\delta v \nsubseteq S \setminus F$, then there exists $t \in u\gamma x\delta v$ such that $t \in F$. So, $(u\gamma x\delta v) \cap F \neq \emptyset$. Since F is a weak **I**-Γ-hyperfilter of S, it follows that $x \in F$, which is a contradiction. So, $u\gamma x\delta v \subseteq S \setminus F$. It means that

$$S\Gamma(S \setminus F)\Gamma S \subseteq S \setminus F.$$

On the other hand, $(S \setminus F] \subseteq S \setminus F$. Therefore, $S \setminus F$ is an **I**-Γ-hyperideal of S. Next, we prove that $S \setminus F$ is prime. Let $u, v \in S$, $\gamma \in \Gamma$ and $u\gamma v \subseteq S \setminus F$. If $u \in F$ and $v \in F$, then, since F is a weak **I**-Γ-hyperfilter of S, we get $(u\gamma v) \cap F \neq \emptyset$, a contradiction. So, $u \in S \setminus F$ or $v \in S \setminus F$. Therefore, $S \setminus F$ is a prime **I**-Γ-hyperideal of S.

Sufficiency. Let $S \setminus F$ is a prime **I**-Γ-hyperideal of S. We assert that F is a weak **I**-Γ-hyperfilter of S. Let $a, b \in F$ and $\gamma \in \Gamma$. If $a\gamma b \cap F = \emptyset$, then $a\gamma b \subseteq (S \setminus F)$. Since $S \setminus F$ is prime, it follows that $a \in S \setminus F$ or $b \in S \setminus F$, which is a contradiction. So, $a\gamma b \cap F \neq \emptyset$. Now, let $a, b, x \in S$, $\gamma, \delta \in \Gamma$ and $(a\gamma x\delta b) \cap F \neq \emptyset$. If $x \in S \setminus F$, then $a\gamma x\delta b \subseteq S\Gamma(S \setminus F)\Gamma S \subseteq S \setminus F$. So, $a\gamma x\delta b \cap F = \emptyset$, which is a contradiction. It implies that $x \in F$. Clearly, $[F) \subseteq F$. Hence, F is a weak **I**-Γ-hyperfilter of S. □

4. Conclusions

In this study, we introduced the notion of **I**-Γ-hyperfilter of an ordered Γ-semihypergroup and then we obtained some useful properties. Results related to productional ordered Γ-semihypergroups were investigated. Moreover, we tried to generalize these results to various types of **I**-Γ-hyperfilters of ordered Γ-semihypergroups. In ordered Γ-semihypergroups there exist different types of **I**-Γ-hyperfilters. We use (completely) prime **I**-Γ-hyperideal to characterize various kinds of **I**-Γ-hyperfilters. From Examples and Definitions, we conclude that weak **I**-Γ-hyperfilters $\subseteq$ strong **I**-Γ-hyperfilters $\subseteq$ **I**-Γ-hyperfilters. For future work, one could extend the existing work to the framework of fuzzy **I**-Γ-hyperfilters, soft **I**-Γ-hyperfilters and rough **I**-Γ-hyperfilters.

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