ABOUT TENSOR PRODUCT OF CONTINUOUS TRACE C*-ALGEBRAS AND SOME APPLICATIONS

Daniel TUDOR[1]

Lucrarea de faţă se concentrează pe un caz particular de C-algebre, şi anume C*-algebrele cu urmă continuă. Este analizat produsul tensorial al C*-algebrelor cu urmă continuă, şi, în diferite cazuri de crossproduse, sunt date aplicaţii ale acestui produs tensorial. Sunt prezentate cazul crossprodusului dual al unui sistem dinamic şi cazul mai general al unui C*-sistem dinamic grupoidal.*

The paper focuses on a special case of C-algebras, the continuous trace C*-algebras. The case of tensor product of the continuous trace C*-algebras is analyzed and some applications of the tensor product in the different cases of crossed products are given. The case of dual crossed product of a dynamical system and the more general case, of the crossed product of a C*-groupoid dynamical system are presented.*

Keywords: C*-algebras, tensor product, crossed product algebras, topological groupoids

1. Introduction

Quantum mechanics, in its most general formulation, is a theory of abstract operators ("observables") acting on an abstract Hilbert space ("state space"), where the observables represent physically observable quantities and the state space represents the possible states of the system under study. The C^*-algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables. For example, in quantum mechanics, the time evolution of a system is given by an action of the set of real numbers on the algebra of observables, and it is known that the theory of C^*-algebras accomodates actions of groups by automorphismes. C^*-algebras are now an important tool in algebraic formulations of quantum mechanics, with implications in quantum field theory. In quantum field theory, a C^*-algebra A with unit element describes a physical system and the self-adjoint elements of A are thought of as the observables, the measurable quantities, of the system. A not commutative, special case of C^*-algebra is a continuous trace C^*-algebra.

[1]Department of Mathematics and Computer Science, Technical University of Civil Engineering, Romania

The notion of continuous trace C^*-algebra has been defined in at least two different ways. The classical one is the definition given by Dixmier, [1, Definition II.9] and it is closer to the usual concept of trace. A positive element a of a C^*-algebra A is a *continuous trace element* if application $\pi \to Tr(\pi(a))$ is finite and continuous on the spectrum $\hat{A}$ of A. The collection of continuous trace elements is the positive part of a two-sided, self-adjoint ideal M of A. In Dixmier's definition, a C^*-algebra A is with continuous trace if the ideal M is dense in A. The closure of M is generally denoted by $J(A)$, so in Dixmier's definition, a C^*-algebra A is with continuous trace if $J(A) = A$. The modern definition has been used in [2, Definition 5.13., p.121] and it will be called, in this paper, the Williams-Raeburn definition. The equivalence of these two definitions has been proved by Dixmier in [1, Proposition 10, II.9]. The Williams-Raeburn definition says that a C^*-algebra A is a continuous trace C^*-algebra if the following two conditions are acomplished : firstly, the spectrum $\hat{A}$ of A is Hausdorff in Jacobson topology and secondly, for every element π_0 of $\hat{A}$, there are a positive element $a \in A^+$ and a neighborhood V of π_0 such that $\pi(a)$ is a rank one projector for all π from V.

For the special case of a tensor product of two continuous trace C^*-algebras, Tomyiama proved in [3, Theorem 2(a)] that C^*-algebra tensor product $A \otimes B$ is a continuous trace C^*-algebra if and only if A and B are both continuous trace C^*-algebras. The direct part of theorem (the "if" part) has been proved based on Dixmier's definition, by A. Wulfsohn, in [4]. The first purpose of this paper is to give another proof for the direct part of Tomyiama's theorem, using a direct verification of the conditions from the Williams-Raeburn definition. The idea of this proof is suggested by the special form of the dual crossed product of a dynamical system (A, G, α) with A a C^*-algebra, G an abelian, locally compact group and $\alpha : G \to Aut(A)$ a continuous homomorphism from G to the group of automorphismes of A with the point- norm topology. This special form of dual crossed product is given in Takai's theorem [5, Theorem 3.4], and expresses the dual crossed product, denoted by $(A \times_{\alpha} G) \times_{\bar{\alpha}} \overline{G}$ as a tensor product $A \otimes K(L^2(G))$. But one of the factors of this tensor product is the most simple example of continuous trace C^*-algebra, $K(L^2(G))$, and from here arises the idea to study the tensor product of two continuous trace C^*-algebras. Otherwise, the first application of the tensor product of continuous trace C^*-algebras will be on the dual crossed product of an abelian, locally compact group with a C^*-algebra. The second application is on the particular case of a crossed product of a locally compact groupoid with a bundle of C^*-algebras. Theorem Fulman-Muhly-

Williams ([6, Theorem1]) that establishes the conditions for the crossed product of a locally compact groupoid with a bundle of *C**-algebras to be a continuous trace *C**-algebra, has been used. In paragraph 3.2. of this paper we analyze the particular case when the bundle of *C**-algebras is a bundle with constant fiber and obtain, using the properties of tensor product of continuous trace *C**-algebras, in Theorem 4, a generalization of a result of Raeburn and Rosenberg (Remark 7).

2. Another proof for direct part of Tomyiama's theorem

Before discussing the proof of direct part of Tomyiama's theorem, some remarks have to be made.

Remark 1. From [1, Proposition 10, II.9] it follows that a continuous trace *C**-algebra is always a CCR-algebra, a *C**-algebra where for every irreducible representation π of *A* on a Hilbert space and for every element *x* from $A, \pi(x)$ is a compact operator.

Remark 2. An unique norm that makes $A \otimes B$ a *C**-algebra doesn't exist, in general, on a tensor product of two *C**-algebras *A* and *B*. In particular, according to [2, Corollary B44, p262] a *C**-algebra with Hausdorff spectrum is a nuclear *C**-algebra. This means that on a tensor product of *A* with any other *C**-algebra, it exists only one norm which gives the structure of *C**-algebra. By the Williams-Raeburn definition, a continuous trace *C**-algebra has Hausdorff spectrum, therefore, without any other explanation, on the tensor product where at least one factor is a continuous trace *C**-algebra there is only one *C**-norm on tensor product.

Remark 3. By Remark 1 and the Williams-Raeburn definition, every continuous trace *C**-algebra is a CCR algebra and has Hausdorff spectrum. This means that two continuous trace *C**-algebras, *A* and *B*, satisfy the hypothesis from [2, Theorem B.37, p. 256], therefore the spectrum of the tensor product $A \otimes B$ is homeomorphic to the cartesian product of the spectra of *A* and *B*, respectively.

PROPOSITION 1. *If A and B are two continuous trace C*-algebras, then the tensor product* $A \otimes B$ *is a continuous trace C*-algebra.*

Proof. To show that $A \otimes B$ is a continuous trace *C**-algebra we directly check the conditions from Williams-Raeburn definition, so we have to prove that the spectrum of $A \otimes B$ is a Hausdorff spectrum and if π_0 is an element of the spectrum of the tensor product $A \otimes B$, there are a positive element of $A \otimes B$,

$y \in (A \otimes B)^+$ and a neighborhood V of π_0, such that $\pi(y)$ is a rank one projector for every $\pi \in V$.

By Remark 3, the spectrum of $A \otimes B$ is homeomorphic to the cartesian product of the spectra of C*-algebras A and B. But A and B are continuous trace C*-algebras, therefore the spectra of A and B are Hausdorff. It follows that the cartesian product $\hat{A} \times \hat{B}$ is also Hausdorff, and the first condition of Williams-Raeburn definition is verified.

Let $(\pi_0, \rho_0) \in \hat{A} \times \hat{B}$ with the unique correspondent the representation $\pi_0 \otimes \rho_0$ from the spectrum of $A \otimes B$. Because A and B are continuous trace C*-algebras, for $\pi_0 \in \hat{A}$ there are $a \in A^+$ and a neighborhood V of π_0 such that $\pi(a)$ is a rank one projector, for every π in V, and similarily for $\rho_0 \in \hat{B}$, there are $b \in B^+$ and a neighborhood W of ρ_0 such that $\rho(b)$ is a rank one projector, for every ρ in W. We will show that $(\pi \otimes \rho)(a \otimes b)$ is a rank one projector in the neighborhood $V \times W$ of $\pi_0 \otimes \rho_0$, for π from the neighborhood V of π_0 and ρ from the neighborhood W of ρ_0.

Hence, using the usual properties of tensor product of two representations of C*-algebras we obtain:

$$(\pi \otimes \rho)^2 (a \otimes b) = (\pi \otimes \rho)(\pi(a) \otimes \rho(b)) = \pi^2(a) \otimes \rho^2(b) = \pi(a) \otimes \rho(b) =$$
$$= (\pi \otimes \rho)(a \otimes b); \quad \text{and}$$

$$[(\pi \otimes \rho)(a \otimes b)]^* = [\pi(a) \otimes \rho(b)]^* = \pi(a)^* \otimes \rho(b)^* = \pi(a) \otimes \rho(b) =$$
$$(\pi \otimes \rho)(a \otimes b).$$

It remains to show that the projector (above proved) $(\pi \otimes \rho)(a \otimes b)$ is rank one, if π and ρ have the range dimension one. For this, we will consider the vectors h_π, and, h_ρ, respectively, from the Hilbert spaces H_π and H_ρ ($\pi : A \to B(H_\pi), \rho : A \to B(H_\rho)$), vectors that generate the range of $\pi(a)$ and the range of $\rho(b)$. That means $\pi(a)h = \lambda h_\pi$, for every $h \in H_\pi$, λ scalar, and $\rho(b)k = \mu h_\rho$, for every $k \in H_\rho$, μ scalar. We will show that the vector $h_\pi \otimes h_\rho$ generates the range of $(\pi \otimes \rho)(a \otimes b)$:

$$[(\pi \otimes \rho)(a \otimes b)](h \otimes k) = [\pi(a) \otimes \rho(b)](h \otimes k) = (\pi(a)h) \otimes (\rho(b)k) =$$
$$= (\lambda h_\pi) \otimes (\mu h_\rho) = \lambda\mu(h_\pi \otimes h_\rho).$$

3. Applications at different kinds of crossed products

3.1. Applications at the dual crossed product of an abelian group and a *C**-algebra

According to [7, §3, p.194], the dual crossed product (or the iterated crossed product), denoted by $(A \times_{\alpha} G) \times_{\overline{\alpha}} \overline{G}$ can be constructed, starting from the dynamical system (A, G, α), containing the abelian, locally compact group *G*, the *C**-algebra *A* and the continuous homomorphism $\alpha : G \to Aut(A)$. This crossed product is corresponding to the dynamical system $(A \times_{\alpha} G, \overline{G}, \overline{\alpha})$, where $\overline{G}$ is the dual of the group *G* and $\overline{\alpha}$ is the dual action induced by α. Takai's theorem ([5, Theorem 3.4.]) establishes that $(A \times_{\alpha} G) \times_{\overline{\alpha}} \overline{G}$ is isomorphic to the tensor product $A \otimes K(L^2(G))$, where $K(L^2(G))$ is the *C**-algebra of compact operators on Hilbert space $L^2(G)$. However, the *C**-algebra of compact operators is the most simple example of continuous trace *C**-algebra. This will be proved in the following proposition, using the Williams-Raeburn definition.

PROPOSITION 2. *If* $K(H)$ *denotes the C*-algebra of compact operators on a Hilbert space H, then* $K(H)$ *is a continuous trace C*-algebra.*

Proof. From [2, A.2., Example A.15., p.210], every irreducible representation of $K(H)$ is equivalent to the identity representation $id : K(H) \to K(H)$, so that the spectrum of $K(H)$ contains only one element, $K(\hat{H}) = \{id\}$. Therefore, the part about the Hausdorff spectrum from the Williams-Raeburn definition will become trivial in this case. On the other hand, to check the second condition from the Williams- Raeburn definition, it is clear that the only one element of the spectrum $id : K(H) \to K(H)$ is a projector, and choosing, for *a* from the definition an operator of the form $T_{h,k}T^*_{h,k}$ (this kind of operator is positive) with $T_{h,k}(l) = h\langle l, k\rangle$ (this kind of operator is known to be compact), then $id(T_{h,k}T^*_{h,k})$ will be a rank one projector, because the operator $T_{h,k}$ is a rank one operator, its range being generated by element *h*.

PROPOSITION 3. *If G is an abelian, locally compact group, $\overline{G}$ its dual, (A,G,α) a dynamical system and A is a continuous trace C*-algebra, then the dual crossed product $(A\times_{\alpha} G)\times_{\overline{\alpha}} \overline{G}$ is a continuous trace C*-algebra.*

Proof. This proposition is a direct consequence of Takai's theorem and the Proposition 1. $K(L^2(G))$ is a continuous trace C*-algebra, and so is A, therefore $A\otimes K(L^2(G))$ will be a continuous trace C*-algebra. But $A\otimes K(L^2(G))$ is isomorphic to $(A\times_{\alpha} G)\times_{\overline{\alpha}} \overline{G}$, hence the dual crossed product will be with continuous trace.

Remark 4. According to Tomyiama's theorem, it is also true that, if the tensor product of two C*-algebras, $A\otimes B$, is a continuous trace C*-algebra, then both A and B are continuous trace C*-algebras. This means, for the dual crossed product, that $(A\times_{\alpha} G)\times_{\overline{\alpha}} \overline{G}$ is a continuous trace C*-algebra if and only if A is. In a few situations, the inverse statement of Proposition 3 can be proved.

To show the special cases of the inverse statement of Proposition 3, we need the following definitions :

Definition 1. (see [8, §3]) If G is a locally compact group, λ a Haar measure on G and $B(G)$ the Borel sets of G, a subfamily U of $B(G)$ is *dense* in $B(G)$ if for every set M from $B(G)$ and for every $\varepsilon > 0$, there is $A\in U$ such that $\lambda(M\cup A - M\cap A) < \varepsilon$. The lowest number, cardinal of a dense subfamily in $B(G)$, is called the *character* of G. The space $(G, B(G), \lambda)$ is separable in Kodaira-Kakutani sense if its character is smaller than or equal to $\aleph_0$.

Remark 5. The space $(G, B(G), \lambda)$ is separable in Kodaira-Kakutani sense if and only if $L^2(G)$ is separable (in classical sense).

The above remark is a short observation from [8, §3].

Definition 2. ([2, Definition 5.48., p.143]) A C*-algebra is *stable* if the tensor product $A\otimes K(H)$ is isomorphic to A, where $K(H)$ denotes the C*-algebra of compact operators on a separable, infinite dimensionally Hilbert space H.

PROPOSITION 4. *If G is an abelian, locally compact, separable in Kodaira-Kakutani sense group,* $\overline{G}$ *the dual of G,* (A, G, α) *a dynamical system and A a stable C*-algebra, then the following statements are equivalent :*

a) the dual crossed product $(A \times_{\alpha} G) \times_{\overline{\alpha}} \overline{G}$ *is a continuous trace C*-algebra ;*

b) A is a continuous trace C-algebra.*

Proof. $a) \Rightarrow b)$ G being separable in Kodaira-Kakutani sense implies that the Hilbert space $L^2(G)$ is separable according to Remark 5. From Takai's theorem, $(A \times_{\alpha} G) \times_{\overline{\alpha}} \overline{G}$ is isomorphic to $A \otimes K(L^2(G))$, but because A is stable, $A \otimes K(L^2(G))$ is isomorphic to A and the hypothesis that the dual crossed product is a continuous trace C*-algebra makes that A is a continuous trace C*-algebra.

$b) \Rightarrow a)$ This implication has been proved in Proposition 3.

Remark 6. Keeping the context of dual crossed product of a dynamical system (A, G, α) and the condition that A is a continuous trace C*-algebra, an analogue of Pontrjagin theorem ([9, 2.5., Theorem 12]) can be stated. If in the Pontrjagin theorem the dual of $\overline{G}$, where $\overline{G}$ is the dual of a locally compact, abelian group, is isomorphic to the initial group G, in our case the spectrum of dual crossed product is homeomorphic to the spectrum of initial C*-algebra. More precisely, the following proposition holds.

PROPOSITION 5. *If* (G, A, α) *is a dynamical system with G an abelian, locally compact group and A a continuous trace C*-algebra, then the spectrum of dual crossed product* $(A \times_{\alpha} G) \times_{\overline{\alpha}} \overline{G}$ *is homeomorphic to the spectrum of A.*

Proof. According to Takai's theorem, $(A \times_{\alpha} G) \times_{\overline{\alpha}} \overline{G}$ is isomorphic to $A \otimes K(L^2(G))$. From Remark 3, the spectrum of $A \otimes K(L^2(G))$ is homeomorphic to $\hat{A} \times K(L^2(G))\hat{}$. But $K(L^2(G))\hat{} = \{id_{K(L^2(G))}\}$, hence the spectrum of $A \otimes K(L^2(G))$ is homeomorphic to $\hat{A} \times \{id_{K(L^2(G))}\}$. $\hat{A} \times \{id_{K(L^2(G))}\}$ can be identified with $\hat{A}$ by application $(\pi, id_{K(L^2(G))}) \to \pi$

for every π in $\hat{A}$, therefore the spectrum of $A \otimes K(L^2(G))$ is homeomorphic to $\hat{A}$.

3.1. Applications on a special case of the Fulman-Muhly-Williams theorem

THEOREM 1 (Fulman-Muhly-Williams) *Let G be a second countable, locally compact groupoid with Haar system* $\{\lambda^u\}_{u \in G^{(0)}}$, *where* $G^{(0)}$ *is the unit space of G,* $\mathcal{A}$ *a bundle of C*-algebras over* $G^{(0)}$, $\sigma : G \to Iso(\mathcal{A})$ *a continuous homomorphism from G into the isomorphism groupoid of* $\mathcal{A}$, *in particular, for* $\gamma \in G$, *the image of* σ *in Iso(*$\mathcal{A}$*), denoted* σ_γ *will be a C*-isomorphism from* $A(s(\gamma))$ *to* $A(r(\gamma))$. *Under the hypothesis that C*-algebra* $C_0(G^{(0)}, \mathcal{A})$ *is a continuous trace C*-algebra and the action of G on X, the spectrum of* $C_0(G^{(0)}, \mathcal{A})$, *is free, the crossed product* $C^*(G, \mathcal{A})$ *is a continuous trace C*-algebra if and only if the action of G on X is proper.*

Note 1. The space *Iso*($\mathcal{A}$) denotes the following known space: *Iso*($\mathcal{A}$)=$\{(u,V,v) \mid V : A(v) \to A(u)$ isomorphism of C*-algebras, $u, v \in G^{(0)}\}$, and r and s from the above theorem are the usual range and the source maps of a groupoid, $r, s : G \to G^{(0)}$, $r(g) = gg^{-1}$ and $s(g) = g^{-1}g$.

The special case analyzed here is the case when the bundle $\mathcal{A}$ of C*-algebras, has a constant fiber, C*-algebra A. In this case, the space denoted *Iso*($\mathcal{A}$), will be the group $Aut(A)$ of automorphisms of A, and the triplet (A, G, σ) will be a C*-groupoid dynamical system. The crossed product, denoted $C^*(G, A)$, can be constructed starting from this triplet, using the same technique as in the general case of [6, Introduction]. The Fulman-Muhly-Williams theorem will be adapted in this way:

THEOREM 2 *If* (A, G, σ) *is a C*-groupoid dynamical system, where A is a C*-algebra, G a locally compact groupoid with a Haar system of measures and* $\sigma : G \to Aut(A)$ *a continuous homomorphism, C*-algebra* $C_0(G^{(0)}, A)$ *is a continuous trace C*-algebra and the action of G on the spectrum* $\hat{A}$ *of A is free, the following statements are equivalent:*

a) the crossed product $C^*(G, A)$ *is a continuous trace C*-algebra;*

b) the action of G on $\hat{A}$ *is proper.*

Using the following results, some comments can be stated about the *C**-algebra $C_0(G^{(0)}, A)$:

PROPOSITION 6 [2, Proposition B.16, p.244] *Let T be a locally compact Hausdorff space and A a C*-algebra. Then, there is an isomorphism* ϕ *of* $C_0(T)\otimes A$ *onto* $C_0(T, A)$ *such that* $\phi(f\otimes a)(t) = f(t)a$ *for* $a\in A$ *and* $f\in C_0(T)$.

THEOREM 3 [10, Theorem 2.15] *If A and B are Morita equivalent C*-algebras, then A has continuous trace if and only if B has continuous trace.*

The unit space $G^{(0)}$ of a locally compact groupoid, is a closed subspace of *G*, and this implies that $G^{(0)}$ is a Hausdorff space. According to Proposition 6, the *C**-algebra $C_0(G^{(0)}, A)$ can be identified with the tensor product $C_0(G^{(0)})\otimes A$. Because $G^{(0)}$ is a Hausdorff space, $C_0(G^{(0)})$ is another classical example of a continuous trace *C**-algebra. For proof, [2, Example 5.18, p. 123] can be used, where it is proved that $C_0(T, K(H))$ is a continuous trace *C**-algebra if *T* is Hausdorff and is Morita equivalent with $C_0(T)$. Using Theorem 3, $C_0(T)$ will be a continuous trace *C**-algebra. With this and Proposition 1, we obtain the following result:

THEOREM 4. *If* (A, G, σ) *is a C*-groupoid dynamical system, where A is a C*-algebra, G a locally compact groupoid with a Haar system of measures and* $\sigma: G\to Aut(A)$ *a continuous homomorphism, C*-algebra A is a continuous trace C*-algebra and the action of G on the spectrum* $\hat{A}$ *of A is free, the following statements are equivalent:*

a) the crossed product $C^*(G, A)$ *is a continuous trace C*-algebra;*

b) the action of G on $\hat{A}$ *is proper.*

Remark 7. The above result extends a similar result from [11, Theorem 1.1.,(3)], obtained by Raeburn and Rosenberg in the particular case when the groupoid *G* is a group.

6. Conclusions

In this paper, we have studied the tensor product of continuous trace C^*-algebras. We presented in Proposition 1 a different proof for direct part of Tomyiama's theorem. In Section 3, we have analyzed posibilities to apply the properties of tensor product to different cases of crossed products. We have analyzed in 3.1. the case of dual crossed product and gave conditions in which this crossed product is a continuous trace C^*-algebra (Proposition 3 and Proposition 4). In the same context of dual crossed product, we have obtained in Proposition 5 a result similar with Pontrjagin theorem for locally compact, abelian group. In paragraph 3.2, we have analyzed the particular case of crossed product of a groupoid with a bundle of C^*-algebras and obtained in Theorem 4 an extension of a result of Raeburn and Rosenberg.

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