PIECEWISE LINEAR APPROXIMATION OF LOGARITHMIC IMAGE PROCESSING MODELS FOR DYNAMIC RANGE ENHANCEMENT

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Modelele logaritmice de prelucrare a imaginilor oferă un cadru potrivit pentru vizualizarea şi îmbunătăţirea unei game variate de imagini digitale. Deşi iniţial aceste modele au fost dezvoltate pentru sisteme în care fenomenele fizice de bază sunt multiplicative, încadrarea ulterioară în structuri algebrice a permis diversificarea aplicaţiilor. În consecinţă, ne putem pune problema construcţiei matematice a unor modele neliniare care să ofere un cadru practic pentru aplicaţii specifice din domeniul prelucrării de imagini. În această lucrare vom deriva un set de condiţii suficiente pentru elaborarea unor asemenea modele care să aibă o structură algebrică de spaţiu vectorial. Pe baza acestora vom construi modele noi, liniare pe porţiuni, care, în plus, să reducă efortul computaţional necesar implementării directe a modelelor neliniare. În final, vom demonstra utilitatea practică a formalismului matematic dezvoltat prin descrierea unei aplicaţii simple de creştere a gamei dinamice a imaginilor achiziţionate cu camere fotografice digitale.

It has been proven that Logarithmic Image Processing (LIP) models provide a suitable framework for visualizing and enhancing digital images acquired by various sources. The underlying initial reason for derivation of such models has been the necessity to deal with multiplicative phenomena. Later, it has been proven that LIP models have a precise mathematical structure and, hence, are suitable for various image processing applications, not necessarily of multiplicative nature. In this paper, we investigate, from a mathematical point of view, the set of sufficient conditions to derive such a non-linear image processing model that complies with the algebraic structure of a vector space. Given this set of conditions, we build new models, that are piecewise linear and reduce the intense computational effort required by the classical models. Finally, we prove the usability of the developed theory by proposing a simple and practical application of digital still camera dynamic range enhancement.

Keywords: non-liniar image processing model, algebraic structure, piecewise liniarization, dynamic range enhancement.

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1. Introduction

In most imaginable circumstances, digital images are obtained by means implying machines with finite power supply; hence digital images are defined over a finite range of values. The image processing algorithms, traditionally, rely on classical real operations for implementations. Under certain circumstances, such a combination, named Classical Linear Image Processing (CLIP) – [1] proves its limitations. For instance, let us mention the upper range overflow, which is brutally solved by truncation. Consequently, more elaborate structures appeared, such as the logarithmic image processing (LIP) models.

The starting point of the logarithmic image processing models lies in the homomorphic theory introduced by Oppenheim [2]. Implementations of the LIP models have been given, to our best knowledge, by Jourlin and Pinoli [3] and respectively by Pătraşcu [4]. Lately, the scheme of a new pseudo-logarithmic model has been proposed by Vertan et al, [5], [6]. Using these models, various applications have been developed: illumination correction [4], contrast enhancement [7], color image enhancement [4], histogram equalization [8], dynamic range enhancement [6], edge detection [5], etc.

The first derivation of such a model, as proposed by Jourlin and Pinoli, has been developed for the case of transmitted light. The mathematical construction begins by defining the addition of two elements according to with the equivalent of a cascade of two initial transparent environments; the multiplication is derived by induction from repeated addition; the consequent properties arise naturally. Unlike his predecessors, Pătraşcu derived its model from a mathematical point of view by enforcing some defined properties to the basic laws (addition and scalar multiplication).

The mathematical construction of such a non-linear model may start by defining the operational laws (the addition and the scalar multiplication) or, equivalently, by determination of a function that maps the investigated model definition set onto the real number algebraic structure. We will focus on the second alternative and we will investigate the restrictions that have to be imposed to the mapping (generative function), such that the new model obeys some practical properties and, thus, leads to a consistent mathematical form.

Accordingly, we shall structure the remaining of the document as follows: we shall discuss the mathematical background of the problem in order to define the set of rules that guaranties the needed structure. We shall continue by considering an example of generative function, which being piecewise linear is simpler to implement. The described formalism is evaluated in the framework of high dynamic range image enhancement. The paper ends with a summary and a discussion on further development.

2. Mathematical background

In order to have practical usage, it is of common sense to impose some properties to any newly determined model. To be more precise, we shall investigate the nature of the definition set, the means of laws determination, the closing properties and we shall discuss the requirements to form a vector space. The mathematical formulation, described in this section, is a particular case of the more general homomorphic theory, particularized to discrete images.

2.1. Defining the set and laws

Let us consider a function, $\varphi : E \to F$. Within this choice, the set E is the image definition set. Typically, if the image values are intensities, like any plane in RGB color representation, the set E has the form $[0,M)$; in the case of YUV (YCbCr) space, for the color differences channels, the set E has a symmetrical form, like $\left(-M/2;+M/2\right)$. Thus, in any circumstance, the set is bounded:

$$\exists m_E = \inf(E), \exists M_E = \sup(E) \tag{1}$$

The function φ defines the model structure and maps the image definition set, E, onto a subset of real numbers, F.

Furthermore, we shall add two operations to the given set, E: addition of two elements of the set, $\oplus$, and multiplication with an outer scalar, $\otimes$. Given a scalar, $\alpha \in K \subseteq R$, and two elements of the set, u and v, we can determine the exact formulas for the mentioned operations using the generative function φ:

$$\varphi(u \oplus v) = \varphi(u) + \varphi(v), \forall u, v \in E \tag{2}$$

$$\varphi(\alpha \otimes u) = \alpha \cdot \varphi(u), \forall u \in E \tag{3}$$

Equations (2) and (3) are the conditions that must be fulfilled by a homomorphism between two similar algebraic structures. In our approach we assume the function φ to be known and we intend to use the mentioned relations to determine the analytic form of the addition and scalar multiplication laws; in such a case, the simplest solution is achieved when the function is a bijection and the laws are uniquely determined. The bijectivity implies surjectivity, which is deemed for solution existence, and injectivity - required for solution uniqueness.

Now let us analyze equation (3). While u may be any element of the finite input set E, α is, typically, a real positive scalar ($K = \Re^+$). Under the assumed bijectivity condition, $\exists u_m, \varphi(u_m) = \inf(F)$ and respectively, $\exists u_M, \varphi(u_M) = \sup(F)$.

If one writes down relation (3) for u_m, one will find that the resulting domain must include any upper vicinity of 0 and, therefore, $\inf(F)=0$; similarly, for supra-unitary constant one will find that $\sup(F)=+\infty$. Because image amplification and attenuation are common applications, we must enforce $F=[0,+\infty)$.

With respect to the bijectivity constraint (and, hence, the existence of φ^{-1}), the definition laws are determined by:

$$\begin{aligned} u \oplus v &= \varphi^{-1}\left(\varphi(u)+\varphi(v)\right) \\ \alpha \otimes u &= \varphi^{-1}\left(\alpha \cdot \varphi(u)\right) \end{aligned} \quad (4)$$

2.2. Closing property

The closing property of both addition and scalar multiplication is of paramount practical importance since the sum of any two images should lead to another valid image and, respectively, any amplified or attenuated image should be an image. Formally, one may write:

$$\begin{aligned} &\forall u, v \in E, z = u \oplus v \Rightarrow z \in E \\ &\forall u \in E, \forall \alpha \in K, z = \alpha \otimes v \Rightarrow z \in E \end{aligned} \quad (5)$$

These properties hold under the assumed bijectivity hypothesis since:

$$z = \varphi^{-1}\left(\varphi(u)+\varphi(v)\right) \; and \; \forall x \in F, \varphi^{-1}(x) \in E \Rightarrow z \in E \quad (6)$$

2.3. Vector space structure

Given the two operative laws, $\oplus, \otimes$, the vector set E and the outer scalar set K, the formal definition of the vector space implies several properties. Addition law must be associative, commutative, must have identity element and inverse element. Distributivity must hold for scalar multiplication over vector addition and for scalar multiplication in the field of scalars. Scalar multiplication must have identity element and to be compatible with multiplication in the field of scalars.

The commutative, associative and distributive properties of the implied laws are important because the order of operations should not matter in a weighted sum of images. Under the assumed hypothesis of bijective application and because the φ^{-1} function maps the $\Re$ structure to the given set, these properties are verified.

The existence of the identity element, u_0, with respect to the addition, implies further conditions over the mapping function, φ. The condition is a consequence of the isomorphic behavior:

$$\forall u \in E, \exists u_0 \Rightarrow u \oplus u_0 = u \big| \varphi(\cdot) \Rightarrow \varphi(u_0) = 0 \tag{7}$$

The inverse element, u^-, for asymmetrical spaces, like RGB, has rather impractical application: "given an intensity value, the inverse element is something that perfectly absorbs the light!?". It is more important in defining the subtraction of one image from another to be consistent with the addition. However, the inverse element makes sense if we discuss about symmetrical color spaces, like U and V planes from YUV. In such a case, the mapping function must take values in a symmetrical interval:

$$\forall u \in E, \exists u^- \Rightarrow u \oplus u^- = u_0 \big| \varphi(\cdot) \Rightarrow \varphi(u) + \varphi(u^-) = 0 \Rightarrow \varphi(u^-) = -\varphi(u) \tag{8}$$

Similarly, the identity element of the scalar multiplication has to be 1:

$$\forall u \in E - \{u_0\}, \exists \alpha_1 \Rightarrow \alpha_1 \otimes u = u \big| \varphi(\cdot) \Rightarrow \alpha_1 = 1 \tag{9}$$

In conclusion, the sufficient conditions that a mapping function, φ, has to fulfill in order to generate a usable non-linear image processing model are:

- The target, should by $F = \Re$. If we use an intensity based image representation, then the inverse element in the additive law does not exists and the subtraction cannot be defined; hence F becomes $\Re^+$.
- φ should be bijective; this implies continuity by the nature of the problem;
- $\varphi(u_0)=0$: commonly the "black image" is represented by $u_0=0$, and therefore: $\varphi(0)=0$.

3. Non-linear image processing models

To the existing date, three non-linear image processing models have been developed. The first two, as mentioned, are logarithmic and have been proposed by Jourlin and Pinoli, [3], and respectively Pătraşcu, [4]. The generative functions for these cases are:

$$\varphi : [0, M] \rightarrow (-\infty, 0), \ \varphi(x) = -\log(1 - x) \tag{10}$$

for the Jourlin and Pinoli model and, respectively, for the Pătraşcu model:

$$\varphi : (-1,1) \to (-\infty,+\infty),\ \varphi(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \tag{11}$$

Obviously, the imposed conditions are verified by both functions; still the Jourlin and Pinoli model does not comply with the $\Re$ target domain and, therefore, its additive law has not an inverse element; thus, it has a cone structure. By comparison, the Pătraşcu model introduces a vector space. Considering the fact that the generative functions are logarithmic, the models are known as logarithmic image processing (LIP) models.

The third model, proposed by Vertan et al. [5], [6], has been called pseudo-logarithmic image model and its mathematical structure was not fully investigated yet. In this case, the generative function is:

$$\varphi : [0,1) \to [0,+\infty),\ \varphi(x) = \frac{x}{1-x} \tag{12}$$

Taking into account that the generative function is a bijection, with values on $F = [0,+\infty)$ and *φ(0)=0,* the model fulfills all the properties of a cone space structure. The extension to a vector space structure is achieved by the use of the following generative function:

$$\varphi : (-1,1) \to (-\infty,+\infty),\ \varphi(x) = \frac{x}{1-|x|} \tag{13}$$

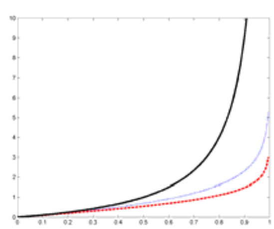

Fig 1a. The generative functions, in the positive range, for Jourlin and Pinoli model (dotted line), Pătraşcu model (dashed line) and Vertan model (solid line)

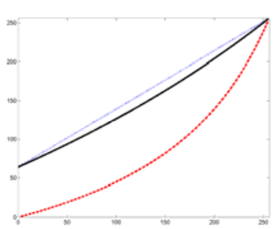

Fig. 1b. Example of addition of element 64 with all the other possible integer values in the range [0,255], using the model: Jourlin and Pinoli (dotted line), Pătraşcu (dashed line) and Vertan (solid line)

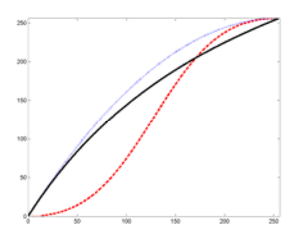

Fig. 1c. Example of scalar multiplication of all the elements in [0,255] range with 2. The operation has been performed with the model: Jourlin and Pinoli (dotted line), Pătraşcu (dashed line) and Vertan (solid line)

The plots of the three mentioned generative functions are shown in figure (1a). Figures (1b) and (1c) show examples of addition and scalar multiplication. As one can notice, all of the models share the same behavior in the positive range. If the Jourlin and Pinoli model and, respectively, Vertan model are extended by odd symmetry to the negative range, then all the models become similar.

4. Piecewise linear approximation of non-linear models

The exercise developed in section 2, beyond pointing to a short-cut to the analysis of known models, gives the user the flexibility to choose the generative function according to his application particularities. In the current work we shall investigate a general problem. All of the known models share the same complex behavior, which leads to the practical problem of lack of efficiency in implementation. Hence, it make sense to try to build a piecewise linear model. Thus, we shall choose a piecewise linear generative function that complies with the rules determined in section 2 and we derive the remaining of the model later.

For simplicity of explanation let us consider a generative function composed of 2 segments. The general approach (*n* segments) can be developed straight-forward. Such a 2 segment function and its inverse have the form:

$$\varphi(x)=\begin{cases} a_1 x\,, x\in[0,x_0) \\ a_2 x + b_2\,, x\in[x_0,M) \\ +\infty,\ x=M \end{cases} \quad , \ \varphi^{-1}(y)=\begin{cases} y/a_1\,, y\in[0,y_0) \\ (y-b_2)/a_2\,, y\in[y_0,y_M) \\ M,\ y=y_M \end{cases} \qquad (14)$$

In the equations above, the offset constants are determined from continuity constraints.

In order to determine the operational laws, one will replace the generative and inverse generative function formulas (equations (14)) in equation (6).

The proposed model is function of the $\{a_1, a_2, x_0\}$ parameters. To determine their values, one may choose a non-linear model as target and perform parameter regression. But in section 2 we showed that it is no need to do that, because the known models span just a little part of the valid functions range. We have some degree of freedom in choosing the parameter set according to the envisaged application. The current exercise aims to efficient implementation. We consider the choice of a_1 and a_2 as power of 2 as being more important, such that the model implementation uses bit shift instead of the expensive multiplication and, especially, division. Under such an approach, the abscissa breaking point, x_0, is a free parameter and is to be found after minimizing the mean squared error (or other similar criteria) in respect to a target model.

Two possible generative functions, obtained after regression from the Vertan model (for 2 and 4 segments), are shown in figure 2.

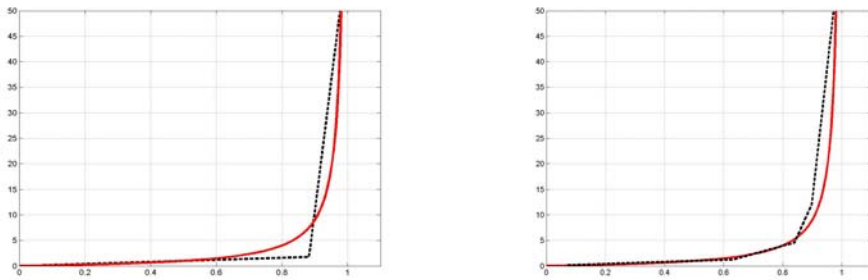

Fig 2. The generative functions for the original non-linear model (Vertan – solid line) and piecewise linear approximations: left - 2 segment and right 4 segment approximation.

5. Dynamic range enhancement

The typical consumer digital still camera outputs images in the target range of [0,255]. By comparison, the human eye is capable of comprising unitary scenes with thousands of different levels. Therefore, there is need for methods to enlarge the dynamic range of digital camera acquired images. Due to the fact that the problem is known, the literature stores many proposed solutions, [9].

The typical approach is to acquire several images of the scene, with different exposures, and to combine them in a high dynamic range (HDR) resulting image. The different exposure ensures that different parts of the gamut are recorded correctly by different images. The combination, which is done by summation, preserves the information variation existing in the input frames. Lately it has been shown that logarithmic (and by extension all non-linear models), provide a better solution to the dynamic range problem than classical real operations. This solution was called log-bracketing [10] and it will be revised, in the light of the proposed approximation in the following sections.

Let us consider, as input data, a set of frames (e.g. 3), f_1, f_2, f_3 acquired with different exposures: $Ev_1=-1$, $Ev_2=0$, $Ev_3=+1$. Ev is a logarithmic measure for relative exposure; $Ev=0$ is given to a picture, where the exposure time and aperture balance the scene illumination and internal camera amplification in order to have a near uniform resulting histogram. An image with $Ev=-1$ is acquired with half of the normal exposure time and, therefore, is underexposed. The $Ev=+1$ is obtained for an image with double of exposure time and, thus, is overexposed.

Each such image correctly records one part of the gamut and is less accurate elsewhere: the underexposed image records correctly the upper part, while the lower one is degraded by quantization error and noise; the normal image is accurate in the center part, while the overexposed image is accurate in the lower part, while the upper range is degraded by near saturation. This information may be encoded by a set of weights, μ. A formal derivation of these weights may be found in [10]. An example of such weights is presented in figure 3.

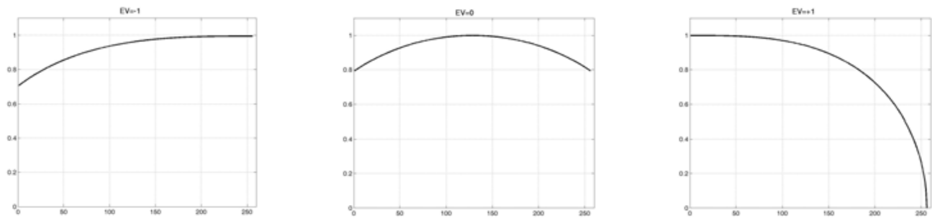

Fig. 3. The weights that encode accuracy of information in an underexposed image (*Ev=-1*, left hand plot), normal image (*Ev=0*, center plot) and overexposed image (*Ev=+1*, right hand plot). The weights correspond to a consumer digital still camera

The mixing of the input frames takes the form of a convex combination:

$$f_{HDR}(l,m) = \frac{\sum_{i=1}^{N} \oplus \left(\mu\left(Ev(i), f_i(l,m)\right) \otimes f_i(l,m)\right)}{\sum_{i=1}^{N} \mu\left(Ev(i), f_i(l,m)\right)} \qquad (15)$$

Fig. 4. Digital still camera acquisition of a natural scene. From left to right: underexposed, normal and overexposed image. All images are 24 bits per pixel (bpp) color images.

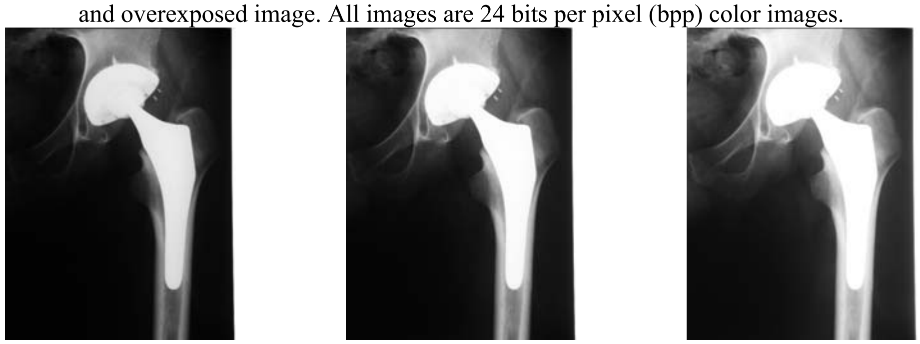

Fig. 5. Digital still camera acquisition of radiographic film image of total hip prostheses (from left to right): underexposed, normal and overexposed image. All images are 8 bpp luminance images of the original 12 bpp equivalent film.

In the equation (15) the right hand multiplications and additions (implied by the sum) are performed in non-linear manner.

6. Results

An example of input images acquired under different exposures, are shown in figures 4 and 5. The resulting HDR images, computed according to the Jourlin, Vertan, and respectively piecewise-linear model (that follows Vertan model) are presented in figures 6 and 7.

In order to evaluate the results, two methods have been proposed: subjective evaluation and objective evaluation. The subjective evaluation derives from the methods proposed in the television standard defined by recommendation ITU-R BT. 500-11.

Given the specific requirements of the tested application (HDR enhancement), we claim that pixel values entropy can be used as objective measure because a HDR image must allocate comparable portions of the visible range to all objects; in such a case, the information (which is measured by entropy) should be maximum. Indeed, as we expect, the HDR image exhibits the maximal range of values, their distribution resembling a uniform one. We measure the entropy effectiveness as:

$$\eta = \frac{H(f)}{H_{\max}}, H(f) = -\sum_{i=0}^{M-1} h_i \log h_i \quad (16)$$

where h_i is the luminance histogram value corresponding to gray level i and M is the maximum number of different pixel values. The maximum value of the entropy $H_{\max}$ is the number of bits per pixel (which is 8 for natural images and 12 for radiographic ones).

Fig. 6. The results for high dynamic range enhancement of the natural scene using (images from left to right) : the Jourlin model, the Vertan, piecewise linear model with 2 segments, the piecewise linear model with 4 segments

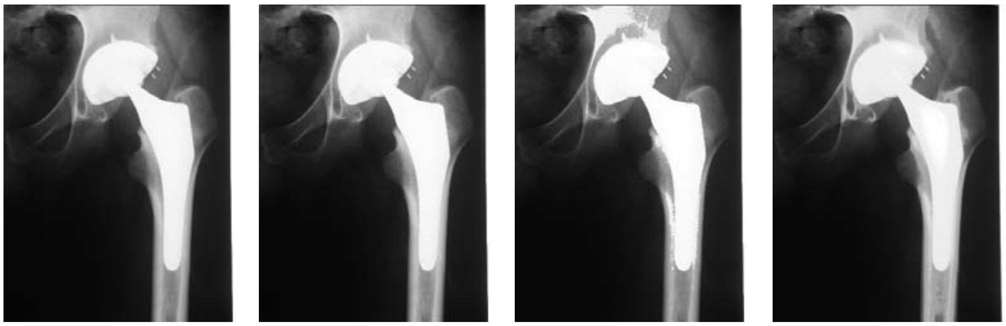

Fig. 7. HDR enhancement for the medical image using (images from left to right): the Jourlin model, the Vertan model, piecewise linear model with 2 segments , piecewise linear model with 4 segments - right hand image

The subjective evaluation was performed by a panel of expert and non-expert observers that graded the quality of obtained HDR images from worst quality (0) to the best quality (5). Table 1 shows the mean opinion score (MOS) and the entropy effectiveness obtained for the proposed experiments.

Table 1

Evaluation measures of HDR images

Image set	Jourlin		Vertan		Linear – 2		Linear – 4	
	MOS	η [%]	MOS	η [%]	MOS	η [%]	MOS	η [%]
Natural-color	4.75	91.2	4.62	89.2	3.75	81.6	4.37	82.8
Medical-gray	4.75	87.7	4.75	87.5	4.25	84.5	4.5	85.5

We skip from evaluation the Pătraşcu model because it is the only symmetrical one. The piecewise linear model followed an asymmetrical behavior (the Vertan model), as required for HDR imaging [10].

The first observation is that entropy, which is a measure of the uniformity of the image histogram, is consistent with subjective evaluation. The next observation is that the Jourlin model is the most appropriate for the current experiment followed closely by the Vertan model. However, piecewise linear approximation also proves its utility because it leads to similar results. Another observation is that a higher number of pieces used in approximation provides more accurate results. Hence, the piecewise approximation exhibits accuracy of results and due to efficient implementation is suitable for practical applications.

7. Conclusions and further work

In this work we investigated the conditions under which a given mapping generates a valid non-linear image processing model. Once the conditions found, we have shown as an argument that all currently proposed models fulfill the mentioned conditions. For any such model, we propose a piecewise linear approximation as an effective computational speedup.

In the second part of the paper, a simple application of dynamic range enhancement of digital still camera has been presented. Using objective and subjective measurements we showed the usability of the proposed approximation.

There are two directions for continuation of the current work. One is related to algebra and it refers to further investigation of the necessary (minimum) conditions that an application has to obey in order to derive a valid image processing model. The other direction refers to more practical issue. The formalism presented in section 2 cleared the path to new non-linear models, even to parametric ones. It is of maximum interest to use such parametric models to the known applications and to investigate their optimization with respect to application-specific objective measures.

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