AN INVESTIGATION ON DERIVATIONS AND HYPERATOM ELEMENTS OF ORDERED SEMIHYPERRINGS

by Yongsheng Rao[1], Qixin Zhou[2], Ruiqi Cai[3], Ya Li[4] and Saber Omidi[5]

In this paper, we continue the study of derivations on an ordered semihyperring $(R, \oplus, \odot, \leq)$. We show that if d is a nonzero derivation on an ordered semihyperring R with no zero divisors and K is a proper right hyperideal of R, then $d(x) \neq 0$ for some $x \in K$. Moreover, we prove that if K is a proper hyperideal of a prime ordered semihyperring R and d is a derivation on R such that $d(m) = 0$ for all $m \in K$, then $d(q) = 0$ for all $q \in R$. Finally, we use the hyperatom elements of an ordered semihyperring to provide an ordered semiring. For an injective strong derivation d of an ordered semihyperring R associated to a d-strongly regular relation such that for every $(u, v) \in \sigma$, v is a hyperatom, there exists a derivation on R/σ.

Keywords: ordered semihyperring; subsemihyperring; derivation; hyperatom element.
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1. Introduction

In 1934, Marty [1] defined the concept of a hypergroup. The notion of semihyperring was proposed by Vougiouklis [2] in 1990. Some applications concerning hyperstructure theory can be found in [3, 4, 5]. Many properties of hyperatom elements were extended to hyper BCK-algebras in [6, 7].

In 2013, Asokkumar [8] introduced derivations for Krasner hyperrings. In [9], Kamali and Davvaz attempted to study differential hyperrings. A study on derivations in prime rings is done in [10]. As one may find, derivations have many applications in coding theory [11, 12].

The concept of ordered semihypergroup is now well-studied in hyperstructure theory in [13, 14, 15, 16, 17] since its introduction in 2011 by Heidari and Davvaz [18]. Some recent studies on ordered semihyperrings are on derivations and pure hyperideals done by Rao et al. in [19, 20] and Shao et al. in [21].

Previous studies on the derivations of ordered semihyperrings [19, 22] and hyperatom elements [6, 7] motivated us to continue the study of the derivation of an ordered semihyperring $(R, \oplus, \odot, \leq)$.

In this note, using the notion of hyperatom element, we establish the relationship between ordered semihyperrings and ordered semirings. Our purpose in the future is to study derivation of fuzzy (prime) hyperideals and fuzzy hyperfilters of ordered semihyperrings.

[1] Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China, e-mail: rysheng@gzhu.edu.cn
[2] Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China, e-mail: 2111906113@e.gzhu.edu.cn
[3] Corresponding author, Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China, e-mail: cairic@e.gzhu.edu.cn
[4] Institute of Computing Science and Technology, Guangzhou University, Guangzhou 510006, China, e-mail: liya@gzhu.edu.cn
[5] Ministry of Education Iran, Department of Education in Tehran, Tehran, Iran, e-mail: omidi.saber@yahoo.com

Definition 1.1. [20] *Take a semihyperring* $(R, \oplus, \odot)$ *and a partial order relation* $\leq$. *Then* $(R, \oplus, \odot, \leq)$ *is called an ordered semihyperring if for any* $q, q', a \in R$,

$$q \leq q' \Rightarrow \begin{cases} q \oplus a \preceq q' \oplus a, \\ q \odot a \preceq q' \odot a, \\ a \odot q \preceq a \odot q'. \end{cases}$$

For every $\emptyset \neq X, Y \subseteq R$, $X \preceq Y$ *is defined by* $\forall x \in X, \exists y \in Y$ *such that* $x \leq y$.

Clearly, $X \subseteq Y$ implies $X \preceq Y$, but the converse is not valid in general.

We will say that $\emptyset \neq K \subseteq R$ is a *left* (resp. *right*) *hyperideal* of R if

(1) for all $a, b \in K$, $a \oplus b \subseteq K$;
(2) $R \odot K \subseteq K$ (resp. $K \odot R \subseteq K$);
(3) $(K] \subseteq K$.

The set $(K]$ is given by

$$(K] := \{r \in R \mid r \leq x \text{ for some } x \in K\}.$$

Let $\emptyset \neq U \subseteq R$. If V is a hyperideal of R and $U \preceq V$, then $U \subseteq V$. In particular, $U \preceq \{0\}$ implies $U = \{0\}$. Indeed: Take any $u \in U$. By hypothesis, there exists $v \in V$ such that $u \leq v$. So, $u \in (V] \subseteq V$ and hence $U \subseteq V$.

$(R, \oplus, \odot, \leq)$ is said to be *prime* if $u \odot R \odot v = 0$ implies $u = 0$ or $v = 0$ for all $u, v \in R$.

A *derivation* [19] on $(R, \oplus, \odot, \leq)$ is a function $d : R \to R$ such that for all $u, v \in R$,

(1) $d(u \oplus v) \subseteq d(u) \oplus d(v)$;
(2) $d(u \odot v) \in d(u) \odot v \oplus u \odot d(v)$;
(3) $u \leq v$ implies $d(u) \leq d(v)$.

2. Main Results

First of all, we study several properties of derivations by explicit examples. Then some results on hyperatom elements in ordered semihyperrings are established.

Theorem 2.1. *Let* d *be a nonzero derivation on an ordered semihyperring* $(R, \oplus, \odot, \leq)$ *with no zero divisors. If* K *is a proper right hyperideal of* R, *then* $d(x) \neq 0$ *for some* $x \in K$.

Proof. Let $d(x) = 0$ for all $x \in K$. For every $q \in R$, we have $x \odot q \in K$. So, $d(x \odot q) = 0$. Now,

$$\begin{aligned} d(x \odot q) &\in d(x) \odot q \oplus x \odot d(q) \\ &= 0 \odot q \oplus x \odot d(q) \\ &= 0 \oplus x \odot d(q) \\ &= x \odot d(q) \end{aligned}$$

It follows that $x \odot d(q) = 0$ for every $q \in R$. By hypothesis, $d(q) = 0$ for all $q \in R$, which is a contradiction. Therefore, d is a nonzero derivation on K. □

Example 2.1. *Consider the ordered semihyperring* $(R, \oplus, \odot, \leq)$ *with symmetrical hyperoperation* $\oplus$*:*

$\oplus$	0	u	v	w
0	0	u	v	w
u	u	$\{u, v\}$	v	w
v	v	v	$\{0, v\}$	w
w	w	w	w	$\{0, w\}$

$\odot$	0	u	v	w
0	0	0	0	0
u	0	u	u	u
v	0	v	v	v
w	0	w	w	w

$$0 \leq u \leq v \leq w.$$

Clearly, the function $d : R \to R$ *defined by*

$$d(x) = \begin{cases} 0, & x = 0 \\ v, & x = u, v \\ w, & x = w \end{cases}$$

is a nonzero derivation on R*. Clearly,* $K = \{0, u, v\}$ *is a right hyperideal and* d *is a nonzero derivation on* K*.*

Theorem 2.2. *Let* K *be a proper hyperideal of a prime ordered semihyperring* $(R, \oplus, \odot, \leq)$*. If* d *is a derivation on* R *such that* $d(m) = 0$ *for all* $m \in K$*, then* $d(q) = 0$ *for all* $q \in R$*.*

Proof. Let $0 \neq m \in K$. By hypothesis, we have $d(m) = 0$. Since K is a hyperideal of R, we get $x \odot m \in K$, for every $x \in R$. Thus,

$$d(x \odot m) = 0.$$

So,

$$\begin{aligned} 0 &\in d(x) \odot m \oplus x \odot d(m) \\ &= d(x) \odot m \oplus x \odot 0 \\ &= d(x) \odot m \oplus 0 \\ &= d(x) \odot m \end{aligned}$$

Hence, $d(x) \odot m = 0$ for all $x \in R$. Thus,

$$d(q \cdot q') \cdot m = 0$$

for any $q, q' \in R$. Now,

$$\begin{aligned} 0 &\in [d(q) \odot q' \oplus q \odot d(q')] \odot m \\ &= d(q) \odot q' \odot m \oplus q \odot d(q') \odot m \\ &= d(q) \odot q' \odot m \oplus q \odot 0 \\ &= d(q) \odot q' \odot m \oplus 0 \\ &= d(q) \odot q' \odot m. \end{aligned}$$

Thus $d(q) \odot q' \odot m = 0$. Since R is prime, we obtain $d(q) = 0$ for all $q \in R$. □

Proposition 2.1. *If $(R, \oplus, \odot, \leq)$ is an additively idempotent commutative ordered semihyperring and $x \in R$, then*

$$d_x(q) = q \odot x,$$

for all $q \in R$, is a derivation on R.

Proof. Fix an element $x \in R$. For every $q, q' \in R$, we have

$$\begin{aligned} d_x(q \oplus q') &= (q \oplus q') \odot x \\ &= q \odot x \oplus q' \odot x \\ &= d_x(q) \oplus d_x(q'). \end{aligned}$$

Also,

$$\begin{aligned} d_x(q \odot q') &= (q \odot q') \odot x \\ &\in (q \odot q') \odot x \oplus (q \odot q') \odot x \\ &= q \odot (q' \odot x) \oplus q \odot (q' \odot x) \\ &= q \odot (x \odot q') \oplus q \odot (q' \odot x) \\ &= (q \odot x) \odot q' \oplus q \odot (q' \odot x) \\ &= d_x(q) \odot q' \oplus q \odot d_x(q'). \end{aligned}$$

If $q \leq q'$ and $x \in R$, then $q \odot x \leq q' \odot x$. So, $d_x(q) \leq d_x(q')$. Therefore, $d_x(q)$ is a derivation on R. □

Definition 2.1. *We say that $m \in R$ is a hyperatom element provided that*

(1) *for any $a \in R$, $a \leq m$ implies $a = 0$ or $a = m$;*
(2) *$m \odot a \neq 0$ implies $(m \odot a) \odot q = a$ for some $q \in R$.*

Example 2.2. *In Example **2**.1, 0 and a are hyperatom elements.*

Proposition 2.2. *We denote by $A(R)$ the set if all hyperatom elements of $(R, \oplus, \odot, \leq)$. If for any $m, m' \in A(R)$, $m \oplus m' \preceq m$ and $m \odot m' \leq m$, then $A(R)$ is a subsemihyperring of R.*

Proof. Clearly, $0 \in A(R)$. Let $m, m' \in A(R)$. By hypothesis, $w \leq m$ for any $w \in m \oplus m'$. Since $m \in A(R)$, we get $w = 0$ or $w = m$. So,

$$m \oplus m' \subseteq \{0, m\}.$$

Hence,

$$m \oplus m' \subseteq A(R).$$

Similarly,

$$m \odot m' \in A(R).$$

If $q \in R$ and $q \leq m \in A(R)$, then $q = 0$ or $q = a$. Thus $q \in A(R)$ and so $(A(R)] \subseteq A(R)$. Therefore, $A(R)$ is a subsemihyperring of R. □

$(R, \oplus, \odot, \leq)$ is said to be a *convex* ordered semihyperring associated to a strongly regular relation σ if $(u, v) \in \sigma$ and $u \leq w \leq v$ implies that $(u, w) \in \sigma$. Now, we are able to make the connections of ordered semihyperrings with ordered semirings.

Theorem 2.3. *Let σ be a strongly regular relation on an ordered semihyperring $(R, \oplus, \odot, \leq)$. Denote by $[r]_\sigma$ the equivalence class of r. Consider the quotient semihyperring $R/\sigma = \{[r]_\sigma \mid r \in R\}$. If for every $(u, v) \in \sigma$, v is a hyperatom, then $(R/\sigma, \uplus, \otimes, \preceq)$ is an ordered semiring.*

Proof. Let $(u, v) \in \sigma$ and $u \leq w \leq v$. We assert that $(u, w) \in \sigma$. Since v is a hyperatom, we get $w = 0$ or $w = v$. Consider the following two situation.

Case 1. $w = 0$.
Since R is positive, we have $u = 0$. Thus $(u, w) = (0, 0) \in \sigma$ in this case.

Case 2. $w = v$.
By hypothesis, we obtain $(u, w) = (u, v) \in \sigma$.

Hence, R is a convex ordered semihyperring.

Clearly, $(R/\sigma, \uplus, \otimes, \preceq)$ is a preordered semiring, where for all $[m]_\sigma, [n]_\sigma \in R/\sigma$,

$$[m]_\sigma \uplus [n]_\sigma = \{[u]_\sigma \mid u \in m \oplus n\},$$

$$[m]_\sigma \otimes [n]_\sigma = [m \odot n]_\sigma,$$

$$[m]_\sigma \preceq [n]_\sigma \Leftrightarrow \forall m' \in [m]_\sigma, \exists n' \in [n]_\sigma \text{ such that } m' \leq n'.$$

Now, we assert that $\preceq$ is antisymmetric. Let $[m]_\sigma \preceq [n]_\sigma$ and $[n]_\sigma \preceq [m]_\sigma$ in R/σ. Take $m \in [m]_\sigma$; then there exists $w \in [n]_\sigma$ such that $m \leq w$. On the other hand, for this $w \in [n]_\sigma$ there exists $v \in [n]_\sigma$ such that $w \leq v$. So, $m \leq w \leq v$. Since R is a convex ordered semihyperring, we get $[m]_\sigma = [n]_\sigma$. □

Corollary 2.1. *Assume that σ is a d-strongly regular relation on $(R, \oplus, \odot, \leq)$ such that for every $(u, v) \in \sigma$, v is a hyperatom. If d is an injective strong derivation on R, then*

$$\varphi : R/\sigma \to R/\sigma$$

defined by

$$\varphi([m]_\sigma) = [d(m)]_\sigma$$

for all $[m]_\sigma \in R/\sigma$, is an injective strong derivation on $(R/\sigma, \uplus, \otimes, \preceq)$.

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