ACCURATE ELEMENT METHOD STRATEGY FOR FINDING QUASI-ANALYTIC SOLUTIONS OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Maty BLUMENFELD[1]

*Integrarea ecuațiilor diferențiale hiperbolice de ordinul unu cu derivate parțiale (PDE) având coeficienți variabili conduce la o ecuație integrală. Aceasta ecuație se integrează pe un domeniu $\mathscr{D}$ divizat în elemente dreptunghiulare prin înlocuirea funcției-soluție necunoscută cu o Funcție Concordantă (CF) [1,2,3] riguros adaptată la ecuația diferențială, reprezentată de un polinom de două variabile de grad mare având un număr mare de termeni. Prin integrare rezultă pe fiecare element câte o soluție cvasi-analitică [1]. Această soluție înlocuită în ecuația diferențială conduce – pentru elementul analizat – la o funcție reziduală care poate fi sintetizată prin valori medii patratice (R_{MS}). Se compară strategia de integrare uzual acceptată (bazată pe un număr mare de elemente de formă apropiată de pătrat) cu o stategie total diferită adaptată metodei de integrare dezvoltată în articol. Aceasta din urmă poate conduce la un număr redus de elemente dreptunghiulare la care (pentru un exemplu analizat) înalțimea elementului (H) este de opt ori mai mare decât baza (B). Raportul optim H/B se obține pe baza valorilor medii patratice (R_{MS}) ale funcției reziduale. Rezultatele foarte bune care se obțin sunt explicate prin legătura care există între utilizarea valorilor R_{MS} şi curbele caracteristice ce se pot trasa pe baza coeficienților variabili ai ecuației diferențiale. Valoarea calculată a funcției-soluție în punctul diametral opus originii dimeniului $\mathscr{D}$ se consideră **exactă dacă 7-8 cifre zecimale sunt riguros confirmate utilizând cel puțin două Funcții Concordante**. Astfel de rezultate se pot obține cu un număr relativ redus de elemente cu dimensiuni mari sau foarte mari.*

The integration of a first order hyperbolic partial differential equation (PDE) with variable coefficients leads to an integral equation. This last equation is integrated by the Accurate Element Method (AEM) on a rectangular domain $\mathscr{D}$ – divided in sub domains (elements) – replacing the unknown solution with a Concordant Function (CF). The CF is a high degree two variables polynomial with a great number of terms, rigorously fitted to the PDE. The integration leads to a quasi-analytic solution [1] valid on a single element. This solution is replaced in the PDE leading on each element to a residual function that can be synthesized by root mean square values (R_{MS}). The paper compares the usual integration strategy based on a great number of square elements with a totally different strategy developed by AEM. This last strategy leads to a small number of rectangular elements that can have (for a particular example) the height (H) eight times greater than the base (B).

[1] Professor Emeritus, Strength of Materials Department, University POLITEHNICA Bucharest, 2010, e-mail:mblum_2006@yahoo.com

The best shape of the elements is found by AEM using the root mean square value (R_{MS}) of the residual function. The very good (if not accurate) results are explained by the connection that exists between the AEM solution and the characteristic curves, which depend on the coefficients of each PDE. The computed value of the function at the corner opposite to the origin of the domain $\mathscr{D}$ (Target Value) is considered as ***accurate when 7-8 decimal digits are rigorously verified by using at least two CFs****. Such results can be obtained using a quite small number of elements with large or very large dimensions.*

1. Integration of PDEs using the Accurate Element Method (*AEM*)

1.1 Global and local coordinates

Suppose a rectangular domain $\mathscr{D}$ on which a linear hyperbolic Partial Differential Equation (*PDE*) has to be integrated. The parameters that describe the *PDE* are expressed in a *global coordinates system* ***X–T*** (Fig.1).

The Accurate Element Method (*AEM*) performs the integration of a *PDE* by dividing the domain $\mathscr{D}$ in a convenient number of rectangular elements. The approach is simplified if each element is analyzed by using a *local coordinate system x-t* (Fig.2). If **B** (Base) and **H** (Height) are the dimensions of the element, the coordinates of the four nodes are:

Node1($x_1=0,t_1=0$);***Node2***($x_2=B,t_2=0$);***Node3***($x_3=0,t_3=H$);***Node4***($x_4=B,t_4=H$) (1.1)

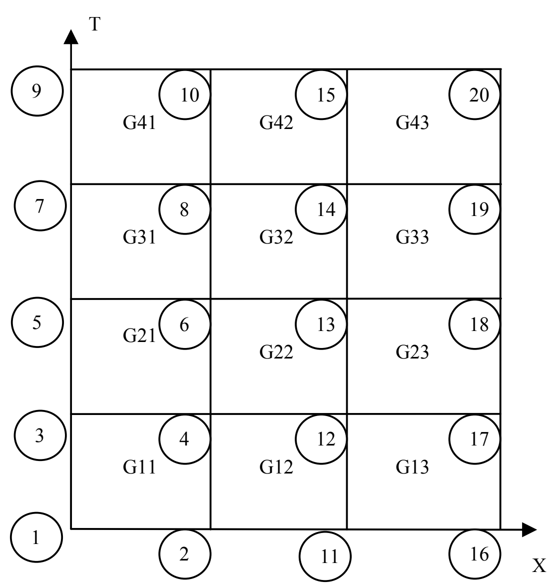

Fig.1

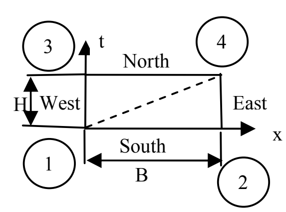

Fig.2

1.2 PDE in global coordinates

A *PDE* can be expressed in global coordinates *X–T* (Fig.1) as

$$M(X,T)\frac{\partial\phi}{\partial X}+N(X,T)\frac{\partial\phi}{\partial T}+P(X,T)\phi+Q_G(X,T)=0 \qquad (1.2)$$

where the coefficients[2] M(X,T), N(X,T), P(X,T) and the free term $Q_G(X,T)$ are two variable polynomials of *X* and *T*.

The case solved here is an *initial-boundary value problem* with known initial and boundary conditions represented by:

1. *Initial conditions* (T=0): $\Psi_G(X)=A_0+A_1X+A_2X^2+A_3X^3+A_4X^4+A_5X^5+...$ (1.3)
2. *Boundary* conditions (X=0): $\Omega_G(T)=B_0+B_1T+B_2T^2+B_3T^3+B_4T^4+B_5T^5...$ (1.4)

1.3 PDE in local coordinates system

The *PDE* (1.2) will be expressed in the *local system* as

$$\boxed{M(x,t)\frac{\partial\phi}{\partial x}+N(x,t)\frac{\partial\phi}{\partial t}+P(x,t)\phi+Q(x,t)=0} \qquad (1.5)$$

where the initial and boundary conditions are given by

Initial conditions (t=0): $\Psi(x)=\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3+\alpha_4x^4+\alpha_5x^5...$ (1.6)

Boundary conditions (x=0): $\Omega(t)=\beta_0+\beta_1t+\beta_2t^2+\beta_3t^3+\beta_4t^4+\beta_5t^5...$ (1.7)

The integral of (1.5) on the rectangular element *ELS* (Fig.2) is given by

$$\int_A\left(M(x,t)\frac{\partial\phi}{\partial x}+N(x,t)\frac{\partial\phi}{\partial t}+P(x,t)\phi\right)dA+\int_A Q(x,t)\,dA=0 \qquad (1.8)$$

where the area $A = B \times H$. This is an *integral equation*, the left side integrals *including the unknown two variables function* $\phi(x,t)$ or its derivatives. In order to perform these integrals *AEM* replaces $\phi(x,t)$ by a *Concordant Function* [1,2,3].

2. Concordant Functions

2.1 Concordant Function: a complete two variables polynomial

The *Concordant Function* (*CF*) – a concept introduced by *AEM* – is a *complete two variables polynomial*, namely it includes all the possible terms that correspond to a chosen degree: *1* constant term + *2* linear terms (*x,t*)+ *3* second degree terms (x^2,xt,t^2) and so an. The total number of terms NT for a complete function results from

$$NT=(G+1)(G+2)/2 \qquad (2.1)$$

where **G** represents the maximum degree of the polynomial function. For instance a five-degree Concordant Function (G=5) having **NT=21 terms,** noted as **CF5-21,**

[2] $M*N>0$

is given in the *local system* by[3]

$$\phi(x,t)=C_1+C_2\mathbf{x}+C_3\mathbf{t}+C_4\mathbf{x^2}+C_5\mathbf{xt}+C_6\mathbf{t^2}+C_7\mathbf{x^3}+C_8\mathbf{x^2t}+C_9\mathbf{xt^2}+C_{10}\mathbf{t^3}+C_{11}\mathbf{x^4}+C_{12}\mathbf{x^3t}+ +C_{13}\mathbf{x^2t^2}+C_{14}\mathbf{xt^3}+C_{15}\mathbf{t^4}+C_{16}\mathbf{x^5}+C_{17}\mathbf{x^4t}+C_{18}\mathbf{x^3t^2}+C_{19}\mathbf{x^2t^3}+C_{20}\mathbf{xt^4}+C_{21}\mathbf{t^5} \quad (2.2)$$

2.2 AEM methodology for finding the coefficients of a Concordant Function

The *Concordant Function* is obtained by *AEM* using a rigorous procedure without any special hypothesis or any approximation. For the particular case (2.2) where 21 coefficients are involved, 21 equations are necessary. The first equation is represented by the integral equation (1.8), consequently 20 more equations remain to be established.

2.2.1 Equations based on the initial and boundary conditions

The first kind of equations is those that impose rigorously the initial-boundary conditions.

a. Initial conditions on the South edge 1-2 (t=0)

On the South edge *1-2* (Fig.2) is imposed the initial condition (1.6), supposed here to be a polynomial. Because in the local coordinates for this edge it corresponds ***t=0***, the CF5-21 given by (2.2) becomes the polynomial

$$\phi(x,t=0) = C_1+ C_2\mathbf{x} + C_4\mathbf{x^2} + C_7\mathbf{x^3} + C_{11}\mathbf{x^4}+ C_{16}\mathbf{x^5} \quad (2.3)$$

If (2.3) and (1.6) are identified it results 6 coefficients

$$C_1 = \alpha_0 \quad ; \quad C_2 = \alpha_1 \quad ; \quad C_4 = \alpha_2 \quad ; \quad C_7 = \alpha_3 \quad ; \quad C_{11} = \alpha_4 \quad ; \quad C_{16} = \alpha_5 \quad (2.4)$$

b. Boundary conditions on the West edge 1-3 (x=0)

Along the West edge 1-3 (Fig.2) where x=0, the CF (2.2) becomes

$$\phi(x=0,t)=C_1+ C_3\mathbf{t}+ C_6\mathbf{t^2}+ C_{10}\mathbf{t^3}+ C_{15}\mathbf{t^4}+ C_{21}\mathbf{t^5} \quad (2.5)$$

If the boundary conditions are continuous for x=0 and t=0, namely $\beta_0 = \alpha_0$, the constant C_1 is already known from (2.4). By identifying (2.5) and the boundary condition (1.7) it results only 5 coefficients

$$C_3 = \beta_1 \quad ; \quad C_6 = \beta_2 \quad ; \quad C_{10} = \beta_4 \quad ; \quad C_{15} = \beta_5 \quad ; \quad C_{21} = \beta_6 \quad (2.6)$$

All the coefficients established until now result directly without any connection to other information. From the 20 necessary equations, 6+5=11 conditions have been already found. The last 9 equations are rigorously established by using a special approach introduced by the *Accurate Element Method* [1,2,3].

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[3] Three types of Concordant Functions will be used below: CF5-21 (G=5, NT=21 terms), CF7-36 (G=7, NT=36 terms) and CF9-55 (G=9, NT=55 terms)

2.2.2 Equations based on the PDE (1.5) and its derivatives

The 11 coefficients established in §2.2.1 were obtained by using the outside information furnished by the initial and boundary condition. No other information originated from the North and/or East neighboring elements will be used. No special hypotheses concerning any type of imposed relations between the coefficients are considered.

The information that is still necessary is taken from inside being **rigorously** furnished only by the governing equation (1.5) itself.

A. Equations based on the PDE

It is obvious that the PDE (1.5) has to be valid at any point inside the integration domain that – in this case – is a rectangular element. The PDE will be applied in at the nodes *2*, *3* and *4* (Fig.2). For instance at the *node 2* (x=B,t=0) the *PDE* (1.5) becomes

$$M(B,0)\left(\frac{\partial\phi}{\partial x}\right)_{x=B,t=0}+N(B,0)\left(\frac{\partial\phi}{\partial t}\right)_{x=B,t=0,}+P(B,0)(\phi)_{x=B,t=0}+Q(B,0)=0 \qquad (2.9)$$

All the functions connected to the *PDE* (1.5) are transferred in local coordinates, their values at the *node 2* being then computed by replacing x=B, t=0. Some terms of (2.9) can be evaluated directly. Besides Q(B,0) one observe that along the axis *x* the function $\phi(x,t=0)$ has to coincide with the initial condition $\Psi(x)$ (1.6), so that $(\phi)_{x=B,t=0}=(\psi)_{x=B}$ and also $\left(\frac{\partial\phi}{\partial x}\right)_{x=B,t=0}=\left(\frac{d\Psi}{dx}\right)_{x=B}$. By deriving (2.2) versus *t* and equating it with the same derivative obtained from (1.2) it results an equation that includes five unknown coefficients

$$\left(\frac{\partial\phi}{\partial t}\right)_{x=B,t=0}=C_2+2BC_4+3B^2C_7+4B^3C_{11}+5B^4C_{16}=$$

$$=-\frac{1}{N(B,0)}\left[M(B,0)\left(\frac{d\Psi}{dx}\right)_{x=B}+P(B,0)(\psi)_{x=B}+Q(B,0)\right]$$

Two more equations that use the *PDE* (1.5) can be written similarly at the nodes *3* and *4*, *but not at the node 1* (x=0,t=0), because for this last node it results from (2.4) and (2.6)

$$M(0,0)\,\alpha_1+N(0,0)\,\beta_1+P(0,0)\,\alpha_0+Q(0,0)=0$$

This equation cannot be accepted, because it represents a condition arbitrarily imposed to the initial and boundary conditions, *whose coefficients do not depend in any way on the PDE (1.5)*.

Because the three equations furnished by the PDE are not enough, one can obtain the six more necessary equations by using the derivatives of (1.5).

B. Equations based on the first derivatives of the PDE

The first order derivatives of the PDE (1.5) versus *x* and *t* are

$$\frac{\partial(\mathrm{PDE})}{\partial x}=M\frac{\partial^2\phi}{\partial x^2}+\frac{\partial M}{\partial x}\frac{\partial\phi}{\partial x}+N\frac{\partial^2\phi}{\partial x\partial t}+\frac{\partial N}{\partial x}\frac{\partial\phi}{\partial t}+P\frac{\partial\phi}{\partial x}+\frac{\partial P}{\partial x}\phi+\frac{\partial Q}{\partial x}=0 \qquad (2.10)$$

$$\frac{\partial(\mathrm{PDE})}{\partial t}=M\frac{\partial^2\phi}{\partial x\partial t}+\frac{\partial M}{\partial t}\frac{\partial\phi}{\partial x}+N\frac{\partial^2\phi}{\partial t^2}+\frac{\partial N}{\partial t}\frac{\partial\phi}{\partial t}+P\frac{\partial\phi}{\partial t}+\frac{\partial P}{\partial t}\phi+\frac{\partial Q}{\partial t}=0 \qquad (2.11)$$

If these derivatives are transferred to the nodes *2,3,4* following a similar approach as above, one obtains *2 equations/node×3 nodes = 6 equations*.

Some of these equations applied in the *node* 4 are based on the unknown function ϕ and its derivatives. Because these parameters are involved in the procedure, AEM is an **implicit method** [1,5] unconditionally stable.

The complete system includes 21 equations: *one* from the integral-equation (1.10), *eleven* from §2.2.1 and *nine* from §2.2.2. By solving this system[4] it *results the function-solution* (2.2). This function will be considered a *quasi-analytic solution* because it is valid on a single sub-domain (element), not on the whole domain $\mathcal{D}$ [1].

It remains to find an answer to a fundamental question [11]: *how good are the results furnished by this solution*?

3. Start and Target Edges

In §2.2 it was shown that for an *initial value problem* there are two edges of the element [*1-2* (South) and *1-3* (West) (fig.2)] where the *initial-boundary conditions are known*. They will be referred as *Start Edges*. The values of the function at the nodes *1,2,3* that are on the *Start Edges,* are also known from the obvious relations

$$\phi_1=\Psi(x=0,t=0)=\Omega(x=0,t=0)\ ;\ \phi_2=\Psi(x=B,t=0)\ ;\ \phi_3=\Omega(x=0,t=H) \qquad (3.1)$$

On the contrary, for the other two edges of the element [*3-4* (North) and *2-4*(East),Fig.2] the function $\tilde{\phi}(x,t)$ that is supposed to verify the PDE is unknown and its value $\boldsymbol{\phi_4(x_4,t_4)}$ at the *node 4* is also unknown. These two last edges will be referred as *Target Edges* and $\boldsymbol{\phi_4}$ as *Target Value*.

Good *Target Edges* solutions are very important for the usual case when the integration is performed on more than one element. Suppose the integration does not stop at the edge 3-4 (Fig.1), but has to be continued along the *T* axis, namely on the following element *G12*. In this case the *North Target Edge 3-4* solution will be used as *initial-condition* on the *South Edge* of *G12.*

[4] For higher degree Concordant Functions the number of equations increases, in which case higher order derivatives of the *PDE* become necessary

4. Residual function and its use

4.1 Residual function

Suppose that for a given *PDE* a function $\widetilde{\phi}(x,t)$ that *fulfils the boundary conditions* exists and is known. The way to verify if $\widetilde{\phi}(x,t)$ represents a solution of the *PDE* is to replace it in (1.5) that leads to the function

$$R(x,t) = M(x,t)\frac{\partial\widetilde{\phi}}{\partial x} + N(x,t)\frac{\partial\widetilde{\phi}}{\partial t} + P(x,t)\widetilde{\phi} + Q(x,t) \qquad (4.1)$$

If R(x,t) – referred as *residual function* – is **zero** the function $\widetilde{\phi}(x,t)$ represents a *particular analytic or exact solution* of the PDE. If the residual function is different from zero, the analysis of its values represents the best way to appreciate the precision of the numerical result. A very small residual will indicate (under certain conditions described below) that $\widetilde{\phi}$ (x,t) is a good solution. An example of the residual variation for a particular case is given in Fig.3 and in Fig.4 (angle of view opposite to Fig.3).

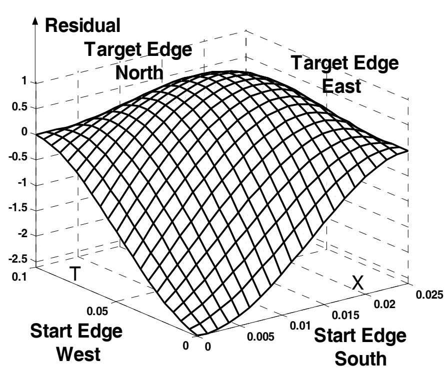

Fig.3

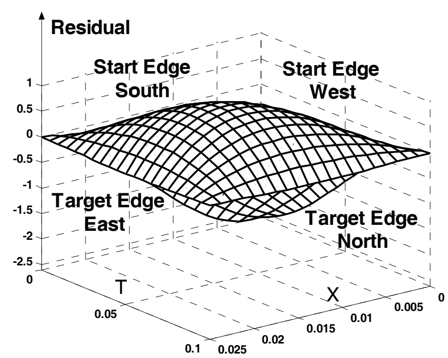

Fig.4

4.2 Residuals at the Target Edges

It is necessary to precise that the equation (1.8) where $\phi(x,t)$ is replaced by a Concordant Function imposes in fact the condition that the integral of the residual is zero. This includes the possibility that on some portions *R(x,t)* may be positive and on other negative. In fact on the Start Edges the residuals $R_{12}(x,t=0)$ and $R_{13}(x=0,t)$ are usually different from zero (see Fig.3), because the CF is rigorously adapted – due to (2.4) and (2.6) – to the imposed boundary conditions **that do not depend on the PDE.** On the contrary due to the integration procedure the residuals on the Target Edges become *close* to zero[5] , as it results from Fig.4.

[5] The visual impresion that the residual is zero is delusive

Therefore the verification of the residuals will be performed **only on the Target Edges** from where the information is transmitted to the neighboring elements

Target Edge Residual 34: $$R_{North} = R(x, t = t_4) = \left(M\frac{\partial\widetilde{\phi}}{\partial x} + N\frac{\partial\widetilde{\phi}}{\partial t} + P\widetilde{\phi} \right)_{t=t4} + Q(x, t = t_4) \quad (4.3)$$

Target Edge Residual 24: $$R_{East} = R(x = x_4, t) = \left(M\frac{\partial\widetilde{\phi}}{\partial x} + N\frac{\partial\widetilde{\phi}}{\partial t} + P\widetilde{\phi} \right)_{x=x4} + Q(x = x_4, t) \quad (4.4)$$

Both residuals (4.3) and (4.4) are **one-dimensional functions,** which simplify the analysis necessary to obtain a numerical criterion that certifies the quality of the solution. These functions can be represented as graphs that allow appreciating how close the *AEM* solution verifies the PDE along *2-4* or *3-4* target edges. Nevertheless it is better to have – instead of a function – a single numerical criterion in order to appreciate the accuracy of the computation.

4.3 Root Mean Square Residuals

The residual R_{North} (4.3) can be computed in a number of points NP having x_i(i=1,2,…NP) abscissas. Based on these values one can calculate a mean square root value given by [1,5]

$$R_{MS,North} = \frac{1}{NP}\sqrt{\sum_{i=1}^{i=NP}[R_{North}(x_i, t = t_4)]^2} \quad (4.5)$$

For the East *Target Edge residual* (Fig.2, edge 2-4), a similar computation is performed along the ordinate *t*, for the constant abscissa $x=x_4$

$$R_{MS,East} = \frac{1}{NP}\sqrt{\sum_{i=1}^{i=NP}[R_{East}(x = x_4, t_i)]^2} \quad (4.6)$$

Both (4.5) and (4.6) values may be compared with an *allowable residual*:

$$R_{MS} < R_{allow} \quad (4.7)$$

The parameter R_{allow} is a conventional value that remains to be established. Some numerical tests solved by the author have shown that a value of R_{MS} smaller than 10^{-9}-10^{-10} indicates a very good result that can be considered as *accurate*, but greater values like $R_{MS} \approx 10^{-7}$-10^{-8} can also be accepted [1,5]. On the contrary, a value like $R_{MS} = 10^{-3}$ shows that *the corresponding solution* $\widetilde{\phi}$ *(x,t) has to be rejected* and a new computation using modified conditions (shape of the element, number of elements or/and a different *Concordant Function*) has to be performed. *The value chosen for* R_{allow} *is obviously disputable*.

The R_{MS} approach gives to the user a powerful and global tool to verify the validity of the whole computation, no matter how many elements or *CF*s are involved. It is important to underline that the numerical tests (4.5) and (4.6) performed on any element are *independent of the results obtained on the previous elements*. Consequently they represent **a verification of the whole procedure, concerned only by the final result, being therefore independent on the various steps covered in order to obtain $\tilde{\phi}$ (x,t)**.

4.4 Accuracy estimation of the Target Value $\phi_4(x_4,t_4)$

The Target Value results from the *CF* (2.2) by replacing $x=x_4$ and $t=t_4$. A first question has to receive an answer: *is the value of $\phi_4(x_4,t_4)$ reliable or not ?* The procedure based on the R_{MS}, namely the relation (4.7), represents a basis for finding the answer. If both values (*North and East*) of the R_{MS} are very small and therefore allowable, one can conclude that the value of $\phi_4(x_4,t_4)$ is reliable and good, but not *how far is from the accurate result*. In fact, because (4.1) includes besides the solution (2.2) also its derivatives, the residual computation **does not verify directly the values of the function.**

An answer concerning the precision level of the *Target Value* $\phi_4(x_4,t_4)$ can be obtained following different ways, from which we mention:

1. Compute the *Target Value* $\phi_4(x_4,t_4)$ by using an increasing number of elements in order to obtain information on the convergence of the results. This can be done by comparing the values obtained for $\phi_4(x_4,t_4)$ using *NE* and [*NE+Δ(NE)*] elements, respectively, which allows to obtain an estimated error given by

$$\textit{Estimated Target Error} = \frac{\phi(x_4,t_4)_{(NE+\Delta NE)} - \phi(x_4,t_4)_{(NE)}}{\phi(x_4,t_4)_{(NE+\Delta NE)} + \phi(x_4,t_4)_{(NE)}} \qquad (4.8)$$

Obviously, this is only an *estimated error* because it is related to another computed value $\phi_4(x_4,t_4)_{(NE)}$, *not to the actual value which is unknown*. This error may be used only as a primary test. Anyway, a lack of convergence can indicate that the integration strategy used is not adequate.

2. Compare the values of $\phi_4(x_4,t_4)$ obtained by using different *Concordant Functions*, for instance *CF5-21* and *CF7-36*. If two Target Values computed using polynomials with different degrees coincide for instance with 6 decimal digits or more the result can be considered as good and the *Estimated Target Error* thus obtained is reliable.

5. Integration examples using a standard strategy

The strategy used for the numerical integration extended on a rectangular mesh is usually based on some trivial ideas:

1. The ratio between the base B and the height H has to be close to unity, therefore the best element is a square.

2 If no reliable method to verify the errors is available, the computation is developed by increasing the number of elements. If the results are convergent

towards a certain value, the computation can be stopped when the relative error between the values of the (S) and (S+1) steps reaches a level that can be considered as allowable. This approach leads to the *estimated error* mentioned above. In the unusual case when the exact value is known, one can calculate at each step an *actual error*.

For all the examples that follow the integration of the PDEs with variable coefficients will be performed on a rectangular domain $\mathscr{D}$ extended along *X* from $X_{Left}=\mathbf{0}$ to $X_{Right}=\mathbf{1}$ and along *T* from $T_{Start}=\mathbf{0}$ up to $T_{Target}=\mathbf{1}$.

5.1 Integration of a PDE for which the solution is known

Example1. Let be the PDE with variable coefficients

$$(1+3X+4T+2X^2+3XT+3T^2)\frac{\partial\phi}{\partial X}+(4+2X+3T+3X^2+2XT+3T^2)\frac{\partial\phi}{\partial T}+ \\ +(1+4X+2T+2X^2+4XT+2T^2)\,\phi+Q_G(X,T)=0 \tag{5.1}$$

that has to be integrated with the following *initial-boundary* conditions:

Initial conditions (T=0): $\Psi(X)=2+3X+X^2$ (5.2)

Boundary conditions (X=0): $\Omega(T)=2+2T+3T^2$ (5.3)

A solution of (5.1) $\overline{\phi}(X,T)$ has been taken at random as a two-dimensional *eleventh-degree complete polynomial with 78 terms*. Replaced in (5.1) the solution $\overline{\phi}(X,T)$ leads to a free term $Q_G(X,T)$ with *105 terms*. The Target value obtained from $\overline{\phi}(X,T)$ is

$$\overline{\phi}(X=1,T=1)=157 \tag{5.4}$$

Due to the great number of terms it was considered that reproducing here $\overline{\phi}(X,T)$ and $Q_G(X,T)$ is not necessary.

The integration will be performed using three different *Concordant Functions*: CF5-21, CF7-36 and CF9-55. Because the degree of the solution $\overline{\phi}(X,T)$ is greater than all the three CFs, the numerical results obtained using a *small number of elements* cannot be accurate [2,3]. The integration will be performed following the standard strategy mentioned above. Because the exact *Target Value* is known from (5.4), in this particular case it is possible to calculate the *actual error*. From the results given in *Table 1* it is useful to observe:

1. For each CF the results are convergent towards the exact value (5.4) when the number of elements increases.

2. The results improve when the degree of the *Concordant Function* increases, if the same number of elements is considered.

3. As it was shown in §2, the *Accurate Element Method* is an *implicit* method, therefore one can use elements having *quite large dimensions that can be considered as improper by other methods*. For instance the computation starts for

each *CF* with a single element having ***B=1*** and ***H=1***. The value obtained in this case is far from (5.4) for CF5-21, but quite good for CF9-55.

4. Some results obtained using CF9-55 can be considered as accurate. The values obtained with 9, 16 and especially 25 elements (such as *157.0000009* or *157.0000001*) are convincing. The results obtained using CF7-36 with more than 64 elements can also be considered as satisfactory.

Table 1

Elements NE=NCOL× NROW	CF5-21		CF7-36		CF9-55	
	ϕ(X=1,T=1)	Actual error	ϕ(X=1,T=1)	Actual error	ϕ(X=1,T=1)	Actual error
1×1=1	164.076	4.5×10^{-2}	155.598469	-8.9×10^{-3}	157.0589174	3.7×10^{-4}
2×2=4	157.992	6.3×10^{-3}	156.963215	-2.3×10^{-4}	157.0002136	1.4×10^{-6}
3×3=9	157.245	1.6×10^{-3}	156.996370	-2.3×10^{-5}	157.0000088	5.6×10^{-8}
4×4=16	157.084	5.3×10^{-4}	156.999326	-4.2×10^{-6}	157.0000009	5.8×10^{-9}
5×5=25	157.035	2.3×10^{-4}	156.999819	-1.1×10^{-6}	157.0000001	9.9×10^{-10}
6×6=35	157.017	1.1×10^{-4}	156.999939	-3.9×10^{-7}	*	*
7×7=49	157.009	6.1×10^{-5}	156.999975	-1.5×10^{-7}	*	*
8×8=64	157.005	3.6×10^{-5}	156.999989	-7×10^{-8}	*	*
9×9=81	157.003	2.3×10^{-5}	156.999994	-3.4×10^{-8}	*	*
10×10=100	157.002	1.5×10^{-5}	156.999997	-1.8×10^{-8}	*	*

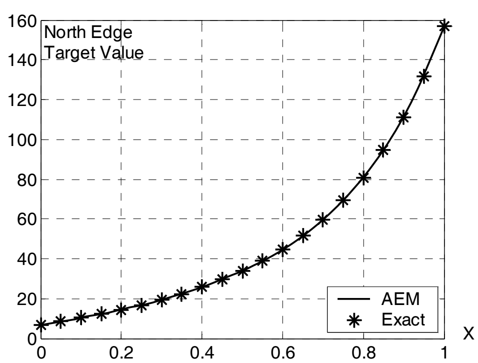

Fig.5

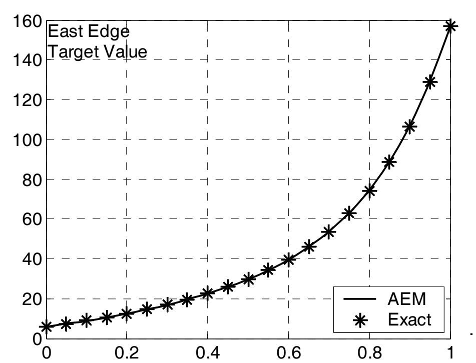

Fig.6

Another way to verify the accuracy of the results is to represent the graphs of the *North Edge Target Values* for *T=1* (Fig.5) and *East Edge Target Values* for *X=1* (Fig.6). Both graphs correspond to the case *4×4=16 elements* (CF7-36, *Table* 1) and show a perfect match between the AEM results and the exact values.

Nevertheless, as it will result from the next example, no hasty conclusion has to be retained concerning the validity of the computation strategy used in this case.

5.2 Integration of a PDE whose solution is not known

Example2. The PDE with variable coefficients

$$\frac{\partial\phi}{\partial X}+(4+2X+3X^2)\frac{\partial\phi}{\partial T}+\phi- \\ -(2.5+19X+22T+13X^2+28XT+18T^2+6X^3+2X^2T+4XT^2+3T^3)=0 \tag{5.5}$$

will be integrated on $\mathscr{D}$ using **CF7-36,** with the following *initial-boundary* conditions taken at random:

Initial conditions (T=0): $\Psi_G(X)=1-0.3X+0.1X^2+0.2X^3+0.03X^4+0.01X^5$ (5.6)

Boundary conditions (X=0): $\Omega_G(T)=1-0.2T+0.3T^2-T^3+0.03T^4+0.01T^5$ (5.7)

The strategy of integration for this PDE – that has as single variable coefficient ($4+2X+X^2$) – is the same used in the previous example. The computation starts also with a single square element (*Table 2, Test 1*). In order to maintain square elements during the further computation, the number of columns (NCOL) and rows (NROW) are multiplied simultaneously by the same number.

Table 2

Test	NE=NCOL× NROW	ϕ(X=1,T=1)	Estim. error	North RMS residual	East RMS residual	Ratio North/East
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1×1=1	9.2710	*	3.1×10^{-3}	8.4×10^{-5}	37
2	2×2=4	9.1849	4.7×10^{-3}	1.7×10^{-2}	2.8×10^{-5}	611
3	3×3=9	8.7845	-2.2×10^{-2}	1.1×10^{-2}	1×10^{-5}	1074
4	4×4=16	8.5328	-1.4×10^{-2}	3.8×10^{-3}	1.4×10^{-6}	2694
5	5×5=25	8.4135	-7×10^{-3}	2.8×10^{-3}	4.8×10^{-6}	575
6	6×6=35	8.6173	1.2×10^{-2}	1.1×10^{-2}	1.2×10^{-5}	905
7	7×7=49	9.1462	3×10^{-2}	9.3×10^{-3}	2.1×10^{-5}	446
8	8×8=64	8.6249	-2.9×10^{-2}	5.1×10^{-2}	5.3×10^{-5}	950
9	9×9=81	6.9851	-1×10^{-1}	6.1×10^{-2}	1.1×10^{-4}	538
10	10×10=100	9.3847	1.5×10^{-1}	2.1×10^{-1}	1.9×10^{-4}	1144

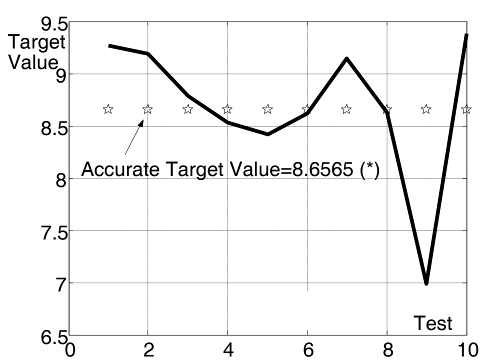

Fig 7

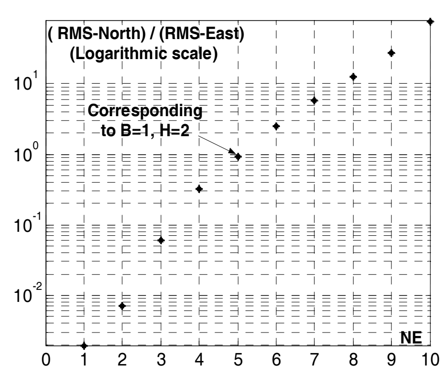

Fig.8

The results obtained in this case following the strategy used for the previous example are disappointing, because the computed *Target Values* (*Table 2, column 3*) are erratic and non-convergent. This erratic behavior is also confirmed by the *Estimated Target Error* (4.8) given in *column 4* and by the values of the North and East *RMS* residuals (Columns 5 and 6). From the graph given in Fig.7 (where the accurate *Target Value* obtained following a different strategy is also indicated) it results clearly that no credible Target Value $\phi(X=1,T=1)$ can be retained. **The standard strategy which recommends the increase of the number of elements has failed**.

6. The integration strategy of the Accurate Element Method

6.1 A deeper analysis concerning the integration of a very simple PDE

The integration strategy that will be further used is based on a more flexible approach made possible by the information concerning the *North* and *East* residuals, furnished by the *Accurate Element Method*. Some important aspects have resulted in [1] from the simplest PDE ($\frac{\partial\phi}{\partial X}+2\frac{\partial\phi}{\partial T}=0$) that has been integrated on the large rectangular domain limited by **X=1, T=10**. The results represented by the *Target Values* are reproduced here in *Table 3*. Because in this case the exact *Target Value* is known, the analysis and the conclusions are reliable. The strategy used for the integration was different from that used here in §5 being performed on a single column, increasing at each test the number of elements. This procedure *leads to a modified shape of the elements*. In fact, the ratio between the height *H* and the base *B* starts for the first element with *H/B=10*, becoming for the last element *ten times smaller*, namely H*/B=1.* The *North* and *East* residuals show an improvement when the number of elements increases from *1* to *4*. For NE=5 it results an impressive improvement because both residuals drop to 10^{-13} while the actual error becomes 1.4×10^{-15}. This indicates that **the accurate solution has been reached**. If the number of elements is further increased, the results remain good but with some greater errors.

Table 3

Exact Target Value ϕ_{TE} = 749.2						
NE	B×H	Target Value $\phi_{TE}(X=1,T=10)$	Residuals			Actual error
			RMS-North	RMS-East	Ratio N/E	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1×10	748.0424382716078	7.6×10^{-4}	3.9×10^{-1}	1/513	-1.5×10^{-3}
2	1×5	747.7472991695568	7.7×10^{-4}	1.1×10^{-1}	1/143	-1.9×10^{-3}
3	1×3.33	749.2102287345520	7.3×10^{-5}	1.2×10^{-3}	1/16.4	1.4×10^{-5}
4	1×2.5	749.2011443978998	1.3×10^{-4}	4×10^{-4}	1/3.1	1.5×10^{-6}
5	**1×2**	**749.2000000000011**	$\mathbf{3\times10^{-13}}$	$\mathbf{3.9\times10^{-13}}$	**≈1**	$\mathbf{1.4\times10^{-15}}$

6	1×1.66	749.2000001699947	1×10^{-7}	4×10^{-8}	2.5	$2.2\times10^{--10}$
7	1×1.43	749.2000000113434	4×10^{-9}	7×10^{-10}	5.7	1.5×10^{-11}
8	1×1.25	749.1999999939001	1.4×10^{-9}	1.1×10^{-10}	12.7	-8.1×10^{-12}
9	1×1.11	749.1999999992441	1.5×10^{-9}	5.5×10^{-11}	27.2	-1×10^{-12}
10	1×1	749.2000000104398	9.1×10^{-10}	1.2×10^{-11}	75.8	1.4×10^{-11}

This analysis can represent a basis for a modified strategy that will be applied in the following examples. The first idea which has to be retained is that **very good results can be obtained by using elements whose shapes are different from a square**; the first problem to be solved is to find the best possible shape. The second idea is that *increasing the number of elements does not lead always to better values*; the problem that results is to *find the number of elements beyond which the values worsens*. The *Accurate Element Method* can give answers to both problems.

6.2 A strategy to find the best possible shape. As it results from *Table 3* the best value was not obtained for the maximum of elements (NE=10), but for a smaller number (NE=5). For NE=10 the element is square, while for NE=5 is rectangular with H/B=2. Or, as it results from the columns (4) and (5), this last element is **the only one for which the North and East residuals are nearly equal.** The equality of the two residuals can be synthesized by their ratio

$$\text{Ratio N/E}=(\text{RMS}_{\text{North}}) / (\text{RMS}_{\text{East}}) \tag{6.1}$$

This parameter that was not used in [1] is added here in *Table 3, column (6)*. The variation of the ratio (6.1) using a logarithmic scale is given in Fig.8.

Obtaining the best possible shape means in fact to **find the ratio B/H (or H/B) for which the Ratio E/N is EQUAL to the unity.** To obtain exactly this value is seldom possible; therefore the above condition may be reasonably changed in "**find the ratio B/H (or H/B) for which the Ratio N/E is CLOSE to the unity".**

6.3 A strategy to find a good result. When the procedure to find the best shape has been successfully accomplished and as a consequence the shape of the elements has been established, one can appreciate the accuracy of the result from the RMS of the *North* and *East* residuals. If this value is considered as satisfactory [according to (4.7)] the computation may be stopped. If not, the number of columns and rows of the mesh will be increased in the same amount, in order to maintain the shape for which the *Ratio N/E* was close to the unity. Maintain the shape does not mean compulsory that the ratio (6.1) remains unmodified. Therefore it is necessary to verify if (6.1) is close to the unity; if not, one has to modify the number of columns or rows for finding the best shape for the new configuration.

This strategy may sometimes be more complex because – if the results are unsatisfactory – **the full procedure can also include the decision to change the *Concordant Function***. Nevertheless all these requirements may be easily implemented in a program that finds a good solution without any intervention of the user. This "automatic" search (as it was programmed by the author) does not give always the best results.

The examples that follow intend to illustrate this strategy.

Example 3. The *Example 2* will be solved using CF7-36 starting as in *Table 2* with a single element. Because the ratio $Ratio_{N/E}$=37 (*Table 4,Column 6*), one choose for the second test two rows for which $Ratio_{N/E}$=1599 that is worse than *Test 1*. This shows that the decision to increase the number of rows was wrong, so that for *Test 3* one increases the number of columns (NCOL=2), while NROW=1. This leads to $Ratio_{N/E}$=27, which is better than for the *Test 1*. If the number of columns is further increased, one obtains for *Test 6* a $Ratio_{N/E}$ near to unity, so that the best possible shape corresponding to a single column has been reached. The value of the $R_{MS}=1.2\times10^{-9}$ can be considered a good result, the corresponding target value being

$$\phi(X=1,T=1)=8.656505067110762 \qquad (6.2)$$

If the user is not satisfied with the R_{MS} obtained for the *Test 6*, the computation can continue by doubling succesivaly the number of columns and rows (*Tests 8 and 9*). For the *Test 9* it results $Ratio_{N/E}$=1/2.6, which is a good value and the mean $R_{MS}=2.55\times10^{-13}$. The Target Value corresponding to *Test 9*

$$\phi(X=1,T=1)= 8.656504813177039 \qquad (6.3)$$

can be considered as accurate. This value was used for the graph given in Fig.7.

Table 4

Test	NE=NCOL× NROW	Target Value ϕ(X=1,T=1)	ResidualMS North	ResidualMS East	Ratio North/East	Relative Error
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	1×1=1	9.271063416732901	3.1×10^{-3}	8.4×10^{-5}	37	7.1×10^{-2}
2	1×2=2	9.266059718797713	4.6×10^{-3}	2.9×10^{-6}	1599	7×10^{-2}
3	2×1=2	8.972009233893001	1.1×10^{-3}	4.1×10^{-5}	27	3.6×10^{-2}
4	4×1=4	8.665058847812112	2.2×10^{-5}	7.1×10^{-7}	31	9.9×10^{-4}
5	6×1=6	8.656504565787088	1.6×10^{-9}	4×10^{-9}	1/2.5	-2.8×10^{-8}
6	**7×1=7**	**8.656505067110762**	$\mathbf{1.3\times10^{-9}}$	$\mathbf{1.1\times10^{-9}}$	**1/1.2**	$\mathbf{2.9\times10^{-8}}$
7	8×1=8	8.656504868028492	3.8×10^{-10}	1.7×10^{-9}	1/4.5	6.3×10^{-9}
8	14×2=28	8.656504814504606	7.4×10^{-12}	2.8×10^{-11}	1/3.8	1.5×10^{-10}
9	28×4=112	8.656504813177039	1.4×10^{-13}	3.7×10^{-13}	1/2.6	*

Remarks. 1. The error of the value (6.2) if compared to (6.3) is 2.9×10^{-8}. For a usual computation such a precision may not be necessary, in which case the *Tests 8 and 9* are useless.

2. Unlike the erratic values obtained using a wrong strategy in *Example 2*, the values given in *Table 4* are strictly convergent when the number of elements increases.

Example 4. The PDE with variable coefficients

$$(5+3T+3T^2)\frac{\partial\phi}{\partial X}+\frac{\partial\phi}{\partial T}+(2+3X+5XT)\phi- \\ -(2.5+19X+22T+13X^2+28XT+18T^2+6X^3+2X^2T+4XT^2+3T^3)=0 \tag{6.4}$$

will be integrated on $\mathscr{D}$ the *initial-boundary* conditions being (2.2) and (2.3).

The integration is performed using successively CF5-21, CF7-36 and CF9-55. The main results obtained using the strategy followed in *Example 3* are given in *Table 5*. Some remarks concerning the results have to be mentioned:

1. In this case the ratio B/H>>1, being therefore reversed as compared to *Example 3*, where B/H<<1.

Table 5

Test	NE=NCOL× NROW	$\frac{B}{H}$	Target Value ϕ(X=1,T=1)	ResidMS North	ResidMS East	Ratio North/East
(1)	(2)	(3)	(4)	(5)	(6)	(7)
CF5-21						
1	1×7=7	7	5.460095975254646	4×10^{-4}	1.4×10^{-4}	2.85
2	2×12=24	6	5.454168955719426	1×10^{-5}	3.6×10^{-6}	1/3.6
3	3×18=54	6	5.453791761125538	1.2×10^{-6}	1.4×10^{-6}	1/1.2
4	4×25=100	6.25	5.453725908386619	2.9×10^{-7}	2.7×10^{-7}	1.07
5	5×32=160	6.4	5.453704642743616	9.8×10^{-8}	7.4×10^{-8}	1.32
6	9×65=585	7.2	5.453690222722832	6×10^{-9}	7.2×10^{-9}	1/1.2
CF7-36						
7	1×5=5	5	5.458669407203526	1.3×10^{-5}	4.2×10^{-5}	1/3.2
8	2×15=30	7.5	5.453691682364898	8.7×10^{-9}	8.7×10^{-9}	≈1
9	3×22=66	7.3	5.453688986594051	2.6×10^{-9}	1.8×10^{-9}	1.44
10	5×32=160	6.4	5.453688728692493	2.35×10^{-10}	2.39×10^{-10}	1.02
11	6×40=240	6.7	5.453688724546589	8.4×10^{-11}	6.8×10^{-11}	1.23
12	10×64=640	6.4	5.453688721408778	4.6×10^{-12}	4.1×10^{-12}	1.12
CF9-55						
13	1×7=7	7	5.453703569440119	1.5×10^{-7}	7×10^{-8}	1/4.7
14	2×14=28	7	5.453688711719918	4.2×10^{-10}	2.3×10^{-10}	1.82
15	3×20=60	6.7	5.453688723145416	2.4×10^{-11}	1.3×10^{-11}	1.85

2. If the values obtained using CF5-21 and CF7-36 are compared it results that the second *CF* leads to similar results with a smaller number of elements. For instance the $R_{MS}\approx 10^{-9}$ is reached by CF5-21 with 585 elements (*Test 6*), while for similar result CF7-36 needs only 66 elements (*Test 9*). The same happens for the

$R_{MS} \approx 10^{-11}$ that is reached by CF7-36 with 240 elements (*Test 11*), while a similar result is obtained by CF9-55 with only 60 elements (Test 15).

3. While the goal of the *Example 3* was to describe and apply the integration strategy developed by the *Accurate Element Method*, from this example results another test that intends to verify the precision of the *Target Value*: *compute the estimated error by comparing the results obtained with different Concordant Functions*. For instance one can compare the following tests that have similar residuals ($R_{MS} \approx 10^{-11}$):

Tests 11 (CF7-21): ϕ(X=1,T=1) = **5.453688724**546589

Test 15 (CF9-55): ϕ(X=1,T=1) = **5.453688723**145416

From the comparison it results:

a. The two values coincide with **9 digits, so that one can considered ϕ(X=1,T=1) = 5.45368872 as reliable**.

b. The relative error between the two values is $\mathbf{1.3\times10^{-10}}$.

This comparison that leads (for similar residuals) to **close results using Concordant Functions with different degrees represents a solid confirmation of the Accurate Element Method as a whole**.

7. The characteristic curves

The strategy for finding the best shape of the element can be explained by the *characteristic curves* of the PDE (5.5) that are obtained by integrating the ordinary differential equation [1,7,8]

$$\frac{dT}{dX} = \frac{N(X,T)}{M(X,T)} = \frac{4+2X+3X^2}{1} = 4+2X+3X^2 \tag{7.1}$$

The integral of (7.1) is

$$T(X) = K + 4X + X^2 + X^3 \tag{7.2}$$

where *K* is an integration constant. In fact (7.2) represents a bunch of parallel curves, corresponding to different values of *K*. Suppose an element whose origin is in global coordinates (Fig.9) at *Node 1* (X_S, T_S). The characteristic curve starting from *Node 1* results from the condition that to X= X_S has to correspond T= T_S

$$T_S = K + 4X_S + X_S^2 + X_S^3 \quad \Rightarrow \quad K = -(4X_S + X_S^2 + X_S^3 - T_S) \tag{7.3}$$

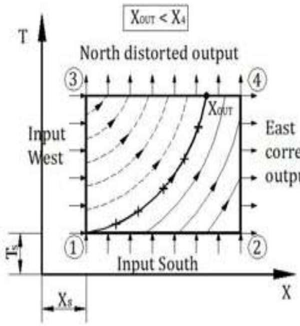

Fig.9a

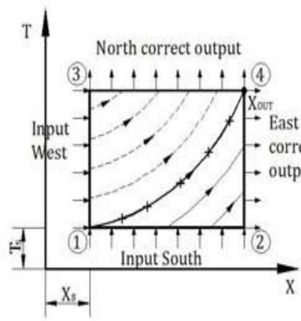

Fig.9b

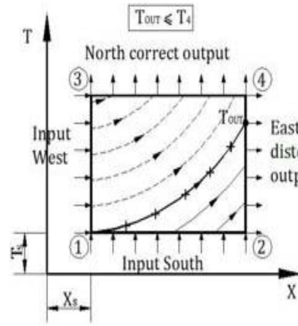

Fig.9c

Suppose the particular case represented in Fig.9b where the curve (7.2) that starts from the node *1* – marked with xxxx – intersects the *Target Edge 3-4* in the node *4*, dividing the element in two sub-domains. In this case the information furnished by the initial condition (*Input South*) is used to perform the integration only on the south-east sub-domain, while the boundary condition (*Input West*) is used only on the north-west sub-domain. Because the two sub-domains are separated by the charactestic curve *1-4* there is no information interference between them. Consequently, the *Concordant Function* receives **correctly** on each sub-domain the information furnished by the initial and boundary conditions, respectively. The information resulted from the integration is therefore also correctly transmited to the North and East output edges. In this case the mean square residuals R_{MS} that correspond to these two edges – both reflecting the same *Concordant Function* and the same PDE – are **correctly evaluated and therefore have to be nearly equal**. This has been observed for the case NE=5 in Table 3 and also in Fig.8.

Suppose now the case represented in Fig.9a, where the characteristic curve intersects the *Target Edge 3-4* at the abscissa X_{OUT} . In this case the information received from the West edge (adapted by the *Concordant Function* to the governing PDE due to the conditions developed in §2.3) is transmitted to the neighboring *North* element through a portion between *Node 3* and X_{OUT}. The information received from the South Edge is less fortunate, because *it is transmitted simultaneously to the East side element but also to the North side element* through the segment between X_{OUT} and *Node 4*. As a consequence, *the North side element will receive distorted information*, composed not only by the correct output between *Node 3* and X_{OUT}, but also by the unexpected (and incorrect) information furnished between X_{OUT} and *Node 4* that arrives from the initial (South Edge) condition. On the contrary the East side element will receive correct information furnished only by the initial (South) condition. It results that **the East side output is correct, while the North side output is incorrect**. This is reflected, numerically, in a Ratio N/E far from unity. This ratio tends to increase (or diminish) when the position of X_{OUT} is closer to the node 3 (Fig.2).

On the contrary, in the third case reflected by Fig.9c, the start curve (xxxx) intersects the *Target Edge 2-4* in a point that has the ordinate T_{OUT}. This time **the correct information is furnished by the North edge; on the East edge the output is distorted** being composed by the (correct) South input information to which the some West input information is forcibly added.

8. Conclusions

The Accurate Element Method based on the strategy "*best shape-tested Target value*" leads towards very good (if not accurate) *Target Values.* But more important is the possibility offered to the user to follow "with opened eyes" the integration procedure, namely to choose/change the type of the *CF* used, to know the level of the possible errors and to decide – based on the requirements of each specific case – when the computation can be stopped. Besides the *Target Value*, it is trivial to obtain any type of graphs, which can be easily drowned because for each element is available a quasi-analytic solution.

Some further developments of the method have already been mentioned in [1].

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