U.P.B. Sci. Bull., Series A, Vol. 84, Iss. 1, 2022 ISSN 1223-7027

APPLICATIONS OF A PERTURBED LINEAR VARIATIONAL PRINCIPLE *VIA P*-LAPLACIAN

Irina MEGHEA[1]

In this paper, starting from Ghossoub-Maurey linear principle, an existence result is obtained and another one on existence and uniqueness for a certain minimization problem involving the p*-Laplacian. This is a generalization of a namely result due to Brezis and Nierenberg which is followed by some applications to problems evolved from modeling of some phenomena of real world. The novelty of this work consists in obtaining the above cited generalization and the proposal of appropriate applications.*

Keywords: Ghossoub-Maurey linear principle, minimization problem, Dirichlet problem, *p*-Laplacian.

1. Introduction

This work is based on a perturbed variational principle. In the frame of Variational Calculus, the elementary proposition

"*If* $\varphi : X \to \mathbf{R}$, X *a real normed space*, *has in* x_0 *a local minimum point* (*hence in particular a global minimum point*) *and it is Gâteaux differentiable at* x_0, *then* $\varphi'_w(x_0) = 0$" (x_0 is *critical point*)

is called **variational principle** ([1]; *w* comes from *weak*).

This is the reason why the neighbour propositions, for instance those in which *X* is replaced by a metric space, or in which the statement "$\varphi'_w(x_0) = 0$" is replaced by "$\|\varphi'_w(x_\varepsilon)\| \le \varepsilon$, $\varepsilon > 0$ any" etc., are called **variational principles (perturbed).** The adjective "perturbed" is imposed by the fact that not the function φ is minimized, but a function of the form $\varphi + \varepsilon\|(\cdot) - x_\varepsilon\|$ (Ekeland, [1]), or of more general form $\varphi + \varepsilon \sum_{n=1}^{\infty} \mu_n d(\cdot, v_n)^2$ (Borwein - Preiss, [2]), or of even more general form $\varphi + f$ (Deville - Godefroy - Zizler, [3]), f having some given properties (the second term is the *perturbation function*).

[1] Assoc. Prof., Department of Mathematical Methods and Methods, Faculty of Applied Sciences, University POLITEHNICA of Bucharest, Romania, e-mail: irina.meghea@upb.ro

The Ekeland principle is a perturbed variational principle discovered in 1972 ([1]) and nowadays, after more than 30 years, it was proved to be the foundation of the modern Variational Calculus (see, for instance, the minimax theorems in Banach spaces or in the Finsler manifolds, in which the key step of demonstration is made by the application of the Ekeland principle). As referring to the applications, these are numerous and diverse: the geometry of Banach spaces, nonlinear analysis, differential equations and partial differential equations, global analysis, probabilistic analysis, differential geometry, fixed point theorems, nonlinear semigroups, dynamical systems, optimization, mathematical programming, optimal control.

We cannot close this part of Introduction without the confession of Ekeland ([4]):

"The grandfather of these all is the celebrated 1961 theorem of Bishop and Phelps that the set of continuous linear functionals on a Banach space E, which attain their maximum on a prescribed non void closed convex bounded subset $X \subset E$, is norm-dense in E^{*}".

This Bishop-Phelps theorem can be found in [5].

A special kind of perturbed variational principle is Ghossoub-Maurey linear principle being one of the Ghossoub-Maurey theorems which can be found in [6]. In this paper, we apply this result to generalize at p-Laplacian a minimization problem from [7]. The author gave in [8] another generalization to p-pseudo-Laplacian of the same minimization problem from [7].

As regarding the p-Laplacian, its huge importance is topical and we can see this fact in [9], [10] and many other recent works. Problems involving p-Laplace operator are subject of intensive studies as they illustrate very well many of phenomena that occur in nonlinear analysis. Among their applications are singular and nonsingular boundary value problems which appear in various branches of mathematical physics. They arise as a model example in the fluid dynamics [11], [12], glaciology [13]; stellar dynamics [14]; in the theory of electrostatic fields [15]; in the more general context in quantum physics ([16], [17]); in the nonlinear elasticity theory as a basic model ([18], [19]); and many others (see e.g. [20]). The partial derivatives equations (PDEs) involving p-Laplacian are considered in differential geometry in the study of critical points for p-harmonic maps between Riemannian manifolds ([21], [22]) and the eigenvalue problems for p-Laplacian on Riemannian manifolds serve for estimations of the diameter of the manifolds [23]. Eigenvalue problems involving p-Laplacian are applied in functional analysis to derive sharp Poincaré and Writinger type inequalities ([24], [25]), Sobolev embeddings and isoperimetric inequalities ([26]), [27], [20]). Geometric properties of p-harmonic functions play significant role in the theory of Carnot-Carathéodory groups like Heisenberg group (see e.g. [28]) and in the analysis on metric spaces (see [29], [30] and references therein).

2. About Ghossoub-Maurey linear principle

In this section, we present the Ghoussoub-Maurey linear principle and highlight some results from which it is obtained together with some auxiliary theorems and propositions with necessary comments.

Definition. Let X be real normed space and C, D with $C \subset D$ nonempty subsets of X^*. C is *strict* w-H_δ *set in* D resp. *strict* w^*-H_δ *set in* D if

$$D \setminus C = \bigcup_{n=1}^{\infty} K_n \,, \tag{1}$$

$\operatorname{dist}(K_n, C) > 0$, K_n convex and weakly compact resp. $*$-weakly compact.

For instance,

Proposition 1 *Any nonempty closed set* C *of a separable reflexive space* X, $C \neq X$, *is strict* w-H_δ *set in* X.

In particular, *if* $\varphi : X \to (-\infty, +\infty]$ *is l. s. c. and proper*, *then the epigraph of* φ *in* $X \times \mathbf{R}$ *is strict* w-H_δ *set in* $X \times \mathbf{R}$.

Let X be reflexive space and C, D subsets of X^*, $C \subset D$. Set

$$M(C, D) := \{x \in X : \exists\, \xi \in C \text{ so that } Jx(\xi) \geq Jx(\eta)\ \forall\, \eta \in D\},$$

otherwise expressed, $M(C, D)$ is the set of x from X for which Jx is upper bounded on D and the least upper bound is attained at a point of C, J the Hahn imbedding of X in X^{**}. So, X being reflexive, J is an isomorphism of vector spaces which preserves the norms.

In the following, *to abridge writing*, *sometimes* x *designates* Jx.

Retain that if C is *-weakly compact, $M(C, D)$ is closed.

Notations. $B_X(x_0, r) \equiv$ the closed ball centred in x_0 of radius r in the normed space X.

$B_X \equiv B_X(0,1)$.

$\overline{E}^* \equiv$ the closure of the subset E from X^* for the *-weak topology.

Pass to the auxiliary propositions.

Proposition 2 *Let* X *be reflexive space*, $D \subset X^*$ *and* $K \subset D$, K *convex* *-*weakly compact*. *If*

$$B_X(x, \alpha) \subset M(K, D),$$

then, *for any* $\varepsilon > 0$,

$$B(D, Jx, \varepsilon) \subset K + \frac{\varepsilon}{\alpha} B_{X^*}.$$

In particular, *when* $C \subset D \subset \overline{\operatorname{conv}}^* C$, *we have*

$$\operatorname{dist}(K, C) = 0.$$

Proposition 3 *Let* X *be reflexive space*, $C \subset X^*$ *nonempty and* $U \subset X$ *nonempty open having the property*

$$\sup Jx(C) < +\infty \ \forall x \in U.$$

Then Jx, *for any* x *from* U, *is upper bounded on* $\overline{\text{conv}}\ ^*$C *and attains its least upper bound*.

Proof. Set $D := \overline{\text{conv}}\ ^*C$. We have

$$\sup Jx\,(C) = \sup Jx\,(D) \qquad (2)$$

[$\xi \in \text{conv}\ C \Rightarrow \xi = \lambda_1 \xi_1 + \lambda_2 \xi_2$, ξ_1, $\xi_2 \in C$, $\lambda_1 + \lambda_2 = 1$, λ_1, $\lambda_2 \geq 0 \Rightarrow Jx\,(\xi) = \lambda_1 Jx\,(\xi_1) + \lambda_2 Jx(\xi_2) \leq \sup Jx\,(C)$; $\xi \in D \Rightarrow \exists\ \xi_n \in \text{conv}\ C$, $\xi_n \xrightarrow{*\text{-weak}} \xi \Rightarrow \xi_n(x) \to \xi(x)$, $\xi_n(x) \leq \sup Jx\,(C)\ \forall n \geq 1$, hence $\xi(x) \leq \sup Jx\,(C)$]
and so (3) $\sup Jx\,(D) < +\infty\ \forall x \in U$, the first assertion is proved.
Pass to the second assertion. Fix x from U, $\exists\ \varepsilon > 0$ with $x + \varepsilon z \in U\ \forall z$ in B_X. Then

$$\sup J(x + \varepsilon z)(D) = \sup\,(Jx + \varepsilon Jz)(D) < +\infty\ \forall z \in B_X\ ((3)),$$

consequently, using once again (3),

$$\sup Jz\,(D) < +\infty\ \forall z \in B_X, \qquad (4)$$

which implies (5) $\sup Jz\,(D) < +\infty\ \forall\ z \in X$ [for z any fixed, $z \neq 0$ replace in (4) z by $\frac{z}{\|z\|}$, $Jy\,(\xi) = \xi(y)$]. Replacing z by $-z$ in (5) one finds (6) $\inf Jz\,(D) > -\infty\ \forall z \in X$. But X is reflexive, hence (5) and (6) express that D is weakly bounded, consequently D is even bounded. D being also *-weak closed, it is *-weak compact ([47], pp. 144), hence the conclusion by applying Weierstrass theorem. □

Remark. The proposition 3 is Lemma 2.7 from [6], Ch. 2. The proof of this one cannot be considered here being not correct. For this reason, the author gave in this place the approipriate demonstration.

Proposition 4 *Let* X *be reflexive space*, $C \subset X^*$ *nonempty and* U *nonempty open from* X *so that*

$$\sup Jx\,(C) < +\infty\ \forall x \in U.$$

If C *is strict* w*-H_δ *set in* $D := \overline{\text{conv}}^*C$, *then the set* $V := \{x \in U : Jx$ *attains* $\sup Jx\,(D)$ *in* $C\}$ *includes a set of* G_δ *type*[2] *dense in* U.

Proposition 5 *Let* X *be reflexive space*, C *subset of* X^* *separable*, $D := \overline{\text{conv}}^*C$ *and* U *nonempty open subset of* X *so that* $\sup Jx\,(C) < +\infty\ \forall x \in U$. *Suppose that* M(C, D) *includes a dense and of* G_δ *type subset of* U.

Then, *for any* $K \subset D$ *-*weakly compact with* $K \cap C = \varnothing$ *and for any* $\varepsilon > 0$, *the set* $G(K, \varepsilon) := \{x \in U : \exists\ r > 0$ *so that* $\overline{B}^*(D, Jx, r) \cap K = \varnothing$ *and* diam $\overline{B}^*(D, Jx, r) < \varepsilon\}$ *is open and dense in* U.

Theorem. *Let* X *be reflexive space*, C *separable subset of* X^* *which is strict* w*-H_δ *set in* $D := \overline{\text{conv}}^*C$ *and* U *open subset of* X *so that* $\sup Jx\,(C) < +\infty\ \forall x \in U$. *Then*

(I) *The set* $\{x \in U : Jx$ *strongly exposes* D *from above at a point of* $C\}$ *is of* G_δ *type and dense in* U;

[2] A set of G_δ type means a set which is a countable intersection of open sets. A set of F_δ type means a set which is a countable union of closed sets.

(II) *If* $\varphi : C \to (-\infty, +\infty]$ *is proper lower semicontinuous and* $\varphi + Jx$ *is*, $\forall x$ *from* X, *bounded from below on* C, *then the set* $\{x \in X : \varphi + Jx$ *strongly exposes* C *from below*$\}$ *is of* G_δ *type and dense in* X ([6]).

Proof. (I). According to the hypothesis, $D \setminus C = \bigcup_{n=1}^{\infty} K_n$, K_n convex *-weakly compact, $\mathrm{dist}(K_n, D) > 0$. $M(C, D)$ includes a subset of G_δ type dense in U (Proposition 4), but then, for each n from **N**, the set $V_n := G(K_1 \cup K_2 \cup ... \cup K_n, \frac{1}{n})$ is open and dense in U (Proposition 5), consequently $\bigcap_{n=1}^{\infty} V_n$ is dense in U (relativized Baire theorem) and it remains only to observe that $\bigcap_{n=1}^{\infty} V_n = \{x \in X : Jx$ strongly exposes D from above at a point of $C\}$.

(II). $C \times \mathbf{R}$, separable subset of $X^* \times \mathbf{R}$, is a strict w*-H_δ set in $D \times \mathbf{R}$ and then, φ being l. s. c., the epigraph epi φ in $C \times \mathbf{R}$ (nonempty set, φ is proper) is a strict w*-H_δ set in $D \times \mathbf{R}$ and hence also in $\overline{\mathrm{conv}}^{\,*}\mathrm{epi}\,\varphi$. $W := \{(x, \alpha) : x \in X, \alpha < 0\}$ is open in $X \times \mathbf{R}$ and $\sup (Jx, \alpha)(\mathrm{epi}\,\varphi) < +\infty$ $\forall (x, \alpha) \in W$ [(Jx, α), continuous linear functional, acts on $C \times \mathbf{R}$ by the rule $(Jx, \alpha)(\xi, \lambda) = Jx(\xi) + \alpha\lambda$]. Indeed, $Jx(\xi) + \alpha\lambda \le Jx(\xi) + \alpha\varphi(\xi)$, $\exists\, a$ in **R** so that $J\frac{x}{\alpha}(\xi) + \varphi(\xi) \ge a$ $\forall \xi \in C$ (the hypothesis), hence $Jx(\xi) + \alpha\varphi(\xi) \le \alpha a$ $\forall \xi \in C$. Show that for each $\varepsilon > 0$ $\exists\, y_0$ with $\|y_0\| \le 2\varepsilon$ and $\varphi + Jy_0$ strongly exposes C from below, which is enough to validate (II). Apply (I), $\exists\, (x_\varepsilon, \alpha_\varepsilon)$ in W so that

$$\|(x_\varepsilon, \alpha_\varepsilon) - (0, -1)\| \le \varepsilon \qquad (7)$$

($(0, -1) \in W$!) and $(Jx_\varepsilon, \alpha_\varepsilon)$ strongly exposes epi φ from above in a point (ξ_0, λ_0). Then $\forall (\xi, \lambda)$ from epi φ with $\xi \neq \xi_0$ we have

$$Jx_\varepsilon(\xi_0) + \alpha_\varepsilon \lambda_0 > Jx_\varepsilon(\xi) + \alpha_\varepsilon \lambda,$$

consequently taking $y_0 := \frac{x_\varepsilon}{\alpha_\varepsilon}$ we have in particular

$$\varphi(\xi_0) + Jy_0(\xi_0) < \varphi(\xi) + Jy_0(\xi) \ \forall \xi \in C \setminus \{\xi_0\},$$

ξ_0 is a strict global minimum point for $\varphi + Jy_0$. Moreover, as it can suppose $\varepsilon < \frac{1}{2}$, we have $\|y_0\| = \frac{\|x_\varepsilon\|}{|\alpha_\varepsilon|} \le 2\varepsilon$, because, *via* (73), $\|x_\varepsilon\| \le \varepsilon$ and $|\alpha_\varepsilon + 1| \le \varepsilon$, hence $\alpha_\varepsilon \in \left(-\frac{3}{2}, -\frac{1}{2}\right)$.

Finally, let $(\xi_n)_{n\geq 1}$ be a minimizing sequence for $\varphi + Jy_0$ on C, then $(\xi_n, \varphi(\xi_n))_{n\geq 1}$ is maximizing sequence for $(Jx_\varepsilon, \alpha_\varepsilon)$ which strongly exposes epi φ in (ξ_0, λ_0), which imposes $\xi_n \to \xi_0$. □

Pass to the linear perturbed variational principle.

Definition. Let X be a real normed space, $f : X \to (-\infty, +\infty]$, C nonempty subset of X and $x_0 \in C$. f *strongly exposes* C *from below in* x_0, when

1° $f(x_0) = \inf f(C) < +\infty$

and

2° $x_n \in C\ \forall n \geq 1,\ f(x_n) \to f(x_0) \Rightarrow x_n \to x_0$.

In the same manner define

f *strongly exposes* C *from above in* x_0.

Ghoussoub-Maurey linear principle. *Let* X *be reflexive separable space and* φ: $X \to (-\infty, +\infty]$ *lower semicontinuous and proper.*

(I) *If* φ *is bounded from below on the closed bounded nonempty subset* C, *the set*

$$\{\xi \in X^* : \varphi + \xi \text{ strongly exposes C from below}\}$$

is of G_δ *type and everywhere dense.*

(II) *If, for any* ξ *from* X^*, $\varphi + \xi$ *is bounded from below*, *the set*

$$\{\xi \in X^* : \varphi + \xi \text{ strongly exposes X from below}\}$$

is of G_δ *type and everywhere dense.*

Ghoussoub-Maurey linear principle devolves from the above theorem, which is more general and from the above auxiliary propositions.

Proof. (I). Set $Y := X^*$, a separable reflexive space ([31], pp.162). Then $Y^* = X$ (identification *via* the Hahn imbedding, X is reflexive). C is separable and strict w*-H_δ set in Y^* (Proposition 1, the weak and *-weak topologies coincide) and hence also in $D := \overline{\text{conv}}^* C$ ($X \setminus C = \bigcup_{n=1}^{\infty} K_n$ with the properties from (1), take the intersection with D). Apply (II) from the above Theorem transcribed with Y replaced by X, this is correct as $\varphi + \xi$, $\xi \in X^* = Y^{**}$, is bounded from below ($|\xi(x)| \leq \|\xi\|\ \|x\|$ and C is bounded).

(II). The epigraph epi φ of φ in $X \times \mathbf{R}$ is strict w-H_δ set in $X \times \mathbf{R}$ (Proposition 1). In the following use the proof for Theorem, assertion (II), begining from (7), epi φ is that considered above. □

Corollary. *Let* X *be reflexive space*, C *a subset of* X^* *separable bounded strict* w*-H_δ *set in* $D := \overline{\text{conv}}^* C$ *and* $\varphi : X \to (-\infty, +\infty]$ *bounded from below l.s.c. proper*. *For any* $\varepsilon > 0$ *there exists* x_0 *in* X *with* $\|x_0\| \leq \varepsilon$ *and* ξ_0 *in* C *so that*

1° $(\varphi + Jx_0)(\xi_0) < (\varphi + Jx_0)(\xi)\ \forall \xi \in C \setminus \{\xi_0\}$,

2° *Any minimizing sequence from* C *for* $\varphi + Jx_0$ *converges to* ξ_0 ([6]).

Proof. C bounded implies $\varphi + Jx$ bounded from below $\forall x \in X$, consequently II, Theorem can intercede to obtain 1° and 2°. □

3. Application of Ghossoub-Maurey linear principle in a minimization problem

We imply Ghoussoub-Maurey linear principle in a minimization problem of the form [7]:

$$C_f := \min\{\frac{1}{p}\|u\|_{1,p}^{p} - \int_\Omega f(u)dx \colon u \in W_0^{1,p}(\Omega), \|u\|_{2^*} = 1\}, \qquad (8)$$

where Ω is an open set of C^1 class in $\mathbf{R}^N$, $N \geq 3$, $\|u\|_{1,p} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm $u \to \left(\|u\|_{L^p(\Omega)}^p + \sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}$ and it is equivalent to the norm $u \to \||\nabla u|\|_{L^p(\Omega)}$, $f \in W^{-1,p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, is the dual of $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$, $2^* = \frac{2N}{N-2}$ – the critical exponent for the Sobolev imbedding.

Let Ω be an open bounded set of C^1 class in $\mathbf{R}^N$, $N \geq 3$. Consider the problem

$$\begin{cases} -\Delta_p u(x) = f(x, u(x)) \text{ on } \Omega \\ u(x) = 0 \text{ on } \partial\Omega \end{cases}, \qquad (9)$$

where $f \colon \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with the growth condition

$$|f(x, s)| \leq c|s|^{p-1} + b(x), \qquad (10)$$

$$c > 0,\ 2 \leq p \leq \frac{2N}{N-2},\ b \in \mathbf{L}^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1.$$

The functional $\varphi \colon W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F(x, u(x))dx \qquad (11)$$

with

$$F(x, s) := \int_0^s f(x,t)\,dt,$$

is of C^1 - Fréchet class and its critical points are the weak solutions of (9).

Let λ_1 be the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ with homogeneous boundary conditions. We have

$$\lambda_1 = \inf \left\{ \frac{\| u \|_{1,p}^{p}}{| i(u) |_{0,p}^{p}} : \ u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\} \tag{12}$$

is the Rayleigh - Ritz quotient and $i : W_0^{1,p}(\Omega) \to \mathbf{L}^p(\Omega)$.

And now we can give an answer for (8).

Proposition 6. *Let be the above assumptions fulfilled and furthermore the growth condition*

$$F(x, s) \le c_1 \frac{s^p}{p} + \alpha(x)s, \tag{13}$$

$0 < c_1 < \lambda_1$ *and* $\alpha \in \mathbf{L}^{q'}(\Omega)$ *for some* $2 \le q \le \frac{2N}{N-2}$ *is verified.*

Then the following assertions hold:

1° *The set of functions* h *from* $W^{-1,p'}(\Omega)$, *having the property*:

the functional $\varphi_h : W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi_h(u) = \frac{1}{p} \| u \|_{1,p}^{p} - \int_\Omega \left[F\left(x, u(x)\right) + h\left(u(x)\right) \right] dx$$

has an attained minimum in only one point,

includes a G_δ *set everywhere dense*;

2° *The set of functions* h *from* $W^{-1,p'}(\Omega)$, *having the property*:

the problem $\begin{cases} -\Delta_p u = f(x, u) + h(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$ *has solutions*,

includes a G_δ *set everywhere dense*;

3° *Moreover*, *if* $s \to f(x, s)$ *is increasing*, *then the set of functions* h *from* $W^{-1,p'}(\Omega)$, *having the property*:

the problem $\begin{cases} -\Delta_p u = f(x, u) + h(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$ *has a unique solution*,

includes a G_δ *set everywhere dense.*

Clarification. This result is a generalization for *p*-Laplacian of the theorem 2.13 from [6] where the problem is proposed with Laplace operator in $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

Proof. It is sufficient to justify 1°. For each *h* from $W^{-1,p'}(\Omega)$, consider the functional ξ_h defined on $W_0^{1,p}(\Omega)$,

$$\xi_h(u) = \int_\Omega h(u(x))\, dx .$$

Observe that $\varphi_h = \varphi + \xi_h$ (see (11)). Consequently, according to (II) of Ghoussoub-Maurey linear principle, if we show that φ_h is bounded from below for any *h* from $W^{-1,p'}(\Omega)$ (this is enough, representation Riesz theorem), then 1° is proved. But, taking into account the Sobolev imbedding and (13), we have $\forall u \in W_0^{1,p}(\Omega)$,

$$(\varphi + \xi_h)(u) = \frac{1}{p}\|u\|_{1,p}^{p} - \int_\Omega F(x,u(x))dx - \int_\Omega h(u(x))dx \geq \frac{1}{p}\|u\|_{1,p}^{p} -$$

$$\left|\int_\Omega F(x,u(x))dx\right| - \|h\|_{W^{-1,p'}}\|u\|_{1,p}^{p} \geq \frac{1}{p}\|u\|_{1,p}^{p} - \int_\Omega |F(x,u(x))|dx -$$

$$\|h\|_{W^{-1,p'}}\|u\|_{1,p} \geq \frac{1}{p}\|u\|_{1,p}^{p} - \frac{c_1}{p\lambda_1}\|u\|_{1,p}^{p} - K\|u\|_{1,p} - \|h\|_{W^{-1,p'}}\|u\|_{1,p} =$$

$$\|u\|_{1,p}\left[\frac{1}{p}\left(1-\frac{c_1}{\lambda_1}\right)\|u\|_{1,p}^{p-1} - K - \|h\|_{W^{-1,p'}}\right] = \|u\|_{1,p}\left[\frac{1}{p}\left(1-\frac{c_1}{\lambda_1}\right)\|u\|_{1,p}^{p-1} - r\right],$$

$r \in \mathbf{R}$, and hence the conclusion because $1 - \frac{c_1}{\lambda_1} > 0$. For other justifications,

$\left|\int_\Omega F(x,u(x))dx\right| \leq \int_\Omega\left(c_1\frac{|u(x)|^p}{p} + \alpha(x)u(x)\right)dx = \frac{c_1}{p}\|i(u)\|_{0,p}^{p} + \int_\Omega \alpha(x)u(x)dx \leq \frac{c_1}{p\lambda_1}\|u\|_{1,p}^{p} + \|\alpha\|_{0,q'}\|u\|_q \leq \frac{c_1}{p\lambda_1}\|u\|_{1,p}^{p} + K\|u\|_{1,p}$ (see q and properties of Sobolev spaces, for instance, in [32]; λ_1 from (12)). $\int_\Omega h(u(x))dx \leq \|h\|_{W^{-1,p'}}\|u\|_{1,p}$ (Riesz representation theorem). □

4. Applications in modeling of some real phenomena

In this section, we involve the result from the above section in the demonstration of the existence and uniqueness of the solutions of some mathematical physics problems issued from modeling real phenomena.

To illustrate our results, we can apply these, for instance, for the problem which appears in astrophysics [14] in relation to Matukuma equations (developed in 1930' to describe the dynamics of a globular cluster of stars); as well as in physical phenomena related to equilibria of anisotropic continuous media. As the model, we can consider the radial solution of the nonlinear eigenvalue problem ([33]):

$$\begin{cases} -\operatorname{div}\left(|\nabla w(x)|^{p-2}\nabla w(x)\right) = \lambda|\nabla w(x)|^{p-2}w(x) \text{ a.e. in } B \\ w = 0 \text{ on } \partial B \end{cases}$$

with B an arbitrary ball.

The conditions from Proposition 6 proven in the previous Section 3, can be *con brio* fulfilled for the above problem and for a bit complicated data imposing some small additional conditions if it is necessary.

Consider also as an example of application of the result from the last section the connection between tug-of-war games in game theory (replacing the role of Brownian motion) and equations involving ∞ or p-Laplacian. This gives a way to a deterministic game interpretation using the equations with p-Laplacian ([34]).

Minimization principles form one of the most wide-ranging means of formulating mathematical models governing the equilibrium configurations of physical systems. Minimization problems that can be analyzed by the calculus of variations serve to characterize the equilibrium configurations of almost all continuous physical systems, ranging through elasticity, solid and fluid mechanics, electro-magnetism, gravitation, quantum mechanics, string theory, and many, many others. Many geometrical configurations, such as minimal surfaces, can be conveniently formulated as optimization problems. Moreover, numerical approximations to the equilibrium solutions of such boundary value problems are based on a nonlinear finite element approach that reduces the infinite-dimensional minimization problem to a finite-dimensional problem ([35]).

We can discuss also on the interesting quasilinear elliptic problem:

$$-\Delta_p u + V(|x|)|u|^{p-2}u = h(u),\ x \in \Omega,$$
$$u = 0 \text{ on } \partial\Omega,$$

where Ω is an open set of C^1 class in $\mathbf{R}^N$, which is a model for describing the stationary state of reaction-diffusion equations in population dynamics, as also plasma physics, condensed matter physics and in cosmology. The existence of solution of this last equation has been studied extensively on $\mathbf{R}^N$ by modern variational methods under various hypotheses on the singular potential V and the nonlinearity h ([9]).

To highlight the application of Proposition 6 to prove the existence and the uniqueness of the solution of the above problem, let us re-write the equation of this one:

$$-\Delta_p u = f(x, u) + h(u) \text{ a. e. on } \Omega,$$

where $f(x, s) = -V(|x|)|s|^{p-2}s$. Observe that f fulfills the growth condition (Section 3, (10)) and $F(x, s) := \int_0^s f(x,t)\,dt$ fulfills the conditions of the cited proposition. The first assertion of Proposition 6 suggests that one can obtain a solution of the mathematical physics equation by using a numerical approach. The assertions 2° and 3° specify the existence and existence together with the uniqueness respectively in relation with properties of h and an additional property of f for the uniqueness.

5. Conclusion

This paper starts with a theoretical part related to Ghoussoub-Maurey linear principle. The series of results to obtain this perturbed linear variational principle is presented here in order to highlight the improvement of one of the propositions from this sequence and to show the basis of this construction.

The novelty of this work consists in obtaining a generalization of a minimization problem from the Laplacian to p-Laplacian by using Ghoussoub-Maurey linear principle. In the third section, it is also proved the existence and the uniqueness of the solution of a kind of Dirichlet problem involving the p-Laplacian.

In the fourth section, we propose some applications of the theorem from the previous section in the demonstration of the existence and the uniqueness of the solution of some problems of mathematical physics with p-Laplacian issued from modeling of real phenomena such as astrophysics, equilibria of anisotropic continuous media, game theory, solid and fluid mechanics, and others. The solution (existence and uniqueness) for a problem describing the stationary state of reaction-diffusion equations in population dynamics, as also plasma physics is developed by using the proposition obtained in this paper by using Ghoussoub-Maurrey linear principle.

REFERENCES

[1]. *I. Ekeland*, „On the variational principle", Cahiers de mathématique de la décision, No. 7217, Université Paris, 1972.

[2]. *J. M. Borwein and D. Preiss*, „A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions", in Trans. Amer. Math. Soc., **vol. 303**, No. 2, 1987, pp. 517-527.

[3]. *R. Deville*, *G. Godefroy and V. Zizler*, „A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions", in J. Func. Anal., **111**, 1993, pp. 197-212.

[4]. *I. Ekeland*, „Nonconvex minimization problems", in Bull. Amer. Math. Soc. **1**, 1979, pp. 443-474.

[5]. *Irina Meghea*, Ekeland variational principle with generalizations and variants, Old City Publishind, Philadelphia, Éditions des Archives Contemporaines, Paris, 2009.

[6]. *N. Ghoussoub*, Duality and perturbation methods in critical point theory, Cambridge University Press, 1993.

[7]. *H. Brezis and L. Nirenberg*, „A minimization problem with critical exponent and non-zero data", in Symmetry in Nature, Scuola Norm. Sup. Pisa, 1989, pp. 129-140.

[8]. *Irina Meghea*, "On some perturbed variational principles: connexions and applications", Rev. Roum. Math. Pures Appl., **54**, 5-6, 2009, pp. 493-511.

[9]. *Y. Su and Z. Feng*, "Lions-type theorem of the p-Laplacian and applications", in Adv. Nonlinear Anal., **10**, 2021, pp. 1178-1200.

[10]. *M. Yano*, AER1418: Variational methods for PDEs, Lecture Notes, University of Toronto, 2020.

[11]. *M. Ružička*, Electrorheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Mathematics, **vol. 1748**, Springer, Berlin, Germany, 2000.

[12]. *Z. Wu, J. Zhao and H. Li*, Nonlinear Diffusion Equations, World Scientific, Singapore, 2001.

[13]. *D. Arcoya, J. Diaz and L. Tello*, "S-shaped bifurcation branch in a quasilinear multivalued-model arising in climatology", in J. Differential Equations, **150**, 1998, pp. 215–225.

[14]. *N. Kawano, E. Yanagida and S. Yotsutani*, "Structure theorems for positive radial solutions to $\mathrm{div}(|Du|^{m-2}Du) + K(|x|)u^q = 0$ in $\mathbf{R}^n$", in J. Math. Soc. Japan, **45**, 1993, pp. 719–742.

[15]. *D. Fortunato, L. Orsina and L. Pisani*, "Born-Infeld type equations for electrostatic fields", in J. Math. Phys., **43**, 2002, pp. 5698–5706.

[16]. *V. Benci, P. D'Avenia*, *D. Fortunato and L. Pisani*, "Solitons in several space dimensions: Derrick's problem and infinitely many solutions", in Arch. Rat. Mech. Anal. **154**, 2000, pp. 297-324.

[17]. *V. Benci, D. Fortunato and L. Pisani*, "Soliton like solutions of a Lorentz invariant equation in dimension 3", in Rev. Math. Phys., **10**, 1998, pp. 315–344.
[18]. *J. M. Ball*, "Some Open Problems in Elasticity, Geometry, Mechanics and Dynamics", Springer, New York, 2002.
[19]. *L. D'Onofrio and T. Iwaniec*, "The p-harmonic transform beyond its natural domain of definition", in Indiana Univ. Math. J., **53**, 2004, pp. 683–718.
[20]. *B. Kawohl*, Rearrangements and convexity of level sets in PDE, Lecture Notes in Mathematics, **vol. 1150**, Springer-Verlag, Berlin-Heidelberg, 1985.
[21]. *A. Cianchi and A. Ferone*, "On symmetric functionals of the gradient having symmetric equidistributed minimizers", in SIAM J. Math. Anal., **38**, 2006, pp. 279–308.
[22]. *M. C. Leung*, "Positive solutions of second order quasilinear equations corresponding to *p*-harmonic maps", in Nonlinear Anal., **31**, 1998, pp. 717–733.
[23]. *J. F. Grosjean*, "*p*-Laplace operator and diameter of manifolds", in Ann. Global Anal. Geom., **28**, 2005, pp. 257–270.
[24]. *P. Drábek and R. Manásevich*, "On the closed solution to some nonhomogeneous eigenvalue problem with *p*-Laplacian", in Differential Integral Equations, **12**, 1999, pp. 773–788.
[25]. *P. Drábek and P. Takáč*, "Poincaré inequality and Palais-Smale condition for the *p*-Laplacian", in Calc. Var. PDE, **29**, 2007, pp. 31-58.
[26]. *M. Belloni, V. Ferrone and B. Kawohl*, "Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators", in Z. Angew. Math. Phys., **54**, 2003, pp. 771-783.
[27]. *L. Damascelli and F. Pacella*, "Monotonicity and symmetry results for *p*-Laplace equations and applications", in Adv. Diff. Equ., **5**, 2000, pp. 1179-1200.
[28]. *Y. Li and W. M. Ni*, "Radial symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$", in Comm. Partial Differential Equations, **18**, 1993, pp. 1043-1054.
[29]. *H. Aikawa and N. Shanmugalingam*, "Hölder estimates of *p*-harmonic extension operators, in J. Diff. Equ., **220**, 2006, pp. 18-45.
[30]. *A. Björn, J. Björn and N. Shanmugalingam*, "The Dirichlet problem for *p*-harmonic functions on metric spaces", in J. Reine Angew. Math., **556**, 2003, pp. 173-203.
[31]. *C. Meghea*, *Irina Meghea*, "Treatise on Differential calculus and Integral calculus for mathematicians, physicists, chemists and engineers in ten volumes", **vol. X**, Old City Publishing, Philadelphia, Éditions des Archives Contemporaines, Paris, 2015.
[32]. *Irina Meghea*, Surjectivité et théorèmes type alternative de Fredholm pour les opérateurs de la forme $\lambda J_\varphi - S$ et quelques problèmes de Dirichlet. In press.
[33]. *T. Adamowicz and Agnieszka Kałamajska*, "On a variant of the maximum principle involving radial *p*-Laplacian with applications to nonlinear eigenvalue problems and nonexistence results", in Top. Meth. In Nonlinear Anal., vol. **34**, 2009, pp. 1-20.
[34]. *L. Diening*, *P. Lindqvist and B. Kawohl*, The *p*-Laplacian operator and applications, Report No. 08/2013, DOI: 10.4171/OWR/2013/08.
[35]. *P. Olver*, The calculus of variations, School of Mathematics, University of Minnesota, Minneapolis.