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HIGHER ORDER GENERALIZED VARIATIONAL INEQUALITIES AND NONCONVEX OPTIMIZATION

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In this paper, we define and consider some new concepts of the higher order generalized convex functions involving two arbitrary functions. Some properties of the generalized convex functions are investigated under suitable conditions. Parallelogram laws are obtained as novel applications of the generalized convex functions characterization of the Banach spaces. It is shown that the optimality conditions of the generalized convex functions are characterized by a class of variational inequalities, which is called the higher order generalized variational inequality. Auxiliary principle technique is used to suggest and analyze some iterative methods for finding the approximate solutions of the higher order general-ized variational inequalities. Convergence criteria of the proposed methods is investigated under mild conditions. Some special cases also discussed as applications of the main results. Results obtained in this paper can be viewed as refinement and improvement of previously known results.

1. **Introduction**

Convexity theory has been generalized and extended in several directions using novel and innovative techniques and ideas to tackle unrelated complicated problems, which occur in different branches of engineering, economics, regional, physics, finance, mathematical science, management sciences and optimization. Polyak [45] introduced the concept of strongly convex functions in optimization, which played the crucial role in the existence theory of complementarity problems, see Karamardian [11]. Noor et al.[22, 25, 26, 28, 29, 31, 32, 33, 34, 37, 40] and Mohsen et al. [14] introduced the concept of higher order strongly convex functions and studied their properties. These results can be viewed as significant refinement of the results of Lin and Fukushima [12] and Alabdali et al. [1] for higher order strongly (uniformly) convex functions. With appropriate choice of non-negative arbitrary functions, one can obtain various known and new classes of convex functions. For the properties of the higher order convex functions and their variant forms, see Adamek [2]and Nikodem et al. [16]. Related to the convexity theory, we have the variational inequality theory, which was introduced by Stampacchia [47] in potential theory. Stampacchia [47] proved that the optimality conditions of differentiable convex functions on the convex set can be represented by the variational inequalities. It is amazing that this simple fact have influenced almost all the branches of pure and applied sciences. Lions and Stampacchia [13] studied the existence of the solution of variational inequalities using the auxiliary principle technique. Variational inequalities contain the Reisz-Frechet representation theorems and Lax-Milgram Lemma as special cases, see Noor et al. [38]. Noor [17] had shown that the minimum of the difference of two convex functions(known as DC-Problem) cam be characterized by the nonlinear variational inequalities and applications in Binghums fluid. General variational inequalities were introduced and investigated by Noor [18] in 1988. It have been shown in [19, 20, 21] that odd order, nonsymmetric and nonpositive obstacle boundary value can be studied in

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the unified framework of the general variational inequalities.

In several cases, the underlying the set may not be a convex set. To overcome this drawback, a set can be made convex function with respect to some arbitrary functions. Jian [10] introduced generalized convex sets and generalized convex functions involving two arbitrary functions. For different appropriate and suitable choice, one can obtain the general convex functions of Youness [51] and Noor [12]. For the applications of general convex sets, see Cristescu at al [7]. Noor [24] proved that the optimality conditions of the differentiable generalized convex functions on the generalized convex set is characterized by the generalized variational inequality. It have shown that various invariants forms of general variational inequalities, considered by Noor [18, 21, 23] can be deduced as special cases of generalized variational inequality.
Continuing these research activities in the convexity theory, we introduce some new classes of the higher order generalized convex functions with respect to two arbitrary functions. Higher order convex generalized convex functions are quite and distinct from the one considered in [25, 26, 27, 28, 29, 31, 32, 33, 34, 40]. This paper is continuous of our recent research in this area. Several new concepts of monotonicity are introduced. As novel applications of higher order generalized convex functions, we derive various parallelogram laws which can be used to characterize the Banach spaces. It is shown that the parallelogram laws for Banach spaces, which are due to Xu [50], Bynum [4] and Chen et al [5, 6], can be obtained as special cases. We have shown that the minimum of a differentiable higher order generalized convex functions on the generalized convex set is characterized by a class of variational inequality. This result inspired us to consider higher order generalized variational inequalities. We have used the technique of the auxiliary principle to suggest some new implicit methods for solving HOGVI. Convergence analysis of the proposed method is investigated under pseudo-monotonicity, which is a weaker condition than monotonicity. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. **Formulations and basic facts**

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the inner product and norm, respectively.

Definition 2.1. [10, 24]. *The set $K \subseteq H$ is said to be a generalized convex set, if there are two functions h and g such as*

$$(1-t)h(u) + tg(v) \in K; \quad \forall u, v \in K, t \in [0;1].$$

We now discuss some special cases of the generalized convex set $K \subseteq D$.
(I). If $g(u) = I(u) = u = h(u)$, the identity operator, then generalized convex set reduces to the classical convex set. Clearly every convex set is a generalized convex set, but the converse is not true.
(II). If $h(u) = I(u) = u$, then the generalized convex set becomes the g-convex set, that is,

Definition 2.2. *The set K is said to be g-convex set, if*

$$(1-t)u + tg(v) \in K \subseteq D, \quad \forall u, v \in K \subseteq D, t \in [0,1],$$

which was introduced and studied by Noor [23]. Cristescu et al [7] discussed various applications of the general convex sets related to the necessity of adjusting investment or development projects out of environmental or social reasons. For example, the easiest manner of constructing this kind of convex sets comes from the problem of modernizing the railway transport system. Shape properties of the general convex sets with respect to a projection are investigated.

(III). If $g(u) = I(u) = u$, then the generalized convex set becomes the h-convex set, that is,

Definition 2.3. *The set K is said to be h-convex set, if*

$$(1-t)h(u) + tv \in K \subseteq D, \qquad \forall u, v \in K \subseteq D, t \in [0,1],$$

which is mainly due to Noor [24].

For the sake of simplicity, we always assume that function $F : D \to R$ and $K \cup h(K) \cup g(K) \subseteq D$. If K is generalized convex set, then this condition becomes $K \subseteq D$.

We now introduce some new classes of higher order generalized convex functions, which is main aim of this paper.

Definition 2.4. *A function F on the generalized convex set $K \subseteq D$ is said to be higher order generalized convex with respect to two arbitrary functions h, g, if there exists a constant $\zeta > 0$, such that*

$$F(h(u) + t(g(v) - h(u))) \leq (1-t)F(h(u)) + tF(g(v))$$
$$-\zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D, t \in [0,1], p \geq 1.$$

A function F is said to higher order generalized concave , if and only if, $-F$ is a higher order generalized-convex function. If $t = \frac{1}{2}$ in Definition 2.4, then one gets the generalized Jensen type property called higher order generalized convex function.

(IV). If $g = I = h$, the identity operator, then Definition 2.4 reduces to:

Definition 2.5. *A function F on the convex set K is said to be higher order strongly convex function, if there exists a constant zeta > 0, such that*

$$F(u + t(v-u)) \leq (1-t)F(u) + tF(v) - \zeta\{t^p(1-t) + t(1-t)^p\}\|v - u\|^p,$$
$$\forall u, v \in K \subseteq D, t \in [0,1], p \geq 1,$$

which were introduced and studied by Mohsen et al [1]. It has been shown that higher order strongly convex functions can be viewed as a significant refinement of the concept introduced by Lin and Fukushima [2]. For the applications of higher order strongly convex functions, see [23, 24, 25] and the references therein.

Remark 2.1. *We would like to mention that for suitable and appropriate choice of the arbitrary functions g, h and parameters p, ζ one can obtain several new and previous known classes of convex functions. This show that the higher order generalized convex functions are unifying ones.*

If a function is both higher order generalized convex and concave, then

Definition 2.6. *A function F on the convex set K_g is said to be a higher order generalized affine convex convex function with respect to an arbitrary functions g, h, if there exists a constant $\zeta > 0$, such that*

$$F(h(u) + t(g(v) - h(u))) = (1-t)F(h(u)) + tF(g(v))$$
$$-\zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D, t \in [0,1], p \geq 1.$$

Definition 2.7. *A function F on the convex set K_g is said to be higher order generalized quasi convex with respect to an arbitrary functions g, h, if there exists a constant $\zeta > 0$ such that*

$$F(h(u) + t(g(v) - h(u)) \leq \max\{F(u), F(g(v))\}$$
$$-\zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p, \forall u, v \in K \subseteq D, t \in [0,1], p \geq 1.$$

Definition 2.8. *A positive function F on the convex set K_g is said to be higher order generalized log-convex with respect to an arbitrary functions g, h, if there exists a constant $\zeta > 0$ such that*

$$F(h(u) + t(g(v) - h(u))(F(h(u)))^{1-t}(F(g(v)))^t$$
$$-\zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p, \forall u, v \in K \subseteq D, t \in [0,1], p \geq 1.$$

From the above definitions, we have

$$\begin{aligned} & F(h(u) + t(g(v) - h(u))) \\ \leq\ & (F(h(u)))^{1-t}(F(g(v))^t - \zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p \\ \leq\ & (1-t)F(h(u)) + tF(g(v)) - \zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p \\ \leq\ & \max\{F(h(u)), F(g(v))\} - \zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D. \end{aligned}$$

This shows that every higher order generalized log-convex function is a higher order generalized convex function and every higher order generalized convex function is a higher order generalized quasi-convex function. However, the converse is not true.

Definition 2.9. *An operator $T : K \to H$ is said to be:*

(1) *higher order monotone, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tg(v), h(u) - g(v)\rangle \geq \alpha\|g(v) - h(u)\|^p, \forall u, v \in K \subseteq D, p \geq 1.$$

(2) *higher order pseudomonotone, if and only if, there exists a constant $\zeta > 0$ such that*

$$\langle Tu, g(v) - h(u)\rangle + \zeta\|g(v) - h(u)\|^p \geq 0$$
$$\Rightarrow$$
$$\langle Tg(v), g(v) - h(u)\rangle - \zeta\|g(v) - h(u)\|^p \geq 0, \forall u, v \in K \subseteq D, p \geq 1.$$

(3) *higher order relaxed pseudomonotone, if and only if, there exists a constant $\mu > 0$ such that*

$$\langle Tu, g(v) - u\rangle \geq 0 \quad \Rightarrow \quad -\langle Tg(v), h(u) - g(v)\rangle + \zeta\|g(v) - h(u)\|^p \geq 0, \forall u, v \in K \subseteq D.$$

Definition 2.10. *A differentiable function F on the convex set K is said to be higher order generalized pseudo convex function, if and only if, if there exists a constant $\zeta > 0$ such that*

$$\langle F'(u), g(v) - h(u)\rangle + \zeta\|g(v) - h(u)\|^p \geq 0 \quad \Rightarrow \quad F(g(v)) \geq F(u), \forall u, v \in K \subseteq D.$$

3. Characterizations of generalized convex functions

In this section, we consider some basic properties of higher order generalized convex functions and their variant forms.

Theorem 3.1. *Let F be a differentiable function on the generalized convex set K. Then the function F is higher order generalized convex function, if and only if,*

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)\rangle + \zeta\|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D, p \geq 1. \quad (1)$$

Proof. Let F be a higher order generalized convex function on the generalized convex set K. Then

$$F(h(u) + t(g(v) - h(u))) \leq (1-t)F(h(u)) + tF(g(v))$$
$$-\zeta\{t^p(1-t) + t(1-t)^p\}\|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D, t \in [0,1], \quad p \geq 1,$$

which can be written as

$$\begin{aligned} F(g(v)) - F(h(u)) \quad \geq \quad & \{\frac{F(h(u) + t(g(v) - h(u)) - F(h(u))}{t}\} \\ & +\zeta\{t^{p-1}(1-t) + (1-t)^p\}\|g(v) - h(u)\|^p. \end{aligned}$$

Taking the limit in the above inequality as $t \to 0$,, we have

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)) \rangle + \zeta \|g(v) - h(u)\|^p, \quad p \geq 1.$$

which is the required result(1).

Conversely, let (1) hold. Then, $\forall u, v \in K, t \in [0,1]$, $h(v_t) = h(u) + t(g(v) - h(u)) \in K$, we have

$$\begin{aligned} F(g(v)) - F(v_t) &\geq \langle F'(v_t), g(v) - h(v_t) \rangle + \zeta \|g(v) - h(v_t)\|^p \\ &\geq (1-t) F'(h(v_t)), g(v) - h(u) \rangle + \mu (1-t)^p \|g(v) - h(u)\|^p. \end{aligned} \quad (2)$$

In a similar way, we have

$$\begin{aligned} F(h(u)) - F(h(v_t)) &\geq \langle F'(v_t), h(u) - h(v_t)) \rangle + \zeta \|h(u) - h(v_t)\|^p \\ &= -tF'(h(v_t)), g(v) - h(u) \rangle + \zeta t^p \|g(v) - h(u)\|^p. \end{aligned} \quad (3)$$

Multiplying (2) by t and (3) by $(1-t)$ and adding the resultant, we have

$$\begin{aligned} F(h(u) + t(g(v), h(u))) &\leq (1-t)F(h(u)) + tF(g(v)) \\ &\quad - \zeta \{t^p(1-t) + t(1-t)^p\} \|g(v) - h(u)\|^p, \end{aligned}$$

showing that F is a higher order generalized convex function. □

Theorem 3.2. *Let F be a differentiable higher order generalized convex function on the general convex set K. Then $F'(.)$ is a higher order generalized monotone operator.*

Proof. Let F be a higher order generalized convex function on the generalized convex set K. Then, from Theorem 3.1. we have

$$F(g(v)) - F(h(u)) \geq \langle F'(u), g(v) - h(u) \rangle + \zeta \|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D, p \geq 1. \quad (4)$$

Changing the role of u and v in (4), we have

$$F(h(u)) - F(g(v)) \geq \langle F'(g(v(), h(u) - g(v)) \rangle + \zeta \|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D, p \geq 1. \quad (5)$$

Adding (4) and (5), we have

$$\langle F'(h(u)) - F'(g(v)), h(u) - g(v) \rangle \geq 2\zeta \|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D, p \geq 1,$$

which shows that $F'(.)$ is a higher order generalized monotone operator. □

We remark that the converse of Theorem 3.2 is not true. However, we have:

Theorem 3.3. *If the differential operator $F'(.)$ of a differentiable higher order generalized convex function F is a higher order generalized monotone operator, then*

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u) \rangle + 2\zeta \frac{1}{p} \|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D. \quad (6)$$

Proof. Let $F'(.)$ be a higher order generalized monotone operator. Then, from (6), we have

$$\langle F'(g(v)), u - g(v) \rangle \geq \langle F'(u), u - g(v)) \rangle + 2\zeta \|g(v) - h(u)\|^p, \quad \forall u, v \in K \subseteq D. \quad (7)$$

Since K is a generalized convex set, $\forall u, v \in K$, $t \in [0,1]$, $h(v_t) = h(u) + t(g(v) - h(u)) \in K$. Taking $h(v) = h(v_t)$ in (7, we have

$$\begin{aligned} \langle F'(h(v_t)), g(u) - h(v_t) \rangle &\leq \langle F'(h(u)), h(u) - h(v_t) \rangle - 2\zeta \|h(v_t) - h(u)\|^p \\ &= -t \langle F'(h(u)), g(v) - h(u) \rangle - 2\zeta t^p \|g(v) - h(u)\|^p, \end{aligned}$$

which implies that

$$\langle F'(h(v_t)), g(v) - h(u) \rangle \geq \langle F'(h(u)), g(v) - h(u) \rangle + 2\zeta t^{p-1} \|g(v) - h(u)\|^P. \quad (8)$$

Consider the auxiliary function

$$\xi(t) = F(h(u) + t(g(v) - h(u)), \forall u, v \in K \subseteq D, t \in [0,1], \quad p \geq 1,$$

from which, we have $\xi(1) = F(g(v)), \quad \xi(0) = F(h(u))$. Then, from (8), we have

$$\xi'(t) = \langle F'(h(v_t)), g(v) - h(u)\rangle \geq \langle F'(h(u)), g(v) - h(u)\rangle + 2\zeta t^{p-1}\|g(v) - h(u)\|^p. \qquad (9)$$

Integrating (9) between 0 and 1, we have

$$\xi(1) - \xi(0) \quad = \quad \int_0^1 g'(t)dt \geq \langle F'(h(u)), g(v) - h(u)\rangle + 2\zeta\frac{1}{p}\|g(v) - h(u)\|^p.$$

Thus it follows that

$$F(g(v)) - F(h(u)) \geq \langle F'(h(u)), g(v) - h(u)\rangle + 2\zeta\frac{1}{p}\|g(v) - h(u)\|^p, \quad \forall u, v \in K, p \geq 1,$$

which is the required (6). □

For $p = 2$, Theorem 3.3 can be viewed as the converse of Theorem 3.2.
We now give a necessary condition for higher order generalized pseudo-convex function.

Theorem 3.4. *Let $F'(.)$ be a higher order generalized relaxed pseudomonotone operator. Then F is a higher order general pseudo-convex function.*

Proof. Let F' be a higher order generalized relaxed pseudomonotone operator. Then,

$$\langle F'(h(u)), g(v) - h(u)\rangle \geq 0, \forall u, v \in K \subseteq D, t \in [0, 1], p \geq 1.$$

implies that

$$\langle F'(g(v)), g(v) - h(u)\rangle \geq \zeta\|g(v) - h(u)\|^p, \forall u, v \in K \subseteq D, t \in [0, 1], p \geq 1. \qquad (10)$$

Since K is a generalized convex set, $\forall u, v \in K \subseteq D, t \in [0, 1], p \geq 1$,
$h(v_t) = h(u) + t(g(v) - h(u)) \in K$. Taking $g(v) = h(v_t)$ in (10), we have

$$\langle F'(h(v_t)), h(v_t) - h(u)\rangle \geq \zeta t^{p-1}\|h(v_t) - h(u)\|^p. \qquad (11)$$

Consider the auxiliary function

$$\xi(t) = F(h(u) + t(g(v) - h(u))) = F(h(v_t)), \quad \forall u, v \in K \subseteq D, t \in [0, 1], p \geq 1,$$

which is differentiable, since F is differentiable function. Then, using (11), we have

$$\xi'(t) = \langle F'(h(v_t)), h(v_t) - h(u))\rangle \geq \zeta t^{p-1}\|g(v) - h(u)\|^p.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \frac{\zeta}{p}\|g(v) - h(u)\|^p,$$

from which it follows that,

$$F(g(v)) - F(h(u)) \geq \frac{\zeta}{p}\|g(v) - h(u)\|^p), \forall u, v \in K \subseteq D, \quad p \geq 1.$$

This shows that F is a higher order generalized pseudo-convex function. □

Definition 3.1. *A function F is said to be sharply higher order generalized pseudo convex, if there exists a constant $\zeta > 0$ such that*

$$\begin{aligned} & \langle F'(h(u)), g(v) - h(u)\rangle \geq 0, \forall u, v \in K \subseteq D, t \in [0, 1], p \geq 1 \\ & \Rightarrow \\ F(g(v)) \quad \geq \quad & F(g(v) + t(h(u) - g(v))) + \zeta\{t^p(1 - t) + t(1 - t)^p\}\|g(v) - h(u)\|^p. \end{aligned}$$

Theorem 3.5. *Let F be a sharply higher order generalized pseudo convex function on K with a constant $\zeta > 0$. Then*

$$\langle F'(g(v)), g(v) - h(u)\rangle \geq \zeta\|g(v) - h(u)\|^p, \forall u, v \in K \subseteq D, p \geq 1.$$

Proof. Let F be a sharply higher order generalized pseudo convex function on K_g. Then

$$F(g(v)) \geq F(g(v)+t(h(u)-g(v)))+\zeta\{t^p(1-t)+t(1-t)^p\}\|g(v)-h(u)\|^p, \quad \forall u,v\in K\subseteq D, t\in[0,1], p\geq 1,$$

from which, we have

$$\{\frac{F(g(v)+t(h(u)-g(v))-F(g(v))}{t}\}+\zeta\{t^{p-1}(1-t)+(1-t)^p\}\|g(v)-h(u)\|^p\geq 0.$$

Taking limit in the above inequality, as $t\to 0$, we have

$$\langle F'(g(v)), g(v)-h(u)\rangle \geq \zeta\|g(v)-h(u)\|^p, \forall u,v\in K\subseteq D, p\geq 1,$$

the required result. □

4. **Parallelogram Laws**

In this section, we discuss some characterizations of uniformly Banach spaces involving the notion of higher order generalized convexity, which can be viewed as novel application. Setting $F(u)=\|u\|^p$ in Definition2.6, we have

$$\begin{aligned}\|h(u)+t(g(v)-h(u))\|^p &= (1-t)\|h(u)\|^p+t\|g(v)\|^p\\ &- \zeta\{t^p(1-t)+t(1-t)^p\}\|g(v)-h(u)\|^p, \forall u,v\in H. \quad (12)\end{aligned}$$

Taking $t=\frac{1}{2}$ in (12) and simple computation, we have

$$\|h(u)+g(v)\|^p+\zeta\|g(v)-h(u)\|^p = 2^{p-1}\{\|h(u)\|^p+\|g(v)\|^p\}, \forall u,v\in H,$$

which is known as the generalized parallelogram for the Banach spaces.

Remark 4.1. *For suitable and appropriate choice of the operators T, g, h, parameters p, ζ and the spaces, one can obtain various known* [4, 5, 6, 16, 50] *and new parallelogram laws, which can be used to characterize the normed spaces. We give only some glimpse of the applications of higher order generalized affine convex functions. It is an interesting problem to explore the applications of these parallelograms in prediction and information theory.*

5. **Higher order generalized variational inequalities**

In this section, we introduce and consider some new classes of higher order generalized variational inequalities. Several iterative methods are suggested and analyzed using the technique of auxiliary principle, which are mainly due to Glowinski et al. [9] and Noor [17]. First of all, we show that the optimality for the differentiable higher order generalized convex functions can be characterized by a class of generalized variational inequalities, which is the main motivation of our next result.

Theorem 5.1. *Let F be a differentiable higher order generalized convex function with modulus $\zeta>0$. If $u\in K$ is the minimum of the function F, then*

$$\langle F'(h(\mu)), (g(v))-h(\mu)\rangle \geq \zeta\|g(v)-h(\mu)\|^p, \quad \forall \mu,v\in K\subseteq D, t\in[0,1], p\geq 1. \quad (13)$$

Proof. Let $\mu\in K$ be a minimum of the function F. Then

$$F(h(\mu))\leq F(g(v)), \quad \forall v\in K. \quad (14)$$

Since K is a generalized convex set, so, $\forall \mu,v\in K\subseteq D, t\in[0,1]$, $g(v_t)=(1-t)h(\mu)+tg(v)\in K$. Taking $g(v)=g(v_t)$ in (14), we have

$$0\leq \lim_{t\to 0}\{\frac{F(h(\mu)+t(g(v)-h(\mu)))-F(h(\mu))}{t}\} = \langle F'(h(u)), g(v)-h(\mu)\rangle. \quad (15)$$

Since F is differentiable higher order generalized convex function, so

$$\begin{aligned} & F(h(\mu)+t(g(v)-h(\mu))) \\ \leq\ & F(h(\mu))+t(F(g(v))-F(h(\mu))-\zeta\{t^p(1-t)+t(1-t)^p\}\|g(v)-h(\mu)\|^p, \end{aligned}$$

from which, using (15), we have

$$\begin{aligned} & F(g(v))-F(h(\mu)) \geq \lim_{t\to 0}\{\frac{F(h(\mu)+t(g(v)-h(\mu)))-F(h(\mu))}{t}\}+\zeta\|g(v)-h(\mu)\|^p \\ & =\langle F'(h(\mu)),g(v)-h(\mu)\rangle+\zeta\|g(v)-h(\mu)\|^p, \end{aligned}$$

the required result (13). □

Remark: We would like to mention that, if $\mu \in K$ satisfies

$$\langle F'(h(\mu)),g(v)-\mu\rangle+\zeta|g(v)-h(\mu)\|^p \geq 0, \quad \forall \mu, v \in K \subseteq D, p \geq 1, \tag{16}$$

then $\mu \in K$ is the minimum of the function F. The inequality (16) is called the higher order generalized variational inequality. We now consider a more generalized variational inequality of which (16 is a special case.

For given three operators T, g, h, we consider the problem of finding $\mu \in K$ for a constant ζ such that

$$\langle T\mu, g(v)-h(\mu)\rangle+\zeta\|g(v)-h(\mu)\|^p \geq 0, \forall \mu, v \in K \subseteq D, \quad p \geq 1, \tag{17}$$

which is called the higher order generalized variational inequality.

For the recent developments in variational inequalities and related optimization theory, see [1, 2, 3, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 40, 41, 42, 43, 44, 46, 47, 48, 49, 50, 51, 52] and the references therein. For suitable and appropriate choice of the parameter μ, p and operators T, g, h, one can obtain several new and known classes of variational inequalities, see [13, 15, 20, 21, 22].

We use auxiliary principle technique to suggest some iterative methods for solving the higher order generalized variational inequalities (17).
For given $\mu \in K$ satisfying (17), consider the problem of finding $w \in K$, such that

$$\langle \rho T w, g(v)-h(w)\rangle+\langle w-\mu, v-w\rangle+\nu\|g(v)-h(w)\|^p \geq 0, \forall \mu, v \in K, p \geq 1, \tag{18}$$

where $\rho > 0$ is a parameter. The problem (18) is called the auxiliary variational inequality.

We not that, if $w(u) = \mu$, then w is a solution of problem (17). This simple observation enables to suggest an iterative method for solving (17).

Algorithm 5.1. *For a given $\mu_0 \in K$, find the approximate solution u_{n+1} by the scheme*

$$\begin{aligned} \langle \rho T\mu_{n+1}, g(v)-h(\mu_{n+1})\rangle &+ \langle \mu_{n+1}-\mu_n, v-\mu_{n+1}\rangle \\ &+ \zeta\|g(v)-h(\mu_{n+1})\|^p \geq 0, \forall \mu, v \in K \subseteq D. \end{aligned} \tag{19}$$

The Algorithm 5.1 is known as the implicit method. Such type of methods have been studied extensively for various classes of variational inequalities.
For the convergence analysis of Algorithm 5.1, we need the following concept.

Definition 5.1. *The operator T is said to be higher order pseudo monotone with respect to $\zeta\|g(v)-h(\mu)\|^p, p > 1$, if*

$$\begin{aligned} & \langle \rho T\mu, g(v)-h(\mu)\rangle+\zeta\|g(v)-h(\mu)\|^p \geq 0, \forall \mu, v \in K, \quad p > 1, \\ & \Longrightarrow \\ & \langle \rho T v, h(v)-g(u)\rangle-\zeta\|g(u)-h(v)\|^p \geq 0, \forall \mu, v \in K, \quad p > 1. \end{aligned}$$

We now study the convergence analysis of Algorithm 5.1.

Theorem 5.2. *Let $\mu \in K$ be a solution of (17) and u_{n+1} be the approximate solution obtained from Algorithm 5.1. If T is a higher order pseudo monotone operator, then*

$$\|\mu_{n+1} - \mu\|^2 \leq \|\mu_n - \mu\|^2 - \|\mu_{n+1} - \mu_n\|^2. \tag{20}$$

Proof. Let $\mu \in K$ be a solution of (17). Then

$$\langle \rho T\mu, g(v) - h(\mu)\rangle + \zeta\|g(v) - h(\mu)\|^p, \forall \mu, v \in K,$$

implies that

$$\langle \rho Tv, g(\mu) - h(v)\rangle - \zeta\|g(u) - h(v)\|^p, \forall \mu, v \in K, \tag{21}$$

Now taking $v = u_{n+1}$ in (21), we have

$$\langle \rho T\mu_{n+1}, h(\mu_{n+1}) - g(\mu)\rangle - \zeta\|h(\mu_{n+1}) - g(\mu)\|^p \geq 0. \tag{22}$$

Taking $v = \mu$ in (19), we have

$$\begin{aligned} \langle \rho T\mu_{n+1}, g(\mu) - h(\mu_{n+1})\rangle &+ \langle \mu_{n+1} - \mu_n, v - \mu_{n+1}\rangle \\ &+ \zeta\|g(u) - h(\mu_{n+1})\|^p \geq 0, \forall \mu, v \in K. \end{aligned} \tag{23}$$

Combining (22) and (23), we have

$$\langle \mu_{n+1} - \mu_n, \mu_{n+1} - \mu\rangle \geq 0.$$

Using the inequality $2\langle a, b\rangle = \|a + b\|^2 - \|a\|^2 - \|b\|^2, \forall a, b \in H$, we obtain

$$\|\mu_{n+1} - \mu\|^2 \leq \|\mu_n - \mu\|^2 - \|\mu_{n+1} - \mu_n\|^2,$$

the required result (20). □

Theorem 5.3. *Let the operator T be a higher order pseudo monotone. Let μ_{n+1} be the approximate solution obtained from Algorithm 5.1 and $\mu \in K$ is the exact solution (17), then* $\lim_{n\to\infty} \mu_n = \mu$.

Proof. Let $\mu \in K$ be a solution of (17). Then, from (20), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{\mu_n\}$ is bounded. From (20), we have

$$\sum_{n=0}^{\infty} \|\mu_{n+1} - \mu_n\|^2 \leq \|\mu_0 - \mu\|^2,$$

from which, it follows that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0. \tag{24}$$

Let $\hat{\mu}$ be a cluster point of $\{\mu_n\}$ and the subsequence $\{\mu_{n_j}\}$ of the sequence u_n converge to $\hat{\mu} \in H$. Replacing μ_n by μ_{n_j} in (19), taking the limit $n_j \to 0$ and from (24), we have

$$\langle T\hat{u}, g(v) - h(\hat{u})\rangle + \mu\|g(v) - h(\hat{u})\|^p, \quad \forall \mu, v \in K, \quad p > 1.$$

This implies that $\hat{u} \in K$ and

$$\|\mu_{n+1} - \mu_n\|^2 \leq \|\mu_n - \hat{\mu}\|^2.$$

Thus it follows from the above inequality that the sequence μ_n has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} = \hat{\mu}$. □

We now use the auxiliary principle technique involving the Bregman distance function.

For a given $\mu \in K$ satisfying (17), find $w \in K$ such that

$$\begin{aligned} &\langle \rho Tw, g(\nu) - h(w))\rangle + \langle \varphi'(h(w)) - \varphi'(\mu), g(\nu) - h(w)\rangle \\ &+\eta\|g(\nu) - h(w)\|^p \geq 0, \forall \mu, v \in K, \quad p \geq 1, \end{aligned}$$

where $\varphi'(\mu)$ is the differential of a general convex function $\varphi(\mu)$ at $\mu \in K$.

Remark 5.1. *The function $\mathcal{B}(w,\mu) = \varphi(w) - \varphi(\mu) - \langle \varphi'(\mu), w-\mu\rangle$ associated with the strongly convex function $\varphi(\mu)$ is called the Bregman function.*

We note that, if $w = \mu$, then clearly w is solution of the problem (17). This fact is used to consider the following iterative method for solving higher order generalized variational inequality(17).

Algorithm 5.2. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho T\mu_{n+1}, g(\nu) - h(\mu_{n+1})\rangle + \langle \varphi'(h(\mu_{n+1})) - \varphi'(g(\mu_n)), g(\nu) - h(\mu_{n+1})\rangle\\&+\eta\|g(\nu) - h(\mu_{n+1})\|^p \geq 0, \quad \forall u,\nu \in K, \quad p > 1,\end{aligned}$$

where $\rho > 0$ is a constant. Algorithm 5.2 is called the proximal method.

We now suggest some iterative methods for solving the problem(17).

Algorithm 5.3. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho\mathcal{T}\mu_n, g(\nu) - h(\mu_{n+1})\rangle + \langle \varphi'(h(\mu_{n+1}) - \varphi'(\mu_n), g(\nu) - h(\mu_{n+1})\rangle\\&+\eta\|g(\nu) - h(\mu_{n+1})\|^p \geq 0, \quad \forall u,\nu \in K, \quad p \geq 1.\end{aligned}$$

Algorithm 5.4. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho T((1-\lambda)\mu_{n+1} + \lambda\mu_n), g(v) - h(\mu_{n+1})\rangle + \langle \mu_{n+1} - \mu, v - \mu_{n+1}\rangle\\&+\nu\|g(v) - h(\mu_{n+1})\|^p \geq 0, \quad \forall u,v \in K, \quad p \geq 1,\end{aligned}$$

which is called the hybrid implicit-explicit iterative method. For $\lambda = \frac{1}{2}$, Algorithm (5.4) reduces to:

Algorithm 5.5. *For a given $\mu_0 \in \mathcal{H}$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho T(\frac{\mu_{n+1} + \mu_n)}{2}, g(v) - h(\mu_{n+1})\rangle + \langle \mu_{n+1} - \mu, v - \mu_{n+1}\rangle\\&+\nu\|g(v) - h(\mu_{n+1})\|^p \geq 0, \quad \forall u,v \in K, \quad p \geq 1.\end{aligned}$$

which is called implicit iterative method.

Remark 5.2. *For suitable and appropriate choice of the operators T, g, h, parameters p, λ, ζ and the spaces, one can obtain various known and new algorithms for solving the problem (17) and related optimization problems. Using the auxiliary principle technique, on can suggest several iterative methods for solving the higher order strongly general variational inequalities and related optimization problems. We have only given a glimpse of the higher order strongly general variational inequalities. It is an interesting problem to explore the applications of such type variational inequalities in various fields of pure and applied sciences.*

Conclusion: In this paper, we have introduced and studied some new classes of generalized convex functions. It is shown that several new classes of strongly convex functions can be obtained as special cases of these generalized convex functions. We have studied the basic properties of these functions. We have derived the some new general parallelogram laws characterizing the Banach spaces, which may have applications in prediction theory and stochastic analysis. We have also considered and analyzed some new class of higher order strongly general variational inequalities. Using the auxiliary principle technique, an implicit iterative method is suggested for finding the approximate solution of higher order general variational inequalities. Using the pseudo-monotonicity of the operator, convergence criteria is discussed. Some special cases are considered as application of the main results. The interested readers may explore the applications and other properties of the higher order strongly general convex functions in various fields of pure and applied sciences. This is an interesting direction of future research.

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