INERTIAL HYBRID PROJECTION METHODS WITH SELECTION TECHNIQUES FOR SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

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In this work, we propose a new hybrid projection method based on inertial effects and selection techniques to solve the split common fixed point problem of demicontractive operators in real Hilbert spaces. The strong convergence of the method is proved by assuming standard assumptions. Additionally, application is given to the multiple-sets split feasibility problem.

Keywords: split common fixed point problem, multiple-sets split feasibility problem, hybrid projection method, inertial method, selection technique.

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1. Introduction

Throughout this article, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces equipped with their own inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint operator A^*. Define $I_1 = \{1, 2, 3, \ldots, s\}$ and $I_2 = \{1, 2, 3, \ldots, t\}$, where s and t are positive integers.

The split common fixed point problem (SCFPP) requires to seek an element $x \in \mathcal{H}_1$ satisfying

$$x \in \bigcap_{i\in I_1} Fix(S_i) \text{ such that } Ax \in \bigcap_{j\in I_2} Fix(T_j), \tag{1}$$

where $Fix(S_i)$ and $Fix(T_j)$ denote the fixed point sets of two classes of nonlinear operators $S_i : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_j : \mathcal{H}_2 \to \mathcal{H}_2$.

Recently, Yao et al. [26] presented two iterative methods with selection techniques for finite families of firmly nonexpansive mappings of the SCFPP (1) and obtain weak and strong convergence theorems. The case $s = t = 1$ was firstly introduced by Censor and Segal [1] and was further studied and extended by many researchers in, for instance, [6, 11, 15, 20, 21, 22, 23, 24, 25, 27].

In optimization theory, to speed up the convergence rate, Polyak [9] firstly introduced the so-called heavy ball method for solving smooth convex minimization problem. In order to improve the convergence rate, Nesterov [8] proposed a modified heavy ball method as follows:

$$\begin{aligned} y_n &= x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} &= y_n - \lambda_n \nabla f(y_n),\ n \in \mathbb{N}, \end{aligned} \tag{2}$$

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where $\theta_n \in [0,1)$ is an extrapolation factor and λ_n is a step-size parameter (sufficiently small) and ∇f is the gradient of a smooth convex function f. Let us recall that the term $\theta_n(x_n - x_{n-1})$ in (2) is known as the inertial step and it plays very important role in improving the performance of the method and has a nice convergence properties, see also [3, 4, 13, 14].

In 2003, Nakajo and Takahashi [7] established strong convergence of the hybrid projection method for nonexpansive mappings in Hilbert spaces. Several authors have presented different methods to solve problems related to fixed point problems; see [2, 10, 12, 16, 18, 19, 28].

Motivated by above research works, we construct inertial hybrid projection method with selection techniques for solving the SCFPP (1) and prove strong convergence theorem of the proposed method under some weakened assumptions.

2. Preliminaries

In this section, we give some mathematical preliminaries which will be used in the sequel. Let $\mathcal{H}$ be a real Hilbert space. We know that the metric projection P_C from $\mathcal{H}$ onto a nonempty, closed and convex subset $C \subseteq \mathcal{H}$ is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|, \quad x \in \mathcal{H}.$$

Next, we have the following equality:

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 \tag{3}$$

for all $x, y \in \mathcal{H}$.

Definition 2.1. *An operator $T : C \to C$ is said to be demicontractive (or k-demicontractive) if there exists a constant $k \in [0,1)$ such that*

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - Tx\|^2,$$

or equivalently,

$$\langle x - Tx, x - x^*\rangle \geq \frac{1-k}{2}\|x - Tx\|^2, \tag{4}$$

for all $(x, x^*) \in C \times Fix(T)$.

We use $\rightharpoonup$ for weak convergence and $\to$ for strong convergence. For a sequence $\{x_n\}$ in $\mathcal{H}$, the weak ω-limit set of $\{x_n\}$ is denoted by $\omega_w(x_n)$. Next, we give some important tools for proving our main results.

Definition 2.2. *Let $T : C \to \mathcal{H}$ be an operator. Then T is said to be demiclosed at $y \in \mathcal{H}$ if, for any sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x \in C$ and $Tx_n \to y$ imply $Tx = y$.*

Lemma 2.1. [17] *Given $x \in \mathcal{H}$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0,$$

for all $y \in C$.

Lemma 2.2. [5] *Given that $x, y, z \in \mathcal{H}$ and $a \in \mathbb{R}$. The set*

$$D := \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

Lemma 2.3. [5] *Let $\{x_n\}$ be a sequence in $\mathcal{H}$ and $u \in \mathcal{H}$. Let $z = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - z\|,$$

for all $n \in \mathbb{N}$. Then $x_n \to z$.

3. Main result

In this section, we study the SCFPP (1) under the following hypothesis:
Let $S_i : \mathcal{H}_1 \to \mathcal{H}_1$ for $i \in I_1$ and $T_j : \mathcal{H}_2 \to \mathcal{H}_2$ for $j \in I_2$ be two finite families of demicontractive operators with constants $\beta_i \in [0,1)$ and $\mu_j \in [0,1)$, respectively, and both $I - S_i$ and $I - T_j$ are demiclosed at zero. Set $\beta = \max_{i\in I_1} \beta_i$ and $\mu = \max_{j\in I_2} \mu_j$. Suppose that

$$\Omega := \left\{ x : x \in \bigcap_{i\in I_1} Fix(S_i) \text{ and } Ax \in \bigcap_{j\in I_2} Fix(T_j) \right\} \neq \emptyset.$$

Next, the following inertial hybrid projection method with selection techniques is constructed to solve SCFPP (1). We also prove strong convergence of the proposed method under standard assumptions.

Algorithm 3.1

Initialization: given initial points $x_0, x_1 \in \mathcal{H}_1$ be arbitrary, $\{\eta_n\} \subset [0,\infty)$ such that $\eta_n \to 0$ as $n \to \infty$, $\{\theta_n\}$ is a real sequence such that $|\theta_n| \le \theta$ for some θ, and set $C_1 = Q_1 = \mathcal{H}_1$ and $n = 1$.
Iterative Steps: Construct $\{x_n\}$ by using the following steps:
Step 1. Select $i_n \in I_1$ and $j_n \in I_2$ such that

$$\|z_n - S_{i_n} z_n\| = \max_{i\in I_1} \|z_n - S_i z_n\| \quad \text{and} \quad \|(I - T_{j_n})Az_n\| = \max_{j\in I_2} \|(I - T_j)Az_n\|,$$

where $z_n = x_n + \theta_n(x_n - x_{n-1})$.
Step 2. Compute

$$y_n = z_n - S_{i_n} z_n + A^*(I - T_{j_n})Az_n.$$

If $y_n = 0$, then stop and $z_n \in \Omega$. Otherwise,
Step 3. Compute

$$w_n = z_n - \tau_n y_n,$$

where $\tau_n = \gamma \frac{\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2}{\|y_n\|^2}$ with $\gamma \in (0, \min\{1-\beta, 1-\mu\})$ is a positive constant.
Step 4. Compute

$$x_{n+1} = P_{C_n \cap Q_n} x_1,$$

where $C_n = \{v \in \mathcal{H}_1 : \|w_n - v\| \le \|z_n - v\| + \eta_n\}$ and
$Q_n = \{v \in Q_{n-1} : \langle x_1 - x_n, x_n - v\rangle \ge 0\}$.
Replace n by $n + 1$ and then repeat **Step 1.**

Lemma 3.1. *$z_n \in \Omega$ if and only if $y_n = 0$ for some $n \in \mathbb{N}$.*

Proof. The sufficiency is obvious. Next, we only need to prove the necessity. Assume that $y_n = 0$ for some $n \in \mathbb{N}$. Then, for any $x^* \in \Omega$, by (4), we obtain

$$\begin{aligned}
0 &= \|y_n\|\|z_n - x^*\| \\
&\ge \langle z_n - S_{i_n} z_n + A^*(I - T_{j_n})Az_n, z_n - x^*\rangle \\
&= \langle z_n - S_{i_n} z_n, z_n - x^*\rangle + \langle A^*(I - T_{j_n})Az_n, z_n - x^*\rangle \\
&= \langle z_n - S_{i_n} z_n, z_n - x^*\rangle + \langle (I - T_{j_n})Az_n, Az_n - Ax^*\rangle \\
&\ge \frac{1-\beta_{i_n}}{2}\|z_n - S_{i_n} z_n\|^2 + \frac{1-\mu_{j_n}}{2}\|(I - T_{j_n})Az_n\|^2.
\end{aligned}$$

According to the definitions of i_n and j_n, it follows from $\beta_{i_n}, \mu_{j_n} \in [0,1)$ that

$$\|z_n - S_i z_n\| \le \|z_n - S_{i_n} z_n\| = 0,\ i \in I_1 \text{ and } \|(I - T_j)Az_n\| \le \|(I - T_{j_n})Az_n\| = 0,\ j \in I_2.$$

Hence, we deduce $z_n \in \bigcap_{i \in I_1} Fix(S_i)$ and $Az_n \in \bigcap_{j \in I_2} Fix(T_j)$. Therefore, it follows that $z_n \in \Omega$. □

Theorem 3.1. *The sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a solution $\bar{z}$ of problem (1), where $\bar{z} = P_\Omega x_1$.*

Proof. **Claim 1.** $\|w_n - z\| \le \|z_n - z\| + \eta_n$ for all $z \in \Omega$.
Indeed, since $z \in \Omega$ and by (4), we have

$$\begin{aligned}\langle y_n, z_n - z\rangle &= \langle z_n - S_{i_n} z_n + A^*(I - T_{j_n})Az_n, z_n - z\rangle \\ &= \langle z_n - S_{i_n} z_n, z_n - z\rangle + \langle (I - T_{j_n})Az_n, Az_n - Az\rangle \\ &\ge \frac{1-\beta}{2}\|z_n - S_{i_n} z_n\|^2 + \frac{1-\mu}{2}\|(I - T_{j_n})Az_n\|^2 \\ &\ge \frac{1}{2}\min\{1-\beta, 1-\mu\}\left(\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2\right). \end{aligned} \tag{5}$$

Using (3) and (5), we derive

$$\begin{aligned}\|w_n - z\|^2 &= \|z_n - z - \tau_n y_n\|^2 \\ &= \|z_n - z\|^2 - 2\tau_n\langle y_n, z_n - z\rangle + \tau_n^2\|y_n\|^2 \\ &\le \|z_n - z\|^2 - \gamma\min\{1-\beta, 1-\mu\}\frac{\left(\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2\right)^2}{\|y_n\|^2} \\ &\quad + \gamma^2\frac{\left(\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2\right)^2}{\|y_n\|^2} \\ &= \|z_n - z\|^2 - \gamma\left(\min\{1-\beta, 1-\mu\} - \gamma\right)\frac{\left(\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2\right)^2}{\|y_n\|^2}.\end{aligned}$$

Since $\gamma \in (0, \min\{1-\beta, 1-\mu\})$, we obtain **Claim 1**.
Claim 2. $\{x_n\}$ is well defined and $\Omega \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$.
From the definition of C_n and Q_n, and by Lemma 2.2, we get $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. By **Claim 1**, we have $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Further, $\Omega \subset C_1 \cap Q_1$ and $x_2 = P_{C_1 \cap Q_1} x_1$ is well defined. Assume that $\Omega \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. This shows that $x_{k+1} = P_{C_k \cap Q_k} x_1$ is well defined. By Lemma 2.1, we have $\langle x_1 - x_{k+1}, x_{k+1} - z\rangle \ge 0$ for all $z \in C_k \cap Q_k$. So, $\langle x_1 - x_{k+1}, x_{k+1} - z\rangle \ge 0$ for all $z \in \Omega$. It implies that $\Omega \subset Q_{k+1}$ and so $\Omega \subset C_{k+1} \cap Q_{k+1}$. Therefore, **Claim 2** is obtained.
Claim 3. $\lim_{n\to\infty}\|x_n - z_n\| = 0$.
Indeed, since Ω is a nonempty, closed and convex, there exists a unique $\bar{z} \in \Omega$ such that $\bar{z} = P_\Omega x_1$. From $x_{n+1} = P_{C_n \cap Q_n} x_1$ and $\Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_1\| \le \|\bar{z} - x_1\| \quad \forall n \in \mathbb{N}. \tag{6}$$

This implies that $\{x_n\}$ is bounded. Using Lemma 2.1 together with the definition of Q_n, we have $x_n = P_{Q_n} x_1$. Since $x_{n+1} \in Q_n$, it implies that

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\| \quad \forall n \in \mathbb{N}. \tag{7}$$

This implies that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. By Lemma 2.1, it follows that

$$\langle x_n - x_{n+1}, x_n - x_1\rangle \le 0.$$

Applying this to (3), we deduce

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 + 2\langle x_n - x_{n+1}, x_n - x_1\rangle \\ &\le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{8}$$

From $\{\theta_n\}$ is bounded sequence and by (8), we have

$$\|x_n - z_n\| = |\theta_n|\|x_n - x_{n-1}\| \le \theta\|\|x_n - x_{n-1}\| \quad \to \quad 0 \quad \text{as} \quad n \to \infty.$$

Claim 4. $\lim_{n\to\infty} \|z_n - S_i z_n\| = \lim_{n\to\infty} \|(I - T_j)Az_n\| = 0$ for all $i \in I_1$ and $j \in I_2$.
From $x_{n+1} \in C_n$, we obtain

$$\begin{aligned}\|w_n - z_n\| &\le \|w_n - x_{n+1}\| + \|x_{n+1} - z_n\| \\ &\le 2\|z_n - x_{n+1}\| + \eta_n \\ &\le 2\|z_n - x_n\| + 2\|x_n - x_{n+1}\| + \eta_n \quad \to \quad 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \frac{\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2}{\|y_n\|} = 0. \tag{9}$$

However, we observe that

$$\begin{aligned}\frac{\left(\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2\right)^2}{\|y_n\|^2} &= \frac{\left(\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2\right)^2}{\|z_n - S_{i_n} z_n + A^*(I - T_{j_n})Az_n\|^2} \\ &\ge \frac{\left(\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2\right)^2}{2\left(\|z_n - S_{i_n} z_n\|^2 + \|A\|^2\|(I - T_{j_n})Az_n\|^2\right)} \\ &\ge \frac{\|z_n - S_{i_n} z_n\|^2 + \|(I - T_{j_n})Az_n\|^2}{2\max\{1, \|A\|^2\}}.\end{aligned} \tag{10}$$

Combining (9) and (10), we immediately obtain

$$\lim_{n\to\infty} \|z_n - S_{i_n} z_n\| = \lim_{n\to\infty} \|(I - T_{j_n})Az_n\| = 0.$$

By the definitions of i_n and j_n, we get **Claim 4**.
Claim 5. $x_n \to \bar{z}$, where $\bar{z} = P_\Omega x_1$. Indeed, from (6) and (7), we get

$$\|x_n - x_1\| \le \|\bar{z} - x_1\| \quad \forall n \in \mathbb{N}. \tag{11}$$

We next show that every weak cluster point of the sequence $\{x_n\}$ belongs to Ω. Let $q \in \omega_w(x_n)$, that is, it has a subsequence $\{x_{n_k}\}$ fulfilling $x_{n_k} \rightharpoonup q$ as $k \to \infty$. By **Claim 3**, we get $z_{n_k} \rightharpoonup q$ as $k \to \infty$. Since A is bounded linear operator, we obtain that $Az_{n_k} \rightharpoonup Aq$ as $k \to \infty$. By the demiclosedness at zero of $I - S_i$ and $I - T_j$, together with **Claim 4**, we have $q \in \Omega$. Applying Lemma 2.3 to the inequality (11), we can conclude that the sequence $\{x_n\}$ converges strongly to $\bar{z} \in \Omega$, where $\bar{z} = P_\Omega x_1$. □

4. Multiple-sets split feasibility problem

Multiple-sets split feasibility problem (MSSFP) is to find a point $x \in \mathcal{H}_1$ such that

$$x \in \bigcap_{i\in I_1} U_i \text{ and } Ax \in \bigcap_{j\in I_2} V_j, \tag{12}$$

where $\{U_i\}_{i\in I_1}$ and $\{V_j\}_{j\in I_2}$ are two finite families of closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Assume that

$$\Phi := \left\{ x : x \in \bigcap_{i\in I_1} U_i \text{ and } Ax \in \bigcap_{j\in I_2} V_j \right\} \neq \emptyset.$$

Next, we present the following algorithm to solve MSSFP (12).

Algorithm 4.1

Initialization: given initial points $x_0, x_1 \in \mathcal{H}_1$ be arbitrary, $\{\eta_n\} \subset [0,\infty)$ such that $\eta_n \to 0$ as $n \to \infty$, $\{\theta_n\}$ is a real sequence such that $|\theta_n| \leq \theta$ for some θ, and set $C_1 = Q_1 = \mathcal{H}_1$ and $n = 1$.

Iterative Steps: Construct $\{x_n\}$ by using the following steps:

Step 1. Select $i_n \in I_1$ and $j_n \in I_2$ such that

$$\|z_n - P_{U_{i_n}} z_n\| = \max_{i\in I_1} \|z_n - P_{U_i} z_n\| \quad \text{and} \quad \|(I - P_{V_{j_n}})Az_n\| = \max_{j\in I_2} \|(I - P_{V_j})Az_n\|,$$

where $z_n = x_n + \theta_n(x_n - x_{n-1})$.

Step 2. Compute

$$y_n = z_n - P_{U_{i_n}} z_n + A^*(I - P_{V_{j_n}})Az_n.$$

If $y_n = 0$, then stop and $z_n \in \Phi$. Otherwise,

Step 3. Compute

$$w_n = z_n - \tau_n y_n,$$

where $\tau_n = \gamma \frac{\|z_n - P_{U_{i_n}} z_n\|^2 + \|(I-P_{V_{j_n}})Az_n\|^2}{\|y_n\|^2}$ with $\gamma \in (0,1)$.

Step 4. Compute

$$x_{n+1} = P_{C_n \cap Q_n} x_1,$$

where $C_n = \{v \in \mathcal{H}_1 : \|w_n - v\| \leq \|z_n - v\| + \eta_n\}$ and $Q_n = \{v \in Q_{n-1} : \langle x_1 - x_n, x_n - v\rangle \geq 0\}$.

Replace n by $n + 1$ and then repeat **Step 1.**

By setting $S_i = P_{U_i}$ and $T_j = P_{V_j}$, then the following results are consequences of Lemma 3.1 and Theorem 3.1, respectively.

Lemma 4.1. $z_n \in \Phi$ *if and only if* $y_n = 0$ *for some* $n \in \mathbb{N}$.

Theorem 4.1. *The sequence* $\{x_n\}$ *generated by Algorithm 4.1 converges strongly to a solution* $\bar{z}$ *of problem* (12), *where* $\bar{z} = P_\Phi x_1$.

5. Conclusions

A new type of hybrid projection method by using inertial effects and selection techniques, Algorithm 3.1, is proven to solve the SCFPP (1). The suggested method's convergence study shows that the sequence generated by Algorithm 3.1 converges strongly to a solution of the problem under some basic control conditions.

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