APPLICATION OF FIXED POINT METHOD FOR SOLVING NONLINEAR VOLTERRA-HAMMERSTEIN INTEGRAL EQUATION

Khosrow MALEKNEJAD[1], Parvin TORABI[2]

There are various numerical methods to solve nonlinear integral equations. Most of them transform the integral equation into a system of nonlinear algebraic equations. It is cumbersome to solve these systems, or the solution may be unreliable. In this paper, we study the application of the fixed point method to solve Volterra-Hammerstein integral equations. This method does not lead to a nonlinear algebraic equations system. We show how the proper conditions guarantee the uniqueness of the solution and how the fixed point method approximates this solution. A bound for the norm of the error is derived and our results prove the convergence of the method. Finally, we present numerical examples which confirm our approach.

Keywords: Fixed point theory; counteractive operator; Volterra-Hammerstein integral equation; Fixed point method; Sinc quadrature.

1. Introduction

Many problems which arise in mathematical physics, engineering, biology, economics and etc., lead to mathematical models described by nonlinear integral equations. (cf. [1],[2],[3]). For instance, the Hammerstein integral equations appear in nonlinear physical phenomena such as electro-magnetic fluid dynamics, reformulation of boundary value problems with a nonlinear boundary condition (see [4]). This equation is as follows

$$x(t) = g(t) + \int_a^b k(t,\tau) H(\tau, x(\tau)) d\tau, \tag{1}$$

for all $t \in I = [a,b]$ which is a Fredholm-type integral equation. Many different methods have been used to approximate the solution of such integral equations. For example we can mention the following approaches. In [5], a variation of Nystrom's method is introduced. The classical method of successive approximations is used in [6]. Some collocation-type methods are developed in [7, 8]. An approach based on single-term Walsh series is proposed in [9]. In [10]

[1] Prof., Dept.of Mathematics, Iran university of science & Technology, Tehran, Iran, e-mail: maleknejad@iust.ac.ir, homepage: http://webpages.iust.ac.ir/maleknejad
[2] Ph.D., Dept.of Mathematics, Iran university of science & Technology, Tehran, Iran, e-mail: p_torabi@iust.ac.ir

Hammerstein equation is solved by using Walsh-Hybrid functions. Some methods based on interpolations, Petrov-Galerkin, a combination of spline-collocation and Lagrange interpolation, and Dabechies wavelets have been introduced in [11, 12, 13, 14].

A Volterra-Hammerstein integral equation is introduced as follows

$$x(t) = g(t) + \int_a^t k(t,\tau)H(\tau, x(\tau))d\tau, \tag{2}$$

for all $t \in I = [a,b]$. Han [15] discusses the asymptotic error expansion of a collocation-type method for Volterra-Hammerstein integral equations. The methods in [15, 7] transform a given integral equation into a system of nonlinear equations, which has to be solved with some kind of iterative method. In [7], the definite integrals involved in the solution may be evaluated analytically only in favorable cases, while in [15] the integrals involved in the solution have to be evaluated at each step of the iteration.

In the methods mentioned above, the integral equation is transformed into a system of nonlinear equations which has to be solved with iterative methods. It is cumbersome to solve these systems, or the solution may be unreliable. To eliminate this problem, we have made an attempt to prepare a numerical scheme to approximate a solution for integral equation (2) based on the fixed point method and some quadrature rules such as sinc quadrature which has exponential rate of accuracy [16, 17]. We studied the appropriate conditions and performance of this method for Fredholm-Hammerstein equation (1) in [18]. This method has two advantages that encourage us to use it. Firstly, there is not any system of nonlinear equations with its relevant difficulties. Furthermore, this method is very simple to apply and to make an algorithm.

The organization of this paper is as follows. First we mention some necessary concepts such as contractive operators and sinc quadrature which we will use later. Then we introduce our numerical technique, and discuss its convergence. Finally, we present some numerical examples to show the efficiency and accuracy of our proposed method.

2. Preliminaries

Let us introduce some necessary concepts and tools which help us to frame our method. They can be found in books on numerical analysis such as [16, 17, 19, 20].

2.1. Contractive operator in Banach spaces

Let V be a Banach space with the norm $\|.\|_V$ and let K be a subset of V. Consider an operator $T : K \to V$ defined on K.

Definition 2.1. We say that an operator $T:K\to V$ is contractive with contractivity constant $\alpha\in[0,1)$ if

$$\|T(x)-T(y)\|_V\le\alpha\|x-y\|_V,\qquad \forall x,y\in K.$$

The operator T is called non-expansive if

$$\|T(x)-T(y)\|_V\le\|x-y\|_V,\qquad \forall x,y\in K,$$

and Lipschitz continuous if there exists a constant $L\ge 0$ such that

$$\|T(x)-T(y)\|_V\le L\|x-y\|_V,\qquad \forall x,y\in K.$$

The following theorem is known as Banach fixed point theorem and plays an important role in guaranteeing the existence and uniqueness of the solution of nonlinear equations.

Theorem 2.2. Assume that K is a nonempty closed set in a Banach space V, and $T:K\to K$ is a contractive mapping with contractivity constant $0\le\alpha<1$. Then the following results hold

(1) Existence and uniqueness: There exists a unique $x^*\in K$ such that

$$x^*=T(x^*).$$

(2) Convergence and error estimates of the iteration: For any $x_0\in K$, the sequence $\{x_n\}\subset K$ defined by $x_{n+1}=T(x_n)$, $n=0,1,\ldots$, converges to x^*:

$$\|x_n-x^*\|_V\to 0,\qquad as\quad n\to\infty.$$

For the error, the following bounds are valid

$$\|x_n-x^*\|_V\le\frac{\alpha^n}{1-\alpha}\|x_1-x_0\|_V, \tag{3}$$

$$\|x_n-x^*\|_V\le\frac{\alpha}{1-\alpha}\|x_n-x_{n-1}\|_V, \tag{4}$$

$$\|x_n-x^*\|_V\le\alpha\|x_{n-1}-x^*\|_V. \tag{5}$$

Proof: [19].

2.2. Sinc quadrature

Based on [17] we introduce a double exponential formula for the numerical evaluation of the indefinite integration of analytic functions over (a,x) where $a<x<b$, by means of the sinc method.

First, we introduce the cardinal function and some of its quadrature properties. We accept the following definition of $\sin c(x)$ from [16]

$$\sin c(x)=\begin{cases}\dfrac{\sin(\pi x)}{\pi x}, & x\neq 0\\ 1, & x=0.\end{cases}$$

Now, for $h > 0$ and integer k, we define k'th sinc function with step size h by

$$S(k,h)(x) = \frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h}.$$

Let $d > 0$, $D_d = \left\{z \in C : |\text{Im}(z)| < d\right\}$, and let D be a simply connected domain with boundary ∂D. Let a and b denote two distinct points of ∂D, and let $t = \phi(z)$ denote a conformal map of D onto the strip region D_d such that

$$\phi((a,b)) = (-\infty,\infty), \qquad \lim_{t\to a}\phi(t) = -\infty, \qquad \lim_{t\to b}\phi(t) = \infty.$$

Then all functions f that are analytic in $\mathbb{C}$, have the cardinal series representation

$$\frac{f(x)}{\phi'(x)} = \sum_{k=-\infty}^{\infty} \frac{f(kh)}{\phi'(kh)} S(k,h)\text{o}\phi(kh),$$

or

$$\frac{f(x)}{\phi'(x)} \approx \sum_{k=-N}^{N} \frac{f(kh)}{\phi'(kh)} S(k,h)\text{o}\phi(kh),$$

for any large N.

Now, in order to have the sinc approximation on a finite interval (a,b) conformal map is employed as $\phi(x) = \ln\left(\frac{x-a}{b-x}\right)$. This map carries the eye-shaped complex domain

$$\left\{z = x + iy : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d \le \frac{\pi}{2}\right\},$$

onto the infinite strip $D_d = \left\{\mu = \alpha + \beta i : |\beta| < d < \frac{\pi}{2}\right\}$, and the basis function on finite interval (a,b) are given by

$$S(k,h)\text{o}\phi(x) = \frac{\sin(\pi(\phi(x)-kh)/h)}{\pi(\phi(x)-kh)/h}.$$

Now, we use the following sinc quadrature formulas to estimate an integral by (see [17] for more details)

$$\int_a^b f(z)dz = h\sum_{k=-N}^{N} \frac{f(z_k)}{\phi'(z_k)} + O(\exp(-\frac{2\pi dN}{\log(2\pi dN/\beta)})), \tag{6}$$

$$\int_a^x f(z)dz = h\sum_{k=-N}^{N} \frac{f(z_k)}{\phi'(z_k)}\eta_{h,k}(x) + O(\frac{\log N}{N}\exp(-\frac{\pi dN}{\log(\pi dN/\beta)})), \tag{7}$$

where $\eta_{h,k}(x) = 1/2 + 1/\pi Si(\pi(x - kh)/h)$ and $Si(t) = \int_0^t \sin(x)/x\, dx$ with $z_k = a + be^{kh}/1 + e^{kh}$ for $k = -N, ..., N$ and $h = 1/N \log(\pi dN/\beta)$.
These quadrature formulas have an exponential rate of accuracy, thus we use them in the present paper.

3. Fixed point method

We consider Hammerstein integral equation (2) and assume that $g \in C[a,b]$ and $k \in L^2[a,b]^2$. Now we define the operator T as follows

$$(Tx)(t) = g(t) + \int_a^t k(t,\tau) H(\tau, x(\tau)) d\tau. \qquad (8)$$

Obviously, the solution of equation (2) is the fixed point of operator T. By choosing the initial function $x_0(t) \in C[a,b]$ we introduce the fixed point iteration

$$x_{n+1}(t) = (Tx_n)(t) = g(t) + \int_a^t k(t,\tau) H(\tau, x_n(\tau)) d\tau, \quad t \in [a,b], \quad n \geq 0. \qquad (9)$$

In the following, we show that under proper assumptions, T has a unique fixed point and the sequence $\{x_n(t)\}_{n=0}^{\infty}$ generated by iteration (9) converges to this unique fixed point. Then to prepare a numerical algorithm, we replace the integral part by a numerical integration. Finally, we find an error bound for the approximation solution.

3.1. Convergence of the method

Here, we show that under proper assumptions the sequence $\{x_n(t)\}_{n=0}^{\infty}$ generated by iteration (9) converges to the unique fixed point of the operator T.

Theorem 3.1. Assume K is a nonempty closed set in a Banach space V, and $T: K \to K$ is continuous. Suppose T^m is a contraction for some positive integer m. Then T has a unique fixed point in K. Moreover, the iteration method

$$x_{n+1} = T(x_n), \; n = 0,1,\ldots,$$

converges.

Proof: The proof is similar to the proof of theorem (2.2.1) of [21]. Since T^m is a contraction then there exists a constant $\alpha \in [0,1)$ such that

$$\|T^m x - T^m y\| \leq \alpha \|x - y\|,$$

for all $x, y \in K$. Consequently, the Banach fixed point theorem implies that operator T^m has a unique fixed point x^* in K, and $T^{mn} y \to x^*$ as $n \to \infty$ for any $y \in K$.
Since

$$Tx^* = T(T^m x^*) = T^{m+1}x^* = T^m(Tx^*),$$

then $Tx^* \in K$ is also a fixed point of T^m, and hence by uniqueness, we have $Tx^* = x^*$, i.e. x^* is a fixed point of T. Moreover, setting $y = T^k x_0$ $(k = 0,1,...,m-1)$ respectively, we see that $T^{mn'+k}x_0 \to x^*$ as $n' \to \infty$ $(k = 0,1,...,m-1)$, and since for every $n \in \mathbb{Z}^+$ there exists a unique $n' \in \mathbb{Z}^+$ and $k \in \{0,1,...,m-1\}$, where $n = mn' + k$, then $x_n \to x^*$ as $n \to \infty$, where x_n is defined by $x_{n+1} = T(x_n)$, $n = 0,1,\ldots$. Finally, the uniqueness of the fixed point for T^m implies the uniqueness of the fixed point for T. □

Theorem 3.2. Consider the operator T introduced by relation (8) and assume $g \in C[a,b]$, $k \in L^2[a,b]^2$, i.e. there exists a constant $M > 0$ where

$$\left(\int_a^b k^2(t,\tau)d\tau\right)^{\frac{1}{2}} \le M < \infty.$$

Also $H(\tau,x)$ satisfies a uniform Lipschitz condition with respect to its second argument

$$\| H(\tau,x) - H(\tau,y)\|_\infty \le h \| x - y \|_\infty, \tag{10}$$

For all $\tau \in [a,b]$ and $x, y \in \mathbb{R}$.

Then T has a unique fixed point in $C[a,b]$. Moreover, the iterative method (9) converges for any initial function $x_0 \in C[a,b]$.

Proof: Suppose that $x, y \in C[a,b]$, then for all $t \in [a,b]$ we derive the following inequality

$$|(Tx)(t) - (Ty)(t)| = \left|\int_a^t k(t,\tau)\big(H(\tau,x(\tau)) - H(\tau,y(\tau))\big)d\tau\right|$$

$$\le \left(\int_a^b k^2(t,\tau)d\tau\right)^{\frac{1}{2}} \| H(\tau,x) - H(\tau,y)\|_\infty (t-a)^{\frac{1}{2}}$$

$$\le Mh(t-a)^{\frac{1}{2}} \| x - y \|_\infty,$$

where Cauchy-Schwartz inequality and Lipschitz condition (10) were used in the recent relation. Also we get

$$|(T^2x)(t) - (T^2y)(t)| = \left|\int_a^t k(t,\tau)\big(H(\tau,(Tx)(\tau)) - H(\tau,(Ty)(\tau))\big)d\tau\right|$$

$$\le Mh\left(\int_a^t |(Tx)(\tau) - (Ty)(\tau)|^2 d\tau\right)^{\frac{1}{2}}$$

$$\le \frac{[Mh(t-a)^{\frac{1}{2}}]^2}{\sqrt{2!}} \| x - y \|_\infty .$$

By a mathematical induction, we obtain

$$|(T^m x)(t) - (T^m y)(t)| \leq \frac{[Mh(t-a)^{\frac{1}{2}}]^m}{\sqrt{m!}} \| x - y \|_\infty .$$

Thus

$$\| (T^m x)(t) - (T^m y)(t) \|_\infty \leq \frac{[Mh(b-a)^{\frac{1}{2}}]^m}{\sqrt{m!}} \| x - y \|_\infty .$$

Since

$$\frac{[Mh(b-a)^{\frac{1}{2}}]^m}{\sqrt{m!}} \to 0 \quad as \quad m \to \infty,$$

the operator T^m is a contraction on $C[a,b]$ when m is chosen sufficiently large. By the theorem (3.1), the operator T has a unique fixed point in $C[a,b]$ and the iteration sequence (9) converges to the solution. □

3.2. Approximation of the integral part

Now, to start iteration (9), we need an initial function. Since $g(t) \in C[a,b]$, it can be chosen as the initial function for iteration, i.e. $x_0 \equiv g$. In each iteration we have to calculate the integral part of operator T . It can be cumbersome if we compute that integral analytically, so we use a quadrature method such as sinc integration to evaluate integral part of the operator T numerically. By substituting sinc quadrature (7) in equation (9) we have

$$\begin{aligned} x_{n+1}(t) &= (Tx_n)(t) = g(t) + \int_a^t k(t,\tau) H(\tau, x_n(\tau)) d\tau \\ &\approx g(t) + h \sum_{k=-N}^{N} \frac{k(t,\tau_k) H(\tau_k, x_n(\tau_k))}{\phi'(\tau_k)} \eta_{h,k}(t), \end{aligned} \tag{11}$$

where τ_k , h , $\phi(\tau)$ and $\eta_{h,k}(t)$ were introduced in subsection (2.2).

In any iteration of relation (11), x_{n+1} arises directly from x_n without solving any large system of nonlinear algebraic equations. This is a great advantage of the proposed method.

3.3. Error bound

In the previous subsection, we proved that if T^m be a contraction then T has a unique fixed point x^*. Also the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (9) converges to x^*. Now we find an error bound for this sequence. Let $\{y_{n'}\}_{n=0}^{\infty}$ be the sequence generated by fixed point iteration for the operator T^m , i.e.

$y_{n'+1} = T^m y_{n'}$, $n' \geq 0$, $y_0 = T^k x_0$ for any x_0. Using Banach fixed point theorem, we have the following error bound

$$\| y_{n'} - x^* \| \leq \frac{\alpha^{n'}}{1-\alpha} \| y_1 - y_0 \|, \ n' > 1.$$

According to the proof of theorem (3.1), we have $x_n = x_{mn'+k} = T^{mn'+k} x_0 = y_{n'}$ then

$$\| x_n - x^* \| = \| y_{n'} - x^* \| \leq \frac{\alpha^{n'}}{1-\alpha} \| y_1 - y_0 \| = \frac{\alpha^{\frac{n-k}{m}}}{1-\alpha} \| x_{m+k} - x_k \|, \qquad (12)$$

where $k \in \{0,1,...,m-1\}$ is the residual of $\dfrac{n}{m}$.

Now, let $x_{n+1}^{(N)}(t)$ be the approximation of $x_{n+1}(t)$ by sinc quadrature in (11) and $x^*(t)$ be the exact fixed point of T. Then, we have the following error bound

$$\| x_{n+1}^{(N)} - x^* \| \leq \| x_{n+1}^{(N)} - x_{n+1} \| + \| x_{n+1} - x^* \|,$$

where the bound of these errors were obtained in (7) and (12).

In the next section the accuracy and efficiency of the method are shown using some numerical examples.

4. Experimental results

In order to test the utility of the proposed numerical method, we give the following examples. In all examples, we choose the tolerance $\varepsilon = 10^{-7}$ to stop the iterations, i.e. fixed point iterations stop when $\| x_n - x_{n-1} \| < \varepsilon$. All routines have been written in Fortran 90.

Example 4.1. Consider the following Volterra-Hammerstein integral equation

$$x(t) = \frac{t}{e^{t^2}} + \int_0^t 2t\tau e^{-x^2(\tau)} d\tau, \ 0 \leq t \leq 1.$$

The exact solution of this equation is $x_{exact}(t) = t$. We try to solve this by our proposed method. By choosing $x_0(t) = \dfrac{t}{e^{t^2}}$ and using sinc quadrature we have

$$x_{n+1}(t) = \frac{t}{e^{t^2}} + \int_0^t 2t\tau e^{-x_n^2(\tau)} d\tau$$

$$\approx \frac{t}{e^{t^2}} + h \sum_{k=-N}^{N} \frac{2t\tau_k e^{-x_n^2(\tau_k)}}{\phi'(\tau_k)} \eta_{h,k}(t), \qquad n \geq 0.$$

Where $h = \pi/6$ and $\tau_k = e^{kh}/1+e^{kh}$ for all $k = -N, ..., N$. The functions $\phi(t)$ and $\eta_{h,k}(t)$ are introduced in subsection 2.2. Table 1 shows the approximation values and errors in some points $t \in [0,1]$.

Table 1

Approximation and error values in some points in example 4.3

t	Exact	N=20		N=50		N=100	
		Appro.	Error	Appro	Error	Appro	Error
0.0	0.0	0.000000E-0	0.000E0	0.000000E-0	0.000E0	0.000000E-0	0.000E0
0.2	2.0E-1	2.000000E-1	1.490E-8	2.000000E-1	0.000E0	2.000000E-1	0.000E0
0.4	4.0E-1	4.000008E-1	7.749E-7	4.000001E-1	1.192E-7	4.000000E-1	2.980E-8
0.6	6.0E-1	6.000094E-1	9.360E-6	6.000015E-1	1.490E-6	6.000004E-1	4.172E-7
0.8	8.0E-1	8.000458E-1	4.584E-5	8.000073E-1	7.331E-6	8.000018E-1	1.848E-6
1.0	1.0	1.000129E0	1.295E-4	1.000021E0	2.062E-5	1.000005E0	5.007E-6

Example 4.2. $x_{exact}(t) = \sqrt{t} + 0.1$ is the exact solution of the following Volterra-Hammerstein integral equation

$$x(t) = \sqrt{t} + 0.1 - t\sin(t) + \int_0^t \frac{\sin(t)x^2(\tau)}{(\sqrt{\tau}+0.1)^2} d\tau, \quad 0 \le t \le 1.$$

By applying the proposed method we approximate the solution of this equation. Approximation of the solution is obtained after 10 iterations with $\|e\|_\infty = 2.161E-7$. Exact and approximation solutions based on sinc quadrature, with $N = 20$ and $N = 50$, are shown in Fig. 1.

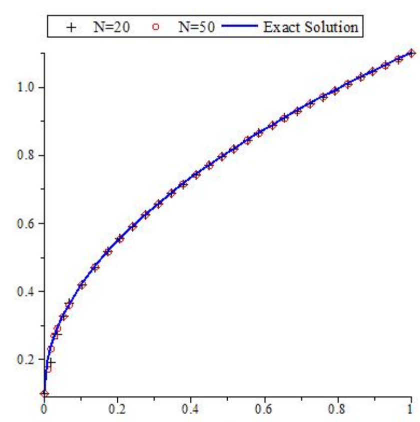

Fig. 1. Approximation and exact solutions for N=20 and N=50 in example 4.2

Example 4.3. Consider the following Volterra-Hammerstein integral equation with the exact solution $x_{exact}(t) = \dfrac{\sin(t)}{\sqrt{t}}$,

$$x(t)=\pi+(t-\pi)\cos(t)-\sin(t)+\frac{\sin(t)}{\sqrt{t}}+\int_0^t \frac{\tau(\tau-t)}{\sin(\tau)}x^2(\tau)d\tau,\ \ 0\le t\le 1.$$

Approximation of the solution is obtained after 17 iterations with $\|e\|_\infty=1.549E-5$. Comparison between the approximation and the exact solution in some points is shown in table 2.

Table 2

Approximation and exact values in some points in example 4.3

t	Appro. values	Exact values
0.001	0.03162434	0.03162277
0.2	0.44423140	0.44423810
0.4	0.61572050	0.61572440
0.6	0.72894610	0.72895030
0.8	0.80202200	0.80202850
1.0	0.84146090	0.84147100

Example 4.4. The following Volterra-Hammerstein integral equation

$$x(t)=\frac{2}{15}t^6-\frac{1}{3}t^4+t^2-1+\int_0^t (t-2\tau)x^2(\tau)d\tau,\ \ 0\le t\le 1.$$

has exact solution $x_{exact}(t)=t^2-1$. Approximation of the solution is obtained after 6 iterations with $\|e\|_\infty=2.342E-5$. Fig. 2 shows $x_i(t)$ in three consecutive iterations.

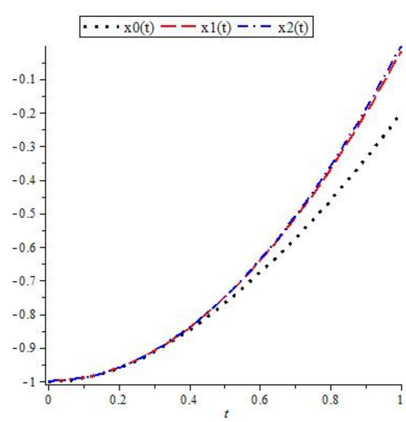

Fig. 2. Graph of $x_i(t)$ in three consecutive iterations in example 4.4

Example 4.5. In this example we solve the following Volterra integral equation which has the exact solution $x_{exact}(t)=t$,

$$x(t)=t\cos(t)+\int_0^t t\ \sin(x(\tau))d\tau,\ \ 0\le t\le 1.$$

For some N , the error of the approximation solution is shown in table 3.

Table 3

Absolute error in some points $0 < t \leq 1$ and for some quadrature nodes N in example 4.5

t	N=25	N=50	N=100
0.001	4.657E-10	4.657E-10	4.657E-10
0.2	3.532E-6	1.371E-6	6.855E-7
0.4	5.811E-6	2.950E-6	1.460E-6
0.6	7.749E-7	4.760E-6	2.444E-6
0.8	1.204E-5	6.855E-6	3.576E-6
1.0	3.988E-5	8.702E-6	4.768E-6

5. Conclusions

We approximated the solution of Volterra-Hammerstein integral equation by the fixed point method. We have shown that some appropriate conditions guarantee the convergence of the method. This method has two advantages. On the one hand, we do not have to deal with any system of nonlinear equations. On the other hand, this method is very simple to apply and to make an algorithm. The numerical results verify that the method is valid. It is worthy to note that this method can be utilized as a very accurate algorithm to solve linear and nonlinear integro-differential equations and functional integral equations arising in physics and other fields of applied mathematics.

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