APPLICATIONS OF TAUBERIAN THEOREMS FOR CESÀRO ORLICZ DOUBLE SEQUENCES

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In this article, we prove the necessary and sufficient Tauberian conditions in $n-$normed spaces for double sequences which are Orlicz $(C, 1, 1)\Delta^r-$summable to be Orlicz Δ^r convergent. We also make an effort to study some (k, l) fold applications of Orlicz $(C, 1, 1)\Delta^r$ summability method for a Tauberian theorem. Finally, we establish some relation between Orlicz $(C, 1, 1)\Delta^r-$summable and Orlicz Δ^r convergent sequences under slowly oscillating conditions over $n-$normed spaces.

Keywords: Orlicz function, difference operator, $n-$normed space, slowly oscillating sequence, $(C, 1, 1)$ summability method, Tauberian theorem.

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1. Introduction and Preliminaries

In [20] Tauber gave the converse of Abel's theorem under certain additional hypothesis known as Tauberian condition(s). Tauberian theorem states that the sequence (x_k) is convergent if it is summable to the same limit with respect to specific summability method and satisfy Tauberian condition(s). Tauberian type theorem has many applications in different fields of mathematics such as probability theory, number theory, complex analysis and the analysis of differential operators (see [1], [2], [9]). Several mathematicians such as Móricz [12], Talo and Başar [18], Talo and Çakan [19] and Tripathy and Dutta [21] have proved Tauberian theorems for different sumability methods.

A function $\mathcal{M} : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function, if it satisfy the following conditions:

(i) $\mathcal{M}$ is convex, (ii) $\mathcal{M}$ is continuous, (iii) $\mathcal{M}$ is non-decreasing with $\mathcal{M}(0) = 0$, $\mathcal{M}(x) > 0$ for $x > 0$ and (iv) $\mathcal{M}(x) \to \infty$ as $x \to \infty$.

Let ω be the space of all real or complex sequences. Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space:

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$$l_{\mathcal{M}} = \left\{ x \in \omega : \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is called an Orlicz sequence space.

The idea of difference sequence spaces was introduced by Kızmaz [8] who studied the difference sequence spaces $l_{\infty}(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_{\infty}(\Delta^r)$, $c(\Delta^r)$ and $c_0(\Delta^r)$. Let r be a non-negative integer and w denotes the set of real or complex sequences. Then for $Z = c, c_0$ and l_{∞}, we have the following sequence spaces

$$Z(\Delta^r) = \{x = (x_k) \in w : (\Delta^r x_k) \in Z\},$$

where $\Delta^r x = (\Delta^r x_k) = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$. Taking $r = 1$, we get the spaces studied by Et and Çolak [4]. Similarly, difference operator on double sequences can be defined as:

$$\Delta x_{k,l} = (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) = x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1},$$

and $\Delta^r x_{k,l} = \Delta^{r-1} x_{k,l} - \Delta^{r-1} x_{k,l+1} - \Delta^{r-1} x_{k+1,l} + \Delta^{r-1} x_{k+1,l+1}$.

Misiak [11] developed the concept of $n-$normed spaces. Many mathematicians have studied this concept and obtained various results (see [5], [6]). A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$, if

$$\lim_{k \to \infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a $n-$normed space X is said to be bounded if for a positive constant M, $\|x_k, z_1, \cdots, z_{n-1}\| \leq M$, for all $z_1, \cdots, z_{n-1} \in X$ and all $k \in \mathbb{N}$. One writes $x_k = O(1)$. For more details about sequence spaces (see [13], [14], [15],[17], [22]) and references therein.

Let $t = (t_{k,l})$ be a double sequence of positive real numbers. A double sequence $x = (x_{mn})$ in a $n-$normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Orlicz Δ^r convergent to $L \in X$, if for $z_1, \cdots, z_{n-1} \in X$

$$\lim_{m,n \to \infty} \left\| \mathcal{M}\left(\frac{|t_{k,l} \Delta^r x_{m,n} - L|}{\rho}\right), z_1, \cdots, z_{n-1} \right\| = 0, \text{ for some } \rho > 0.$$

A double sequence $(x_{m,n})$ in a $n-$normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Orlicz $(C,1,1)\Delta^r-$ summable to $L \in X$, if for $z_1, \cdots, z_{n-1} \in X$, $\lim_{m,n \to \infty} \|\sigma_{m,n}^{\mathcal{M},\Delta^r} - L, z_1, \cdots, z_{n-1}\| = 0$ or

$$\lim_{m,n \to \infty} \left\| \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} \mathcal{M}\left(\frac{|t_{k,l} \Delta^r x_{k,l} - L|}{\rho}\right), z_1, \cdots, z_{n-1} \right\| = 0, \text{ for some } \rho > 0.$$

In other words, we write $\mathcal{M}\big(\frac{|t_{k,l}\Delta^r x_{m,n}|}{\rho}\big) \xrightarrow{\|\cdot,\cdots,\cdot\|_X} \mathcal{M}\big(\frac{L}{\rho}\big)(C,1,1)$. One can say that $\sigma_{m,n}^{\mathcal{M},\Delta^r}$ is an Orlicz $(C,1,1)\Delta^r-$ mean and we write $\sigma_{m,n}^{\mathcal{M},\Delta^r} = \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}\mathcal{M}$

$\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right)$, for all non negative integers m and n.

Let $(u_m), (v_m), (r_n)$ and (s_n) be increasing sequences of positive integers such that $u_m < v_m$ and $u_m \to \infty$ as $m \to \infty$, $r_n < s_n$ and $r_n \to \infty$ as $n \to \infty$, also

$$\liminf_{m\to\infty}\frac{v_m}{u_m} > 1 \quad \text{and} \quad \liminf_{n\to\infty}\frac{s_n}{r_n} > 1. \tag{1}$$

If (1), then we say that $(u,v),(r,s) \in \Omega$.

A double sequence $(x_{m,n})$ in a $n-$normed space $(X, \|\cdot,\cdots,\cdot\|)$ is said to be Orlicz $(C,1,1)\Delta^r-$ slowly oscillating if for some $\rho > 0$,

$$\inf_{\substack{(u,v)\in\Omega,\\(r,s)\in\Omega}}\limsup_{m,n\to\infty}\left\|\frac{1}{(v_m-u_m)(s_n-r_n)}\sum_{k=u_m+1}^{v_m}\sum_{l=r_n+1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho}\right),\right.$$

$$(2)$$

$$\left. z_1,\cdots,z_{n-1}\right\| = 0.$$

Remark 1.1. For $(u_m) = m, v_m = [\lambda m], r_n = n$ and $s_n = [\lambda n]$ with $\lambda > 1$, where [.] means the integer part, then (2) becomes

$$\inf_{\lambda>1}\limsup_{m,n\to\infty}\left\|\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{k=m+1}^{[\lambda m]}\sum_{l=n+1}^{[\lambda n]}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho}\right),\right.$$

$$(3)$$

$$\left. z_1,\cdots,z_{n-1}\right\| = 0.$$

Also, for $0 < \lambda < 1$, (2) becomes

$$\inf_{0<\lambda<1}\limsup_{m,n\to\infty}\left\|\frac{1}{(m-[\lambda m])(n-[\lambda n])}\sum_{k=[\lambda m]+1}^{m}\sum_{l=[\lambda n]+1}^{n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n} - t_{k,l}\Delta^r x_{k,l}|}{\rho}\right),\right.$$

$$(4)$$

$$\left. z_1,\cdots,z_{n-1}\right\| = 0.$$

Remark 1.2. A sequence $(x_{m,n}) \in X$ is Orlicz Δ^r-slowly oscillating in $n-$norm, if for $\rho > 0$,

$$\inf_{\lambda>1}\limsup_{m,n\to\infty}\max_{\substack{m<k\leq[\lambda m],\\n<l\leq[\lambda n]}}\left\|\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho}\right), z_1,\cdots,z_{n-1}\right\| = 0. \tag{5}$$

Equivalently

$$\inf_{0<\lambda<1} \limsup_{m,n\to\infty} \max_{\substack{[\lambda m]<k\leq m,\\ [\lambda n]<l\leq n}} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n} - t_{k,l}\Delta^r x_{k,l}|}{\rho}\right), z_1, \cdots, z_{n-1} \right\| = 0. \quad (6)$$

(5) is satisfied if and only if

$$\left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho}\right), z_1, \cdots, z_{n-1} \right\| \to 0, \text{ whenever } z_1, \cdots, z_{n-1} \in X \quad (7)$$

and $1 < \frac{k}{m} \to 1(k, m \to \infty), 1 < \frac{l}{n} \to 1(l, n \to \infty)$.

For every $\epsilon > 0$, there exists $Q = Q(\epsilon), P = P(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$, as close to 1, such that for some $\rho > 0$, (7) becomes:

$$\left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho}\right), z_1, \cdots, z_{n-1} \right\| \leq \epsilon, \quad (8)$$

whenever $z_1, \cdots, z_{n-1} \in X$, $Q \leq m < k \leq \lambda m$ and $P \leq n < l \leq \lambda n$.

Remark 1.3. The two sided condition of Hardy-type [7] for double sequence $(x_{m,n})$ is given by

$$mn\mathcal{M}\left(\frac{|t_{k,l}\Delta^{r+1} x_{m,n}|}{\rho}\right) = O(1). \quad (9)$$

If condition (9) holds, then $(x_{m,n})$ is said to be Orlicz Δ^r-slowly oscillating in $n-$norm.

The sequence $(\sigma_{m,n}^{(k,l)\mathcal{M},\Delta^r})$ gives the (k,l) fold application of Orlicz $(C,1,1)\Delta^r$ method and is given by

$$\sigma_{m,n}^{(k,l)\mathcal{M},\Delta^r} = \begin{cases} \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{i,j}^{(k-1,l-1)\mathcal{M},\Delta^r}, & k,l \geq 1; \\ \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n}|}{\rho}\right), & k,l = 0, \end{cases}$$

where (k,l) are the elements of the set $\{(k,l)\} = \{(1,1), (2,2), (3,3), \cdots\}$. A sequence $(x_{m,n})$ is Orlicz $(H,k,l)\Delta^r-$summable to L, if $(x_{m,n})$ is transformed into a convergent sequence after $(k,l)-$fold iteration process. In other words if

$$\lim_{m,n\to\infty} \left\| \sigma_{m,n}^{(k,l)\mathcal{M},\Delta^r} - L, z_1, \cdots, z_{n-1} \right\| = 0,$$

for all $z_1, \cdots, z_{n-1} \in X$ and we write $\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n}|}{\rho}\right) \xrightarrow{\|\cdot,\cdots,\cdot\|_X} \mathcal{M}\left(\frac{L}{\rho}\right)(H,k,l)$. It is trivial that for $k,l > 0$, Orlicz $(H,k,l)\Delta^r$ summability of a sequence implies its Orlicz $(H,k+1,l+1)\Delta^r$ summability to the same value. Also Orlicz $(H,0,0)\Delta^r$ summability for $r = 0$, $\rho = 1, t_{k,l} = 1$ and $\mathcal{M}(x) = x$ is the ordinary convergence in $n-$norm and Orlicz $(H,1,1)\Delta^r$ summability method is equivalent to the Orlicz $(C,1,1)\Delta^r$ summability method.

2. **Main Results**

Theorem 2.1. *Let* $\mathcal{M}$ *be an Orlicz function and* $t = (t_{k,l})$ *be a double sequence of positive real numbers. If a double sequence* $(x_{m,n}) \in X$ *is Orlicz* Δ^r *convergent to* $L \in X$, *then it is Orlicz* $(C, 1, 1)\Delta^r - summable$ *to* $L \in X$.

Proof. Let a sequence $(x_{m,n})$ is Orlicz Δ^r convergent to L, then for each $\epsilon > 0$ and $z_1, \cdots, z_{n-1} \in X$, there exists $m_0, n_0 \in \mathbb{N}$ such that

$$\left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n} - L|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq \epsilon/2,$$

for all $m > m_0$ and $n > n_0$. Also, there exists $M > 0$, such that

$$\left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n} - L|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq M,$$

for all $m > m_0$ and $n > n_0$. Thus, for all $z_1, \cdots, z_{n-1} \in X$, we have

$$\left\| \sigma_{m,n}^{\mathcal{M},\Delta^r} - L, z_1, \cdots, z_{n-1} \right\| = \left\| \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho} \right), z_1, \cdots, \right.$$

$$\left. z_{n-1} \right\| \leq \frac{1}{mn} \sum_{k=1}^{m_0} \sum_{l=1}^{n_0} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| + \frac{1}{mn} \sum_{k=m_0+1}^{m} \sum_{l=n_0+1}^{n}$$

$$\left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq \frac{m_0 n_0 M}{mn} + \frac{\epsilon}{2}.$$

Since $\lim\limits_{m,n \to \infty} \frac{m_0 n_0 M}{mn} = 0$, there exist $m_1, n_1 \in \mathbb{N}$ such that $\left| \frac{m_0 n_0 M}{mn} \right| \leq \epsilon/2$, whenever $m > m_1$ and $n > n_1$. Hence, there exist $Q = \max\{m_0, m_1\}$ and $P = \max\{n_0, n_1\}$ such that for $m > Q$ and $n > P$, $\left\| \sigma_{m,n}^{\mathcal{M},\Delta^r} - L, z_1, \cdots, z_{n-1} \right\| \leq \epsilon$. This concludes the proof. $\square$

Lemma 2.1. *Let* (u_m) *and* (r_n) *be two increasing sequences of positive integers such that* $u_m \to \infty$ *as* $m \to \infty$ *and* $r_n \to \infty$ *as* $n \to \infty$. *If the sequence* $(x_{m,n}) \in X$ *is Orlicz* $(C, 1, 1)\Delta^r - summable$ *to* $L \in X$, *then* $(\sigma_{u_m,r_n}^{\mathcal{M},\Delta^r})$ *converges to* L.

Proof. Suppose that $(x_{m,n})$ is Orlicz $(C, 1, 1)\Delta^r -$summable to L. We can also write $\sigma_{m,n}^{\mathcal{M},\Delta^r} \xrightarrow{\|\cdot,\cdots,\cdot\|_X} L$, since $(\sigma_{u_m,r_n}^{\mathcal{M},\Delta^r})$ is a subsequence of $(\sigma_{m,n}^{\mathcal{M},\Delta^r})$. Hence, $\sigma_{u_m,r_n}^{\mathcal{M},\Delta^r} \xrightarrow{\|\cdot,\cdots,\cdot\|_X} L$. $\square$

Lemma 2.2. *Let* $\mathcal{M}$ *be an Orlicz function and* $(x_{m,n})$ *be a double sequence in a* $n-normed$ *space* $(X, \|\cdot, \cdots, \cdot\|)$ *which is Orlicz* $(C, 1, 1)\Delta^r - summable$ *to* $L \in X$. *If* $(u, v), (r, s) \in \Omega$, *then the sequence*

$$\left(R_{u,v,r,s} \right)_{m,n}^{\mathcal{M},\Delta^r} = \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{k=r_n+1}^{s_n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho} \right)$$

also converges to L.

Proof. $\left\|\left(R_{u,v,r,s}\right)_{m,n}^{\mathcal{M},\Delta^r} - L, z_1, \cdots, z_{n-1}\right\|$

$$
= \quad \left\|\left(R_{u,v,r,s}\right)_{m,n}^{\mathcal{M},\Delta^r} - \sigma_{v_m,s_n}^{\mathcal{M},\Delta^r} + \sigma_{v_m,s_n}^{\mathcal{M},\Delta^r} - L, z_1, \cdots, z_{n-1}\right\|
$$

$$
= \quad \left\|\frac{1}{(v_m-u_m)(s_n-r_n)}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) - \frac{1}{(v_m-u_m)(s_n-r_n)}\sum_{k=1}^{u_m}\right.
$$

$$
\sum_{l=1}^{r_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) - \frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) + \frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}
$$

$$
\left.\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho}\right), z_1, \cdots, z_{n-1}\right\|
$$

$$
= \quad \left\|\frac{v_m r_n + u_m s_n - u_m r_n}{(v_m-u_m)(s_n-r_n)}\cdot\frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) - \frac{u_m r_n}{(v_m-u_m)(s_n-r_n)}\right.
$$

$$
\times\frac{1}{u_m r_n}\sum_{k=1}^{u_m}\sum_{l=1}^{r_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) + \frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho}\right), z_1,
$$

$$
\left.\cdots, z_{n-1}\right\|
$$

$$
\leq \quad \left\|\frac{v_m r_n + u_m s_n}{(v_m-u_m)(s_n-r_n)}\cdot\frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right), z_1, \cdots, z_{n-1}\right\| + \left\|-\right.
$$

$$
\frac{u_m r_n}{(v_m-u_m)(s_n-r_n)}\cdot\frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) - \frac{u_m r_n}{(v_m-u_m)(s_n-r_n)}
$$

$$
\times\frac{1}{u_m r_n}\sum_{k=1}^{u_m}\sum_{l=1}^{r_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) + \frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho}\right), z_1,
$$

$$
\left.\cdots, z_{n-1}\right\| \leq \left\|\frac{v_m r_n + u_m s_n}{(v_m-u_m)(s_n-r_n)}\cdot\frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right), z_1, \cdots, \right.
$$

$$
\left.z_{n-1}\right\| + \frac{u_m r_n}{(v_m-u_m)(s_n-r_n)}\left\|\frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right) + \frac{1}{u_m r_n}\sum_{k=1}^{u_m}\right.
$$

$$
\sum_{l=1}^{r_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l}|}{\rho}\right), z_1, \cdots, z_{n-1}\right\| + \left\|\frac{1}{v_m s_n}\sum_{k=1}^{v_m}\sum_{l=1}^{s_n}\mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho}\right),\right.
$$

$$
\left.z_1, \cdots, z_{n-1}\right\|
$$

$$
\leq \quad \frac{v_m r_n + u_m s_n}{(v_m-u_m)(s_n-r_n)}\left\|\sigma_{v_m,s_n}^{\mathcal{M},\Delta_r}, z_1, \cdots, z_{n-1}\right\| + \frac{u_m r_n}{(v_m-u_m)(s_n-r_n)}\left\|\sigma_{v_m,s_n}^{\mathcal{M},\Delta_r}\right.
$$

$$+\sigma^{\mathcal{M},\Delta^r}_{u_m,r_n}, z_1, \cdots, z_{n-1}\Big\| + \Big\|\sigma^{\mathcal{M},\Delta_r}_{v_m,s_n} - L, z_1, \cdots, z_{n-1}\Big\|.$$

Since $(u_m), (v_m), (r_n)$ and (s_n) are increasing sequences then, in the limiting case $\frac{v_m r_n + u_m s_n}{(v_m - u_m)(s_n - r_n)}\Big\|\sigma^{\mathcal{M},\Delta_r}_{v_m,s_n}, z_1, \cdots, z_{n-1}\Big\| = 0$. Now, we have

$$\Big\|\left(R_{u,v,r,s}\right)^{\mathcal{M},\Delta^r}_{m,n} - L, z_1, \cdots, z_{n-1}\Big\| \leq \frac{u_m r_n}{(v_m - u_m)(s_n - r_n)}\Big\|\sigma^{\mathcal{M},\Delta^r}_{v_m s_n} - \sigma^{\mathcal{M},\Delta^r}_{u_m r_n}, z_1, \cdots,$$

$$z_{n-1}\Big\| + \Big\|\sigma^{\mathcal{M},\Delta^r}_{v_m s_n} - L, z_1, \cdots, z_{n-1}\Big\|. \tag{10}$$

By using (1), we have $\limsup\limits_{m\to\infty} \frac{u_m}{(v_m - u_m)} = \left(\liminf\limits_{m\to\infty} \frac{v_m}{u_m} - 1\right)^{-1} < \infty$

and $\limsup\limits_{n\to\infty} \frac{r_n}{(s_n - r_n)} = \left(\liminf\limits_{n\to\infty} \frac{s_n}{r_n} - 1\right)^{-1} < \infty$.

Also, by the Orlicz $(C,1,1)\Delta^r-$summability of $(x_{m,n})$ to L and Lemma 2.1, we get that the sequence $\left(R_{u,v,r,s}\right)^{\mathcal{M},\Delta^r}_{m,n}$ converges to L. $\qquad\square$

Theorem 2.2. *Let $\mathcal{M}$ be an Orlicz function, $t = (t_{k,l})$ be a double sequence of positive real numbers and $(x_{m,n})$ be a double sequence in a $n-$normed space $(X, \|\cdot, \cdots, \cdot\|)$. If $(x_{m,n})$ is Orlicz $(C,1,1)\Delta^r-$summable to $L \in X$, then $(x_{m,n})$ is Orlicz Δ^r convergent to L if and only if*

$$\inf_{\substack{(u,v)\in\Omega, \\ (r,s)\in\Omega}} \limsup_{m,n\to\infty} \Big\| \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{l=r_n+1}^{s_n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho}\right),$$

$$z_1, \cdots, z_{n-1}\Big\| = 0, \text{ for } z_1, \cdots, z_{n-1} \in X \text{ and } \rho > 0.$$

Proof. Let the double sequence $(x_{m,n})$ be convergent to L. Hence, the sequence $(\sigma^{\mathcal{M},\Delta^r}_{m,n})$ is convergent to L. Suppose $(u_m), (v_m), (r_n)$ and (s_n) are arbitrary sequences such that $(u,v), (r,s) \in \Omega$ satisfy (2). Now by Lemma 2.2, for all $z_1, \cdots, z_{n-1} \in X, \rho = 2\rho_1$, we have

$$\lim_{m,n\to\infty} \Big\| \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{l=r_n+1}^{s_n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho}\right), z_1, \cdots,$$

$$z_{n-1}\Big\| \leq \lim_{m,n\to\infty} \Big\| \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{l=r_n+1}^{s_n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho_1}\right), z_1, \cdots,$$

$$z_{n-1}\Big\| + \lim_{m,n\to\infty} \Big\| \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{l=r_n+1}^{s_n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n} - L|}{\rho_1}\right), z_1, \cdots,$$

$$z_{n-1}\Big\| = 0.$$

Conversely, assume that (2) holds. Then for given $\epsilon > 0$, there exist sequences $(u_m), (v_m), (r_n)$ and (s_n) satisfying $(u,v), (r,s) \in \Omega$ such that for

$z_1, \cdots, z_{n-1} \in X$,

$$\limsup_{m,n\to\infty} \left\| \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{l=r_n+1}^{s_n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, \right.$$

(11)

$$\left. z_{n-1} \right\| \leq \epsilon.$$

Hence, for $\rho = 2\rho_1$ and by using (11) and Lemma 2.2, we have

$$\limsup_{m,n\to\infty} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{m,n} - L|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$$

$$\leq \limsup_{m,n\to\infty} \left\| \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{l=r_n+1}^{s_n} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - L|}{\rho_1} \right), z_1, \cdots, \right.$$

$$\left. z_{n-1} \right\| + \limsup_{m,n\to\infty} \left\| \frac{1}{(v_m - u_m)(s_n - r_n)} \sum_{k=u_m+1}^{v_m} \sum_{l=r_n+1}^{s_n} \right.$$

$$\left. \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho_1} \right), z_1, \cdots, z_{n-1} \right\| \leq \epsilon.$$

Thus, the sequence $(x_{m,n})$ is Orlicz Δ^r convergent to $L \in X$. $\qquad\square$

Corollary 2.1. *Let $(x_{m,n}) \in X$ be Orlicz $(C,1,1)\Delta^r-$summable to $L \in X$. If $(x_{m,n})$ is Orlicz $(C,1,1)\Delta^r$ slowly oscillating in $n-$norm, then $(x_{m,n})$ is Orlicz Δ^r convergent to L.*

Proof. Let a double sequence $x = (x_{m,n})$ be Orlicz $(C,1,1)\Delta^r$ slowly oscillating in $n-$norm. Then, we have

$$\left\| \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{k=m+1}^{[\lambda m]} \sum_{l=n+1}^{[\lambda n]} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$$

$$\leq \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{k=m+1}^{[\lambda m]} \sum_{l=n+1}^{[\lambda n]} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, \right.$$

$$\left. z_{n-1} \right\| \leq \max_{\substack{m<k\leq[\lambda m],\\ n<l\leq[\lambda n]}} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|.$$

Taking limsup to both sides of above inequality as $m, n \to \infty$, gives

$$\limsup_{m,n\to\infty} \left\| \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{k=m+1}^{[\lambda m]} \sum_{l=n+1}^{[\lambda n]} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), \right.$$

$$\left. z_1, \cdots, z_{n-1} \right\| \leq \limsup_{m,n\to\infty} \max_{\substack{m<k<[\lambda m],\\ n<l<[\lambda n]}} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|.$$

(12)

Now by considering the inf to both sides of (12), for $\lambda > 1$ we have

$$
\inf_{\lambda > 1} \limsup_{m,n \to \infty} \left\| \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{k=m+1}^{[\lambda m]} \sum_{l=n+1}^{[\lambda n]} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), \right.
$$

$$
\left. z_1, \cdots, z_{n-1} \right\| = 0. \tag{13}
$$

Since (13) is a special case of condition (2). Hence, by Theorem 2.2, we can conclude that the sequence $(x_{m,n}) \in X$ is Orlicz Δ^r convergent to L. $\qquad \square$

Corollary 2.2. *Let* $(x_{m,n}) \in X$ *be Orlicz* $(C,1,1)\Delta^r-$*summable to* $L \in X$. *If* $mn\mathcal{M}\left(\frac{|t_{k,l}\Delta^{r+1}x_{m,n}|}{\rho} \right) = O(1)$, *then* $(x_{m,n})$ *is Orlicz* Δ^r *convergent to* L.

Proof. Suppose that for a sequence $(x_{m,n}) \in X$, the condition $mn\mathcal{M}\left(\frac{|t_{k,l}\Delta^{r+1}x_{m,n}|}{\rho} \right)$ $= O(1)$ holds. Then for some $M > 0$ and for every $z_1, \cdots, z_{n-1} \in X$, we have

$$
\left\| mn\mathcal{M}\left(\frac{|t_{k,l}\Delta^{r+1}x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq M.
$$

Hence, $\left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{k,l} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$

$$
= \left\| \sum_{i=m+1}^{k} \sum_{j=n+1}^{l} \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{i,j} - t_{k,l}\Delta^r x_{i-1,j-1}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq \sum_{i=m+1}^{k}
$$

$$
\sum_{j=n+1}^{l} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^{r+1}x_{i,j}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq \sum_{i=m+1}^{k} \sum_{j=n+1}^{l} \frac{M}{ij} \leq \sum_{i=m+1}^{k} \sum_{j=n+1}^{l}
$$

$$
\frac{M}{mn} = \left(\frac{k}{m} - 1 \right)\left(\frac{l}{n} - 1 \right)M \to 0 \text{ as } \frac{k}{m} \to 1 \text{ and } \frac{l}{n} \to 1.
$$

Hence, (5) is satisfied. Thus, $(x_{m,n})$ is Orlicz Δ^r convergent to L using Corollary 2.1. $\qquad \square$

Lemma 2.3. *Let* $\mathcal{M}$ *be an Orlicz function. If* $(x_{m,n}) \in X$ *is Orlicz* $(C,1,1)\Delta^r$ *slowly oscillating in* $n-$*norm, then* $\left\| \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} kl\mathcal{M}\left(\frac{|t_{k,l}\Delta^{r+1}x_{k,l}|}{\rho} \right), z_1, \cdots, \right.$

$\left. z_{n-1} \right\| \leq M$, *for all* $z_1, \cdots, z_{n-1} \in X$.

Proof. Let a double sequence $(x_{m,n})$ be Orlicz $(C,1,1)\Delta^r$ slowly oscillating in $n-$norm. For $\lambda > 1$, we define

$$
\delta_{m,n}(x,\lambda) = \max_{\substack{m < k \leq [\lambda m], \\ n < k \leq [\lambda n]}} \left\| \sum_{i=m+1}^{k} \sum_{j=n+1}^{l} \mathcal{M}\left(\frac{|t_{k,l}\Delta^{r+1}x_{i,j}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|.
$$

Thus, we have $\left\| \sum\limits_{k=1}^{m} \sum\limits_{l=1}^{n} kl\mathcal{M}\left(\dfrac{|t_{k,l}\Delta^{r+1}x_{k,l}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\infty}$

$\left\| \sum\limits_{\frac{m}{2^{i+1}} \leq k < \frac{m}{2^i}, \frac{n}{2^{j+1}} \leq l < \frac{n}{2^j}} kl\mathcal{M}\left(\dfrac{|t_{k,l}\Delta^{r+1}x_{k,l}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| \leq \left(\sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\infty} \dfrac{mn}{2^i.2^j} \right)$

$\delta_{\left[\frac{m}{2^i+1}, \frac{n}{2^j+1} \right]}(x, \lambda) \leq mnM.$ This concludes the proof. $\qquad\qquad \square$

Theorem 2.3. *Let $\mathcal{M}$ be an Orlicz function, $t = (t_{k,l})$ be a double sequence of positive real numbers and $(x_{m,n}) \in X$ be Orlicz $(H, k, l)\Delta^r-$summable to $L \in X$. If $(x_{m,n})$ is Orlicz slowly oscillating in $n-$norm, then $(x_{m,n})$ is Orlicz Δ^r convergent to L.*

Proof. Firstly, we show that a sequence $(\sigma_{m,n}^{\mathcal{M},\Delta^r})$ is slowly oscillating in $n-$norm. For this consider $\epsilon > 0$ be given. Then there exists $Q = Q(\epsilon), P = P(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$, such that (8) is satisfied. Let $Q \leq m < k \leq \lambda m$ and $P \leq n < l \leq \lambda n$. Then by using Orlicz $(C, 1, 1)\Delta^r-$summability, we have

$\left\| \sigma_{k,l}^{\mathcal{M},\Delta^r} - \sigma_{m,n}^{\mathcal{M},\Delta^r}, z_1, \cdots, z_{n-1} \right\|$

$= \left\| \dfrac{1}{kl} \sum\limits_{i=1}^{k} \sum\limits_{j=1}^{l} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j}|}{\rho} \right) - \dfrac{1}{mn} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$

$= \left\| \dfrac{1}{kl} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j}|}{\rho} \right) + \dfrac{1}{kl} \sum\limits_{i=m+1}^{k} \sum\limits_{j=n+1}^{l} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j}|}{\rho} \right) - \dfrac{1}{mn} \sum\limits_{i=1}^{m} \right.$

$\left. \sum\limits_{j=1}^{n} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\| = \left\| \dfrac{mn-kl}{(kl)(mn)} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j}|}{\rho} \right) \right.$

$\left. + \dfrac{1}{kl} \sum\limits_{i=m+1}^{k} \sum\limits_{j=n+1}^{l} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$

$= \left\| \dfrac{kl-mn}{(kl)(mn)} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{m,n} - t_{k,l}\Delta^r x_{i,j}|}{\rho} \right) + \dfrac{1}{kl} \sum\limits_{i=m+1}^{k} \sum\limits_{j=n+1}^{l} \right.$

$\left. \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$

$= \left\| \dfrac{kl-mn}{(kl)(mn)} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \sum\limits_{p=i+1}^{m} \sum\limits_{q=j+1}^{n} \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^{r+1} x_{p,q}|}{\rho} \right) + \dfrac{1}{kl} \sum\limits_{i=m+1}^{k} \sum\limits_{j=n+1}^{l} \right.$

$\left. \mathcal{M}\left(\dfrac{|t_{k,l}\Delta^r x_{i,j} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$

$\leq \dfrac{kl-mn}{(kl)(mn)} \left\| \sum\limits_{p=2}^{m} \sum\limits_{q=2}^{n} (p-1)(q-1)\mathcal{M}\left(\dfrac{|t_{k,l}\Delta^{r+1} x_{p,q}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|$

$$+ \quad \frac{1}{kl} \sum_{i=m+1}^{k} \sum_{j=n+1}^{l} \left\| \mathcal{M}\left(\frac{|t_{k,l}\Delta^r x_{i,j} - t_{k,l}\Delta^r x_{m,n}|}{\rho} \right), z_1, \cdots, z_{n-1} \right\|.$$

Now, by using (8) and Lemma 2.3, we have $\left\| \sigma_{k,l}^{\mathcal{M},\Delta^r} - \sigma_{m,n}^{\mathcal{M},\Delta^r}, z_1, \cdots, z_{n-1} \right\| =$
$\left(1 - \frac{mn}{kl}\right)M + \left(1 - \frac{m}{k}\right)\left(1 - \frac{n}{l}\right)\epsilon < (\lambda - 1)M + (\lambda - 1)(\lambda - 1)\epsilon < (\lambda - 1)(M + (\lambda - 1)\epsilon) < \epsilon$, whenever $z_1, \cdots, z_{n-1} \in X, Q \le m < k \le \lambda m$ and $P \le n < l \le \lambda n$, provided that $1 < \lambda < (1+\epsilon)/(M+\epsilon)$. Hence, we obtained the slow oscillation of $(\sigma_{m,n}^{\mathcal{M},\Delta^r})$. By applying the same procedure above, we have

$$\sigma_{m,n}^{(c,d)\mathcal{M},\Delta^r} \text{ is slowly oscillating for all integers } c, d > 0. \tag{14}$$

Since, by given condition

$$\sigma_{m,n}^{(k,l)\mathcal{M},\Delta^r} \xrightarrow{\|\cdot,\cdots,\cdot\|_X} L, \tag{15}$$

we have $\sigma_{m,n}^{(k-1,l-1)\mathcal{M},\Delta^r} \xrightarrow{\|\cdot,\cdots,\cdot\|_X} L(H,1,1)$. Now, by taking $c = k-1$ and $d = l-1$ in (14) and by using Corollary 2.1 with respect to sequence $\sigma_{m,n}^{(k-1,l-1)\mathcal{M},\Delta^r}$, gives

$$\sigma_{m,n}^{(k-1,l-1)\mathcal{M},\Delta^r} \xrightarrow{\|\cdot,\cdots,\cdot\|_X} L. \tag{16}$$

Now by taking into account (15) and (16) and continuing in the same way, we have $\sigma_{m,n}^{\mathcal{M},\Delta^r} \xrightarrow{\|\cdot,\cdots,\cdot\|_X} L$. Hence by the consequence of Corollary 2.1, we can conclude that a double sequence $(x_{m,n})$ is Orlicz Δ^r convergent to L. $\qquad\square$

3. Conclusions

In this paper we have presented some Tauberian theorems for Orlicz $(C,1,1)\Delta^r$ summability method and Orlicz $(H,k,l)\Delta^r$ summability method in $n-$normed spaces for double sequences. We have also introduced slowly oscillating conditions for Orlicz $(C,1,1)\Delta^r-$ summable double sequences and Orlicz $(H,k,l)\Delta^r-$ summable double sequences in $n-$normed spaces. Tauberian theorems has many applications in other disciplines also. Likewise, in mathematical physics [3] we use Tauberian Theorems for stabilizing the solutions of the Cauchy problem for the heat kernel equation, to handle multicomponent gas diffusion and to solve asymptotic Cauchy problem for a free Schrödinger equation in the norms of different Banach spaces. In applied mathematics, the general version of Fourier Tauberian theorems for monotone function combined with Berezin's inequality gives new version of the Li-Yau estimates for the counting function of the Dirichlet Laplacian [16].

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