APPROXIMATE K-G-DUALS IN HILBERT SPACES

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In this paper we provide some characterizations of K-g-frames in Hilbert spaces and then we give an equivalent condition for the subsequence of a K-g-frame to make it a K-g-frame. Finally, we obtain some new results of approximate K-g-duals in $\mathcal{B}(\mathcal{H}, \mathcal{H}_j)$, the collection of all bounded linear operators from the Hilbert space $\mathcal{H}$ to its closed subspace $\mathcal{H}_j$.

Keywords: g-frames, K-g-frames, approximate K-g-duals, redundancy.
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1. Introduction and Preliminaries

The concept of frames in Hilbert spaces was introduced by Duffin and Schaeffer [8] to study some problems in nonharmonic Fourier series and then reintroduced by Daubechies et al. [7] to study the connection with wavelet and Gabor systems. For special applications, various generalizations of frames were proposed, such as quasi-affine frames by Hernández et. al. [12] to characterize various affine-like and Gabor systems to determine their frame properties, frame of subspace and fusion frames by Casazza et. al. [2, 4] to deal with hierarchical data processing, g-frames by Sun [14] as generalization of frames, K-frames by Găvruta [10] to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces. The concept of K-g-frames, which is more general than that of K-frames, was considered in [1, 15, 16]. After that, some properties of K-frames were extended to K-g-frames by Hua and Huang [13].

One of the main reason for considering frames and any type of generalization of frames, is that they allow each element in the space to be non-uniquely represented as a linear combination of the frame elements, by using their duals; however, it is usually complicated to calculate a dual frame explicitly. For example, in practice, one has to invert the frame operator, in the canonical dual frames, which is difficult when the space is infinite-dimensional. One way to avoid this difficulty is to consider approximate duals. The concepts of approximately dual frames have been studied since the work of Gilbert et al. [11] in the wavelet setting, see for example Feichtinger et al. [9] for Gabor systems and reintroduced in a systematic way by Christensen and Laugesen [6] for dual frame pairs, to obtain important applications of Gabor systems, wavelets and in the general frame theory.

In this paper, the importance of studying K-g-frames is pointed out; with this motivation, we obtain new K-g-frames and approximate K-g-duals and derive some results for the approximate duality of K-g-frames and their redundancy.

In the rest of this section, we will review some notions related to frames, K-frames and K-g-frames. Some properties of K-g-frames, such as the advantage of K-g-frames and

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the redundancy are found in Section 2. In Section 3 we define approximate duality of K-g-frames, notice some important properties of approximate K-g-duals, and extend some results of approximate duality of frames to K-g-frames. We also use some ideas of [6] to Propositions 3.1 and 3.2. Section 4 concludes the paper.

Throughout this paper, J is a subset of the integers set $\mathbb{Z}$; $\mathcal{H}$ is a separable Hilbert space; $\{\mathcal{H}_j\}_{j\in J}$ is a sequence of closed subspaces of $\mathcal{H}$; $\mathcal{B}(\mathcal{H},\mathcal{H}_j)$ is the collection of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_j$, with $\mathcal{B}(\mathcal{H},\mathcal{H})$ denoted as $\mathcal{B}(\mathcal{H})$; for $K\in\mathcal{B}(\mathcal{H})$, $\mathcal{R}(K)$ is the range of K, $I_{\mathcal{R}(K)}$ is the identity operator on $\mathcal{R}(K)$, the adjoint of K is K^*, and the number of elements in $I\subset J$ is $|I|$. The space $l^2\left(\{\mathcal{H}_j\}_{j\in J}\right)$ is defined by

$$l^2\left(\{\mathcal{H}_j\}_{j\in J}\right)=\left\{\{f_j\}_{j\in J}:\ \ f_j\in\mathcal{H}_j, \|\{f_j\}_{j\in J}\|^2=\sum_{j\in J}\|f_j\|^2<+\infty\right\}, \tag{1}$$

with the inner product given by

$$\langle\{f_j\}_{j\in J},\ \{g_j\}_{j\in J}\rangle=\sum_{j\in J}\langle f_j,\ g_j\rangle. \tag{2}$$

Then $l^2\left(\{\mathcal{H}_j\}_{j\in J}\right)$ is a Hilbert space with pointwise operations. In the sequel, some terminology related to Bessel and g-Bessel systems, frames, g-frames and K-frames is recalled.

A sequence $\{f_j\}_{j\in J}$ contained in $\mathcal{H}$ is called a Bessel system for $\mathcal{H}$, if there exists a positive constant B such that, for all $f\in\mathcal{H}$, $\sum_{j\in J}|\langle f,\ f_j\rangle|^2\leq B\|f\|^2$; the constant B is called a Bessel bound of the system. If, in addition, for $K\in\mathcal{B}(\mathcal{H})$, there exists a lower bound $A>0$ such that, for all $f\in\mathcal{H}$, $A\|K^*f\|^2\leq\sum_{j\in J}|\langle f,\ f_j\rangle|^2$, the system is called a K-frame for $\mathcal{H}$. The constants A and B are called K-frame bounds.

Remark 1: If $K=I_{\mathcal{H}}$, the K-frames are called ordinary frames.

Recall that if $\{f_j\}_{j\in J}$ is a frame for $\mathcal{H}$, the frame operator $S:\mathcal{H}\to\mathcal{H}$, defined by $Sf=\sum_{j\in J}\langle f,\ f_j\rangle f_j$, is bounded, invertible and self-adjoint. This provides every element $f\in\mathcal{H}$ with the expansions

$$f=\sum_{j\in J}\langle f,\ S^{-1}f_j\rangle f_j=\sum_{j\in J}\langle f,\ f_j\rangle S^{-1}f_j. \tag{3}$$

The frame $\{S^{-1}f_j\}_{j\in J}$ is called the canonical dual frame of $\{f_j\}_{j\in J}$.

A sequence $\{\Lambda_j\in\mathcal{B}(\mathcal{H},\mathcal{H}_j):j\in J\}$ is called a g-Bessel system for $\mathcal{H}$ with respect to $\mathcal{H}_j$ if there exists a positive constant B such that, for all $f\in\mathcal{H}$

$$\sum_{j\in J}\|\Lambda_j f\|^2\leq B\|f\|^2. \tag{4}$$

The constant B is called a g-Bessel bound of the system. If, in addition, there exists a lower bound $A>0$ such that, for all $f\in\mathcal{H}$, $A\|f\|^2\leq\sum_{j\in J}\|\Lambda_j f\|^2$, the system is called a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. The constants A and B are called g-frame bounds. If $A=B$, the g-frame is said to be a tight g-frame. For more information on frame theory, basic properties of the K-frames and g-frames, we refer to [5, 10, 14]. Now, we introduce the pseudo-inverse operator and the concept of K-g-frames, which is more general than the concept of g-frames.

Definition 1.1. [5, p. 56] *Let $\mathcal{H}_1$ be a Hilbert space. Suppose that $U:\mathcal{H}\to\mathcal{H}_1$ is a bounded linear operator with closed range $\mathcal{R}(U)$. Then there exists a bounded linear operator $U^\dagger:\mathcal{H}_1\to\mathcal{H}$ for which $UU^\dagger f=f,\ \forall f\in\mathcal{R}(U)$. The operator $U^\dagger$ is called the pseudo-inverse operator of U.*

Definition 1.2. [1, Theorem (2.5)] *Let $K \in \mathcal{B}(\mathcal{H})$ and $\Lambda_j \in \mathcal{B}(\mathcal{H}, \mathcal{H}_j)$ be given, for any $j \in J$. A sequence $\{\Lambda_j\}_{j\in J}$ is called a K-g-frame for $\mathcal{H}$ with respect to $\{H_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|K^*f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \ \ \forall f \in \mathcal{H}. \tag{5}$$

The constants A and B are called the lower and upper bounds of the K-g-frame, respectively. A K-g-frame $\{\Lambda_j\}_{j\in J}$ is said to be tight if there exists a constant $A > 0$ such that

$$\sum_{j\in J} \|\Lambda_j f\|^2 = A\|K^*f\|^2, \ \ \forall f \in \mathcal{H}. \tag{6}$$

Remark 2: If $K = I_{\mathcal{H}}$, the K-g-frames are just the ordinary g-frames.

Now we introduce some of the main operators associated with a K-g-frame. Suppose that $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Obviously, it is a g-Bessel sequence, so we can define the bounded linear operator $T_\Lambda : \ell^2\left(\{\mathcal{H}_j\}_{j\in J}\right) \to \mathcal{H}$ as follows:

$$T_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j, \ \ \forall \{g_j\}_{j\in J} \in \ell^2\left(\{\mathcal{H}_j\}_{j\in J}\right). \tag{7}$$

The operator T_Λ is called the synthesis operator (or pre-frame operator) for the K-g-frame $\{\Lambda_j\}_{j\in J}$. The adjoint operator

$$T_\Lambda^* : \mathcal{H} \to l^2(\{\mathcal{H}_j\}_{j\in J}), \quad T_\Lambda^* f = \{\Lambda_j f\}_{j\in J}, \ \ \forall f \in \mathcal{H}, \tag{8}$$

is called the analysis operator for the K-g-frame $\{\Lambda_j\}_{j\in J}$. The frame operator for the K-g-frame $\{\Lambda_j\}_{j\in J}$ is defined as $S_\Lambda = T_\Lambda T_\Lambda^*$, therefore

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \ \ \forall f \in \mathcal{H}. \tag{9}$$

2. Some properties of K-g-frames in Hilbert spaces

The importance of studying K-g-frames is that they are more general than g-frames in the sense that the lower frame bound holds only for the elements in the range of K. Also, as we will see in the following examples, we can construct a K-g-frame with the help of a g-Bessel sequence which is not a g-frame.

Example 1: Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and $\mathcal{H}_j := \overline{span}\{e_j, e_{j+1}\}, \ \ j = 1, 2, 3, \cdots$. Define the operator $\Lambda_j : \mathcal{H} \to \mathcal{H}_j$ as follows:

$$\Lambda_j f = \langle f, \ e_j + e_{j+1}\rangle(e_j + e_{j+1}), \ \ \forall f \in \mathcal{H}.$$

Then,

$$\begin{aligned}
\sum_{j=1}^{\infty} \|\Lambda_j f\|^2 &= 2\sum_{j=1}^{\infty} |\langle f, \ e_j + e_{j+1}\rangle|^2 \\
&\leq 2\sum_{j=1}^{\infty} (|\langle f, \ e_j\rangle| + |\langle f, \ e_{j+1}\rangle|)^2 \\
&\leq 4\sum_{j=1}^{\infty} |\langle f, \ e_j\rangle|^2 + 4\sum_{j=1}^{\infty} |\langle f, \ e_{j+1}\rangle|^2 \\
&\leq 8\|f\|^2.
\end{aligned}$$

That is, $\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence. However, $\{\Lambda_j\}_{j\in J}$ does not satisfy the lower g-frame condition, because if we consider the vectors $g_m := \sum_{n=1}^{m}(-1)^{n+1}e_n,\ \ m \in \mathbb{N}$, then $\|g_m\|^2 = m$, for all $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$, we see that

$$\langle g_m, e_j + e_{j+1}\rangle = \begin{cases} 0, & j > m \\ (-1)^{m+1}, & j = m \\ 0, & j < m \end{cases}$$

Therefore,

$$\sum_{j=1}^{\infty}\|\Lambda_j g_m\|^2 = 2\sum_{j=1}^{\infty}|\langle g_m,\ e_j + e_{j+1}\rangle|^2 = 2 = \frac{2}{m}\|g_m\|^2,\ \ \forall m \in \mathbb{N},$$

that is, $\{\Lambda_j\}_{j\in J}$ does not satisfy the lower g-frame condition. Now define

$$K : \mathcal{H} \to \mathcal{H},\ \ Kf = \sum_{j=1}^{\infty}\langle f,\ e_j\rangle(e_j + e_{j+1}),\ \ \forall f \in \mathcal{H}.$$

Then $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$, because

$$\begin{aligned}
\|K^*f\|^2 &= \langle f,\ KK^*f\rangle = \left\langle f,\ \sum_{j=1}^{\infty}\langle K^*f,\ e_j\rangle(e_j + e_{j+1})\right\rangle \\
&= \sum_{j=1}^{\infty}\langle f,\ e_j + e_{j+1}\rangle\langle Ke_j,\ f\rangle = \sum_{j=1}^{\infty}\langle f,\ e_j + e_{j+1}\rangle\langle e_j + e_{j+1},\ f\rangle \\
&= \sum_{j=1}^{\infty}|\langle f,\ e_j + e_{j+1}\rangle|^2 \le \sum_{j=1}^{\infty}\|\Lambda_j f\|^2 \le 8\|f\|^2,
\end{aligned}$$

as desired.

Example 2: Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and

$$\mathcal{H}_j := \overline{span}\{e_{3j-2}, e_{3j-1}, e_{3j}\},\ \ j = 1, 2, 3, \cdots.$$

Define the operator $\Lambda_j : \mathcal{H} \to \mathcal{H}_j$ as follows:

$$\Lambda_1 f = \langle f,\ e_1\rangle e_1 + \langle f,\ e_2\rangle e_2 + \langle f,\ e_3\rangle e_3 \ \text{ and } \ \Lambda_j f = 0,\ \text{ for } \ j \ge 2.$$

By a simple computation $\{\Lambda_j\}_{j\in J}$ is not a g-frame for $\mathcal{H}$ with respect to $\mathcal{H}_j$, because, if we take $f = e_4$, then

$$\|f\|^2 = 1 \ \text{ and } \ \sum_{j=1}^{\infty}\|\Lambda_j f\|^2 = \|\Lambda_1 e_4\|^2 = 0.$$

Define now the operator $K : \mathcal{H} \to \mathcal{H}$ as follows:

$$Ke_1 = e_1,\ \ Ke_2 = e_2 \ \text{ and } \ Ke_j = 0,\ \text{ for } \ j \ge 3.$$

It is easy to see that, $K^*e_1 = e_1,\ \ K^*e_2 = e_2$ and $K^*e_j = 0,$ for $j \ge 3$. We show that $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\mathcal{H}_j$. In fact, for any $f \in \mathcal{H}$, we have

$$\|K^*f\|^2 = \|\sum_{j=1}^{\infty}\langle f,\ e_j\rangle K^*e_j\|^2 = |\langle f,\ e_1\rangle|^2 + |\langle f,\ e_2\rangle|^2,$$

and

$$\begin{aligned}\sum_{j=1}^{\infty}\|\Lambda_j f\|^2 &= \|\Lambda_1 f\|^2 = \|\langle f,\ e_1\rangle e_1 + \langle f,\ e_2\rangle e_2 + \langle f,\ e_3\rangle e_3\|^2 \\ &= |\langle f,\ e_1\rangle|^2 + |\langle f,\ e_2\rangle|^2 + |\langle f,\ e_3\rangle|^2 \geq \|K^* f\|^2.\end{aligned}$$

Therefore, for any $f \in \mathcal{H}$

$$\|K^* f\|^2 \leq \sum_{j=1}^{\infty}\|\Lambda_j f\|^2 \leq \|f\|^2,$$

as desired. An obvious natural question is whether there exists a class of operators K which can guarantee the existence of g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. The following simple proposition answer this query.

Proposition 2.1. *Let $\{\Lambda_j \in \mathcal{B}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ be a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Then $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ if K^* is bounded below. Furthermore, if $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ with K-g-frame bound A_1, then $\{\Lambda_j\}_{j\in J}$ is a tight g-frame with g-frame bound A_2 if and only if the right inverse of the operator K is $\frac{A_1}{A_2}K^*$.*

Proof. Since K^* is bounded below, by definition, there exists a constant $C > 0$ such that,

$$\|K^* f\| \geq C\|f\|,\ \ \forall f \in \mathcal{H}.$$

Therefore, for all $f \in \mathcal{H}$,

$$AC^2\|f\|^2 \leq A\|K^* f\|^2 \leq \sum_{j\in J}\|\Lambda_j f\|^2 \leq B\|f\|^2,$$

that is, $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Now if $\{\Lambda_j\}_{j\in J}$ is a tight g-frame with bound A_2, then

$$\sum_{j\in J}\|\Lambda_j f\|^2 = A_2\|f\|^2,\ \ \forall f \in \mathcal{H}.$$

Since $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame with bound A_1, we have $A_1\|K^* f\|^2 = A_2\|f\|^2,\ \forall f \in \mathcal{H}$, that is,

$$\langle KK^{\star} f,\ f\rangle = \langle \frac{A_2}{A_1} f,\ f\rangle,\ \ \forall f \in \mathcal{H}.$$

Therefore, $KK^* = \frac{A_2}{A_1}I_{\mathcal{H}}$, i.e., the right inverse of K is $\frac{A_1}{A_2}K^*$. The converse is straightforward. □

One of the important property of the frame theory is the possibility of redundancy. For example, in [3, Theorem (3.2)] the authors have provided sufficient conditions on the weights in a fusion frames to remain a fusion frames, when some elements are removed. The following proposition is a generalization of [5, Proposition (1.5.6)].

Proposition 2.2. *Let $K \in \mathcal{B}(\mathcal{H})$, such that K^* is bounded below with a constant $C > 0$ and let $\{\Lambda_j \in \mathcal{B}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ be a normalized K-g-frame with lower bound $A > \frac{1}{C}$, i.e., $\|\Lambda_j\| = 1, \forall j \in J$. Then for each subset $I \subset J$ with $|I| < AC^2$, the family $\{\Lambda_j\}_{j\in J\setminus I}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ with lower K-g-frame bound $AC^2 - |I|$.*

Proof. Given $f \in \mathcal{H}$,

$$\sum_{j\in I}\|\Lambda_j f\|^2 \leq \sum_{j\in I}\|\Lambda_j\|^2\|f\|^2 = |I|\|f\|^2.$$

Thus

$$\begin{aligned}\sum_{j\in J\setminus I}\|\Lambda_j f\|^2 &\geq A\|K^*f\|^2-|I|\|f\|^2\\ &\geq AC^2\|f\|^2-|I|\|f\|^2\\ &=(AC^2-|I|)\|f\|^2.\end{aligned}$$

□

Note that, because of the generality of K-g-frames, the frame operator of a K-g-frame is not invertible on $\mathcal{H}$ in general, but it is invertible on a subspace $\mathcal{R}(K)\subset\mathcal{H}$. In the following Theorem, we provide an equivalent condition for the subsequence of a K-g-frame to make it a K-g-frame.

Theorem 2.1. *Let $I\subset J$ be given. Suppose that $\{\Lambda_j\in\mathcal{B}(\mathcal{H},\mathcal{H}_j):j\in J\}$ is a K-g-frame with bounds A,B and K-g-frame operator $S_{\Lambda,J}$. Then the following statements are equivalent:*

(i) *$I_{\mathcal{R}(K)}-S^{-1}_{\Lambda,J}S_{\Lambda,I}$ is boundedly invertible on $\mathcal{R}(K)$,*
(ii) *The sequence $\{\Lambda_j\}_{j\in J\setminus I}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ with lower K-g-frame bound $\frac{B^{-1}}{\|S^{-1}_{\Lambda,J}\|^2\|K^*(I_{\mathcal{R}(K)}-S^{-1}_{\Lambda,J}S_{\Lambda,I})^{-1}\|^2}$.*

Proof. Denote the frame operator of the K-g-frame $\{\Lambda_j\}_{j\in J\setminus I}$ by $S_{\Lambda,J\setminus I}$. Since

$$S_{\Lambda,J\setminus I}=S_{\Lambda,J}-S_{\Lambda,I}=S_{\Lambda,J}(I_{\mathcal{R}(K)}-S^{-1}_{\Lambda,J}S_{\Lambda,I}),$$

we have that, $\{\Lambda_j\}_{j\in J\setminus I}$ is a K-g-frame if and only if $S_{\Lambda,J\setminus I}$ is boundedly invertible and hence if and only if $I_{\mathcal{R}(K)}-S^{-1}_{\Lambda,J}S_{\Lambda,I}$ is boundedly invertible.
Now, assume that $I_{\mathcal{R}(K)}-S^{-1}_{\Lambda,J}S_{\Lambda,I}$ is invertible. Since $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ with bounds A and B, for any $f\in\mathcal{H}$,

$$\begin{aligned}f&=S^{-1}_{\Lambda,J}S_{\Lambda,J}f\\ &=S^{-1}_{\Lambda,J}\left(\sum_{j\in I}\Lambda_j^*\Lambda_j f+\sum_{j\in J\setminus I}\Lambda_j^*\Lambda_j f\right)\\ &=S^{-1}_{\Lambda,J}S_{\Lambda,I}f+\sum_{j\in J\setminus I}S^{-1}_{\Lambda,J}\Lambda_j^*\Lambda_j f.\end{aligned}$$

Hence we have, $(I_{\mathcal{R}(K)}-S^{-1}_{\Lambda,J}S_{\Lambda,I})f=\sum_{j\in J\setminus I}S^{-1}_{\Lambda,J}\Lambda_j^*\Lambda_j f$.

Therefore we obtain

$$\begin{aligned}
\|(I_{\mathcal{R}(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})f\| &= \|\sum_{j\in J\setminus I} S_{\Lambda,J}^{-1}\Lambda_j^*\Lambda_j f\| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} |\langle \sum_{j\in J\setminus I} S_{\Lambda,J}^{-1}\Lambda_j^*\Lambda_j f,\ g\rangle| \\
&= \sup_{g\in\mathcal{H},\|g\|=1} |\sum_{j\in J\setminus I} \langle \Lambda_j f,\ \Lambda_j S_{\Lambda,J}^{-1} g\rangle| \\
&\le \sup_{g\in\mathcal{H},\|g\|=1} \sum_{j\in J\setminus I} \|\Lambda_j f\| \|\Lambda_j S_{\Lambda,J}^{-1} g\| \\
&\le \sup_{g\in\mathcal{H},\|g\|=1} \left(\sum_{j\in J\setminus I} \|\Lambda_j f\|^2\right)^{\frac{1}{2}} \left(\sum_{j\in J\setminus I} \|\Lambda_j S_{\Lambda,J}^{-1} g\|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{B}\|S_{\Lambda,J}^{-1}\| \left(\sum_{j\in J\setminus I} \|\Lambda_j f\|^2\right)^{\frac{1}{2}},
\end{aligned}$$

where the last inequality is deduced by (5). Therefore,

$$\|(I_{\mathcal{R}(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})f\| \le B^{1/2}\|S_{\Lambda,J}^{-1}\|(\sum_{j\in J\setminus I} \|\Lambda_j f\|^2)^{\frac{1}{2}}. \tag{10}$$

It follows that $I_{\mathcal{R}(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I}$ is well defined in $\mathcal{H}$. If $I_{\mathcal{R}(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I}$ is invertible on $\mathcal{H}$, then for any $f \in \mathcal{H}$ we have

$$\|K^* f\| \le \|K^*(I_{\mathcal{R}(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})^{-1}\| \cdot \|(I_{\mathcal{R}(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})f\|. \tag{11}$$

From (10) and (11) it follows that

$$\frac{B^{-1}}{\|S_{\Lambda,J}^{-1}\|^2\|K^*(I_{\mathcal{R}(K)} - S_{\Lambda,J}^{-1}S_{\Lambda,I})^{-1}\|^2}\|K^* f\|^2 \le \sum_{j\in J\setminus I} \|\Lambda_j f\|^2,\ \ \forall f \in \mathcal{H},$$

which completes the proof. □

One of the main problem in the K-g-frame theory is that the interchangeability of two g-Bessel sequence with respect to a K-g-frame is different from a g-frame. The following characterization of K-g-frames is given [1, Theorem (2.5)].

Proposition 2.3. *Let $K \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(i) *$\{\Lambda_j\}_{j\in J}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$;*
(ii) *$\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ and there exists a g-Bessel sequence $\{\Gamma_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$ such that*

$$Kf = \sum_{j\in J} \Lambda_j^*\Gamma_j f,\ \ \forall f \in \mathcal{H}. \tag{12}$$

The positions of the two g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ in (12) are not interchangeable in general. However, there exists another type of dual such that $\{\Lambda_j\}_{j\in J}$ and a sequence derived by $\{\Gamma_j\}_{j\in J}$ are interchangeable in the subspace $\mathcal{R}(K)$ of $\mathcal{H}$. For $K \in \mathcal{B}(\mathcal{H})$, if $\mathcal{R}(K)$ is closed, then the pseudo-inverse $K^\dagger$ of K exists.

Theorem 2.2. [13, Theorem (3.3)] *Suppose that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences as in (12). Then there exists a sequence $\{\Theta_j\}_{j\in J} = \{\Gamma_j(K^\dagger\mid_{\mathcal{R}(K)})\}_{j\in J}$ derived by $\{\Gamma_j\}_{j\in J}$ such that*

$$f = \sum_{j\in J} \Lambda_j^* \Theta_j f, \ \ \forall f \in \mathcal{R}(K). \tag{13}$$

Moreover, $\{\Lambda_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ are interchangeable for any $f \in \mathcal{R}(K)$.

3. Approximate K-g-duals

Motivated by the concept of approximate dual of frames in [6], we define approximate dual of K-g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in J}$.

By Theorem 2.2, since $K^\dagger\mid_{\mathcal{R}(K)}: \mathcal{R}(K) \to \mathcal{H}$, we obtain $\Theta_j : \mathcal{R}(K) \to \mathcal{H}_j$. For any $f \in \mathcal{R}(K)$, we have

$$\sum_{j\in J} \|\Theta_j f\|^2 = \sum_{j\in J} \|\Gamma_j K^\dagger f\|^2 \le B\|K^\dagger f\|^2 \le B\|K^\dagger\|^2\|f\|^2. \tag{14}$$

That is, $\{\Theta_j\}_{j\in J}$ is a g-Bessel sequence for $\mathcal{R}(K)$ with respect to $\{\mathcal{H}_j\}_{j\in J}$. Let T_Θ be the synthesis operator of $\{\Theta_j\}_{j\in J}$. Consider two mixed operators $T_\Lambda T_\Theta^*$ and $T_\Theta T_\Lambda^*$ as follows:

$$T_\Lambda T_\Theta^* : \mathcal{R}(K) \to \mathcal{H}, \ \ T_\Lambda T_\Theta^* f = \sum \Lambda_j^* \Theta_j f, \ \ \forall f \in \mathcal{R}(K),$$

$$T_\Theta T_\Lambda^* : \mathcal{H} \to \mathcal{R}(K), \ \ T_\Theta T_\Lambda^* f = \sum_j \Theta_j^* \Lambda_j f, \ \ \forall f \in \mathcal{H}.$$

Based on Theorem 2.2 and on the definition of dual and approximate duality of frames stated in [6], we introduce the following notions:

Definition 3.1. *Consider two g-Bessel sequences $\{\Lambda_j \in \mathcal{B}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$ and $\{\Theta_j \in \mathcal{B}(\mathcal{H}, \mathcal{H}_j) : j \in J\}$.*

(i) *The sequences $\{\Lambda_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ are said to be K-g-dual frames if $T_\Lambda T_\Theta^* = I_{\mathcal{R}(K)}$ or $T_\Theta T_\Lambda^*|_{\mathcal{R}(K)} = I_{\mathcal{R}(K)}$. In this case, we say that $\{\Theta_j\}_{j\in J}$ is a K-g-dual of $\{\Lambda_j\}_{j\in J}$,*

(ii) *The sequences $\{\Lambda_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ are said to be approximately K-g-dual frames if $\|I_{\mathcal{R}(K)} - T_\Lambda T_\Theta^*\| < 1$ or $\|I_{\mathcal{R}(K)} - T_\Theta T_\Lambda^*|_{\mathcal{R}(K)}\| < 1$. In this case, we say that $\{\Theta_j\}_{j\in J}$ is an approximate K-g-dual of $\{\Lambda_j\}_{j\in J}$.*

A well-known algorithm to find the inverse of an operator is the Neumann series algorithm. Since $\|I_{\mathcal{R}(K)} - T_\Lambda T_\Theta^*\| < 1$, $T_\Lambda T_\Theta^*$ is invertible with

$$(T_\Lambda T_\Theta^*)^{-1} = (I_{\mathcal{R}(K)} - (I_{\mathcal{R}(K)} - T_\Lambda T_\Theta^*))^{-1} = \sum_{n=0}^{\infty} (I_{\mathcal{R}(K)} - T_\Lambda T_\Theta^*)^n.$$

Therefore, every $f \in \mathcal{R}(K)$ can be reconstruct as

$$f = \sum_{n=0}^{\infty} (I_{\mathcal{R}(K)} - T_\Lambda T_\Theta^*)^n T_\Lambda T_\Theta^* f. \tag{15}$$

The following Propositions is the analogon of Prop. 3.4 and Prop. 4.1 in [6] to obtain new natural and approximate K-g-duals.

Proposition 3.1. *If $\{\Theta_j\}_{j\in J}$ is an approximate K-g-dual of $\{\Lambda_j\}_{j\in J}$, then $\{\Theta_j(T_\Lambda T_\Theta^*)^{-1}\}$ is a K-g-dual of $\{\Lambda_j\}_{j\in J}$.*

Proof. It is easy to see that $\{\Theta_j(T_\Lambda T_\Theta^*)^{-1}\}_{j\in J}$ is a g-Bessel sequence and

$$\begin{aligned} f=(T_\Lambda T_\Theta^*)(T_\Lambda T_\Theta^*)^{-1}f &= \sum_{j=0}^{\infty}\Lambda_j^*\Theta_j(T_\Lambda T_\Theta^*)^{-1}f \\ &= \sum_{j=0}^{\infty}\Lambda_j^*(\Theta_j\sum_{n=0}^{\infty}(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^n f). \end{aligned}$$

Therefore, $\{\Theta_j(T_\Lambda T_\Theta^*)^{-1}\}=\{\Theta_j\sum_{n=0}^{\infty}(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^n\}_{j\in J}$ is a K-g-dual of $\{\Lambda_j\}_{j\in J}$. □

Now, for each $N\in\mathbb{N}$, define $\gamma_j^{(N)}=\sum_{n=0}^{N}\Theta_j(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^n$ and $T_N:\mathcal{H}\to\mathcal{H}$ by $T_N=\sum_{n=0}^{N}(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^n$. Then $\gamma_j^{(N)}=\Theta_jT_N,\ \forall j\in J$. The sequence $\{\gamma_j^{(N)}\}_{j\in J}$ is obtained from the g-Bessel sequence $\{\Theta_j\}_{j\in j}$ by means of a bounded operator, therefore, it is a g-Bessel sequence. For each $f\in\mathcal{R}(K)$,

$$T_\Lambda T_\Theta^* T_N f=\sum_{j=0}^{\infty}\Lambda_j^*\Theta_jT_Nf=\sum_{j=0}^{\infty}\Lambda_j^*\gamma_j^{(N)}f=T_\Lambda T_\Gamma^* f,$$

where T_Γ is the synthesis operator of $\{\gamma_j^{(N)}\}_{j\in J}$. Thus,

$$\begin{aligned} T_\Lambda T_\Gamma^* f &= T_\Lambda T_\Theta^* T_N f \\ &= [I_{\mathcal{R}(K)}-(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)]\sum_{n=0}^{N}(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^n f \\ &= [I_{\mathcal{R}(K)}-(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^{N+1}]f, \end{aligned}$$

by telescoping. Therefore,

$$\|I_{\mathcal{R}(K)}-T_\Lambda T_\Gamma^*\| = \|(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^{N+1}\| \tag{16}$$

$$\leq \|I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)\|^{N+1}. \tag{17}$$

If $\{\Theta_j\}_{j\in J}$ is an approximate K-g-dual of $\{\Lambda_j\}_{j\in J}$, then

$$\|I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)\|<1. \tag{18}$$

By (16) and (18), we obtain $\|I_{\mathcal{R}(K)}-T_\Lambda T_\Gamma^*)\|<1$, that is, $\{\gamma_j^{(N)}\}_{j\in J}$ is an approximate K-g-dual of $\{\Lambda_j\}_{j\in J}$.

We summarize what we have proved:

Proposition 3.2. *Let $\{\Theta_j\}_{j\in J}$ be an approximate K-g-dual of $\{\Lambda_j\}_{j\in J}$. Then*

$$\left\{\sum_{n=0}^{N}\Theta_j(I_{\mathcal{R}(K)}-T_\Lambda T_\Theta^*)^n\right\}_{j\in J}$$

is an approximate K-g-dual of $\{\Lambda_j\}_{j\in J}$.

Next we state and prove the following theorem.

Theorem 3.1. *Let $\{\Lambda_j\in\mathcal{B}(\mathcal{H},\mathcal{H}_j):j\in J\}$ be a K-g-frame and $\{\Theta_j\in\mathcal{B}(\mathcal{H},\mathcal{H}_j):j\in J\}$ be a g-Bessel sequence. Let also $\{f_{i,j}\}_{i\in I_j}$ be a frame for $\mathcal{H}_j$ with bounds A_j and B_j for every $j\in J$ such that, $0<A=\inf_{j\in J}A_j\leq\sup_{j\in J}B_j=B<\infty$. Then $\{\Theta_j\}_{j\in J}$ is an approximate K-g-dual of $\{\Lambda_j\}_{j\in J}$ if and only if $E=\{\Theta_j^*f_{i,j}\}_{i\in I_j,j\in J}$ is an approximate dual of $F=\{\Lambda_j^*\widetilde{f}_{i,j}\}_{i\in I_j,j\in J}$, where $\{\widetilde{f}_{i,j}\}_{i\in I_j}$ is the canonical dual of $\{f_{i,j}\}_{i\in I_j}$.*

Proof. For each $f \in \mathcal{H}$ we have

$$\sum_{j\in J}\sum_{i\in I_j}|\langle f,\ \Theta_j^* f_{i,j}\rangle|^2 = \sum_{j\in J}\sum_{i\in I_j}|\langle \Theta_j f,\ f_{i,j}\rangle|^2 \le \sum_{j\in J} B_j\|\Theta_j f\|^2 \le B\sum_{j\in J}\|\Theta_j f\|^2.$$

This implies that E is a Bessel sequence for $\mathcal{H}$. Similarly, F is also Bessel sequence for $\mathcal{H}$. Moreover, for each $f \in \mathcal{H}$ we have

$$\begin{aligned} T_\Theta T_\Lambda^* f &= \sum_{j\in J}\Theta_j^*\Lambda_j f = \sum_{j\in J}\Theta_j^*\sum_{i\in I_j}\langle \Lambda_j f,\ \widetilde{f}_{i,j}\rangle f_{i,j} \\ &= \sum_{j\in J, i\in I_j}\langle f,\ \Lambda_J^*\widetilde{f}_{i,j}\rangle \Theta_j^* f_{i,j} = T_E T_F^* f. \end{aligned}$$

So, $\|I - T_\Theta T_\Lambda^*\| < 1$ if and only if $\|I - T_E T_F^*\| < 1$. This concludes the proof. □

4. **Conclusions**

We characterized K-g-frames in Hilbert spaces (as some type of frame generalization) and provided two examples. We have given a condition that turns a subsequence of K-g-frame into a K-g-frame. A method to obtain new natural and approximate K-g-duals of a K-g-frame is pointed out. The relation between an approximate K-g-dual of a K-g-frame and an approximate dual of a frame is also discussed. Our future goal is to look into those properties of K-g-orthonormal bases that help us to study the behavior of K-g-frames.

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