COINCIDENCE POINT THEOREMS IN S-METRIC SPACES USING INTEGRAL TYPE OF CONTRACTION

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In the present paper, we consider a coupled coincidence point results in partially ordered S-metric spaces using integral type of contraction as well as the mixed monotone property of the mappings. We also generalize the famous Branciari fixed point theorem [A. Branicari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Sci. 29 (9), 531-536, (2002)] from the metric spaces to the framework of partially ordered S-metric spaces.

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MSC 2000: 47H 10, 54H 25.

1. Introduction and Preliminaries

The famous Banach contraction [8] principle is one of the most important and most cited result in the fixed point theory. After that a huge number of papers was appeared, which represented a generalization of this important result (see [5], [34], [35]). A Branciari [10] generalized the Banach contraction principle in the context of integral type of contraction. Namely, Branciari in [10] proved the following result.

Theorem 1.1. *Let (X, d) be a complete metric space, $k \in (0, 1)$, and let $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le k \int_0^{d(x,y)} \varphi(t)dt$$

where $\varphi : [0, +\infty) \to [0, +\infty]$ is a Lebesgue-integrable mapping which is summable (i.e., with a finite integral) on each compact subset of $[0, +\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

Also, in literature exists a large number of generalizations of metric spaces, such as 2-metric spaces ([17], [18])), D^*-metric spaces ([32]), partial metric spaces ([1], [6], [9], [11], [36]), cone metric spaces ([21]), S-metric spaces [29], [30], b-metric spaces [15], $G-$metric

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spaces ([4], [7], [12], [25], [33]) and G_b-metric spaces ([2]). For the coupled questions in these ordered spaces see ([3], [14], [19], [24].)

Definition 1.1. [30] *Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.*
(S1) $0 < S(x,y,z)$ *for all* $x,y,z \in X$ *with* $x \neq y \neq z$,
(S2) $S(x,y,z) = 0$ *if* $x = y = z$,
(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ *for all* $x,y,z,a \in X$.
The pair (X,S) is called an S-metric space.

Example 1.1. [30] *Let $X = \mathbb{R}^2$ and d be an ordinary metric on X. Put $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x,y,z \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by x,y,z. Then S is an S-metric on X.*

Definition 1.2. [31] *Let (X,S) be an S-metric space and $A \subseteq X$.*
(1) *If for every $x \in X$ there exists $r > 0$ such that $B_s(x,r) \subseteq A$, then the subset A is called an open subset of X.*
(2) *Subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.*
(3) *A sequence $\{x_n\}$ in X convergent to x if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$ and we denote by $\lim_{n\longrightarrow\infty} x_n = x$.*
(4) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for each $n,m \geq n_0$, $S(x_n,x_n,x_m) < \varepsilon$.*
(5) *The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.*
(6) *Let τ be the set of all $A \subseteq X$ witch $x \in A$ if and only if there exists $r > 0$ such that $B_s(x,r) \subseteq A$. Then τ is a topology on X.*

Lemma 1.1. [31] *Let (X,S) be an S-metric space. If there exist the sequences $\{x_n\}, \{y_n\}$ such that $\lim_{n\longrightarrow\infty} x_n = x$ and $\lim_{n\longrightarrow\infty} y_n = y$, then $\lim_{n\longrightarrow\infty} S(x_n,x_n,y_n) = S(x,x,y)$.*

Lemma 1.2. [16] *Let (X,S) be an S-metric space. Then*

$$S(x,x,z) \leq 2S(x,x,y) + S(y,y,z) \text{ and } S(x,x,z) \leq 2S(x,x,y) + S(z,z,y) \text{ for all } x,y,z \in X.$$

Also, $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$ by [29].

Definition 1.3. [20] *In each partially ordered set $(X,\preceq)$ the elements $x,y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.*

Definition 1.4. *Let X be a nonempty set. Then $(X,S,\preceq)$ is called an ordered S-metric space if: (X,S) is a S-metric space and $(X,\preceq)$ is a partially ordered set.*

Definition 1.5. [9] *Let $(X,\preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, i.e., for any $x,y \in X$,*

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \Rightarrow F(x_1,y) \preceq F(x_2,y),$$
$$y_1, y_2 \in X,\ y_1 \preceq y_2 \Rightarrow F(x,y_1) \succeq F(x,y_2).$$

Definition 1.6. [13] *Let $(X,\preceq)$ be a partially ordered set and suppose $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property if F is* *monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, i.e., for any* $x, y \in X$,

$$x_1, x_2 \in X,\ g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y),$$
$$y_1, y_2 \in X,\ g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 1.7. [9] *An element* $(x, y) \in X \times X$ *is called a coupled coincidence point of the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *if* $F(x, y) = gx$ *and* $F(y, x) = gy$*, and their common coupled fixed point if* $F(x, y) = gx = x$ *and* $F(y, x) = gy = y$.

Definition 1.8. [22] *Let* X *be a nonempty set. Then we say that the mappings* $F : X \times X \to X$ *and* $g : X \to X$ *are commutative if* $gF(x, y) = F(gx, gy)$.

Definition 1.9. [22] *An element* $(x, y) \in X \times X$ *is called a coupled fixed point of mapping* $F : X \times X \to X$ *if* $F(x, y) = x$ *and* $F(y, x) = y$*. Obviously,* (x, y) *is a coupled fixed point if and only if* (y, x) *is such.*

Definition 1.10. *Let* (X, S) *and* (X', S') *be two* S*-metric spaces, and let* $f : (X, S) \to (X', S')$ *be a function. Then* f *is said to be continuous at a point* $x \in X$ *if and only if for every sequence* x_n *in* X, $S(x_n, x_n, x) \to 0$ *implies* $S'(f(x_n), f(x_n), f(x)) \to 0$*. A function* f *is continuous at* X *if and only if it is continuous at all* $x \in X$.

Lemma 1.3. [23] *Let* $\{r_n\}_{n\in\mathbb{N}}$ *be a non-negative sequence such that* $\lim\limits_{n\to\infty} r_n = r$*. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\, dt = \int_0^{r} \varphi(t)\, dt,$$

where $\varphi : [0, +\infty) \to [0, +\infty]$ *is Lebesgue integrable, summable on each compact subset of* $[0, +\infty)$ *and* $\int_0^{\varepsilon} \varphi(t)dt > 0$ *for each* $\varepsilon > 0$.

Lemma 1.4. [23] *Let* $\{r_n\}_{n\in\mathbb{N}}$ *be a non-negative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\, dt = 0$$

if and only if $\lim\limits_{n\to\infty} r_n = 0$*, where* $\varphi : [0, +\infty) \to [0, +\infty]$ *is Lebesgue integrable, summable on each compact subset of* $[0, +\infty)$ *and* $\int_0^{\varepsilon} \varphi(t)dt > 0$ *for each* $\varepsilon > 0$*. For other details also see* [4].

2. **Main results**

Firstly, we introduce the following two new notions in the framework of partially ordered $S-$metric spaces.

Definition 2.1. *We say that partially ordered* $S-$*metric space* $(X, S, \preceq)$ *is regular if it has the following properties:*

($r_{\preceq}$) *if for non-decreasing sequence* $\{x_n\}$ *holds* $S(x_n, x_n, x) \to 0$ *as* $n \to \infty$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$;

($r_{\succsim}$) *if for non-increasing sequence* $\{y_n\}$ *holds* $S(y_n, y_n, y) \to 0$ *as* $n \to \infty$*, then* $y_n \succeq y$ *for all* $n \in \mathbb{N}$.

Definition 2.2. *Let* $(X, S, \preceq)$ *be a* $S-$*metric space. Mappings* $f, g : X \to X$ *are called* $S-$*compatible if*

$$S\left(f\left(g\left(x_n\right)\right), f\left(g\left(x_n\right)\right), g\left(f\left(x_n\right)\right)\right) \to 0 \quad \text{as } n \to \infty,$$

holds whenever $f(x_n), g(x_n)$ are sequences in X such that

$$\lim_{n\to\infty} S(f(x_n), f(x_n), t) = \lim_{n\to\infty} S(g(x_n), g(x_n), t),$$

for some $t \in X$.

Example 2.1. *Let $X = [0,1]$ and let $S : X^3 \to [0,\infty)$ be the $S-$metric defined as follows:*

$$S(x,y,z) = |x-z| + |y-z|,$$

for all $x,y,z \in X$ with $x \preceq y \preceq z$ or $x \succeq y \succeq z$ where $a \preceq b$ if and only if $b \leq a$ for all $a,b \in X$. It is clear that $(X,S,\preceq)$ is a partially ordered complete $S-$metric space. Define the self mappings f and g on X by $f(x) = \frac{1}{8}x$ and $g(x) = x^2$. The pair (f,g) is $S-$compatible but it is not commuting.

Assertion a similar to the following lemma were used (and proved) in the course of proofs of several fixed point results in various papers for framework of metric spaces (for some details see [26]-[28]).

Lemma 2.1. *Let (X,S) be a $S-$metric space and let $\{x_n\}$ be a sequence in it such that*

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S(x_{m_k}, x_{m_k}, x_{n_k}), S(x_{m_k}, x_{m_k}, x_{n_k+1}), S(x_{m_k-1}, x_{m_k-1}, x_{n_k}),$$
$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}), S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}), ...$$

Proof. Suppose that the sequence $\{x_n\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ i $\{x_{n_k}\}$, such that for every $k \in \mathbb{N}$ i $n_k > m_k > k$ the following is satisfied:

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon \text{ and } S(x_{m_k}, x_{m_k}, x_{n_k-1}) < \varepsilon.$$

Then, using Lemma 1.3, Lemma 1.6 and (S3) we have

$$\begin{aligned} \varepsilon &\leq S(x_{m_k}, x_{m_k}, x_{n_k}) \\ &\leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}) \\ &< \varepsilon + 2S(x_{m_k}, x_{m_k}, x_{m_k-1}), \end{aligned}$$

and

$$\varepsilon \leq \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) \leq \varepsilon.$$

Therefore $\lim_{k\to\infty} S(x_{n_k}, x_{n_k}, x_{m_k}) = \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon$. Further, as

$$|S(x_{n_k}, x_{n_k}, x_{m_k}) - S(x_{n_{k+1}}, x_{n_{k+1}}, x_{m_k})| \leq 2S(x_{n_{k+1}}, x_{n_{k+1}}, x_{n_k})$$

we obtain that

$$\lim_{k\to\infty} S(x_{n_{k+1}}, x_{n_{k+1}}, x_{m_k}) = \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_{k+1}}) = \varepsilon.$$

Analogous, it can be proved that

$$S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k}\right), S\left(x_{m_k-1}, x_{m_k-1}, x_{n_k+1}\right), S\left(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}\right), \ldots$$

tend to ε. □

Our first result is a generalization of the Branciari's fixed point theorem in the context of partially ordered $S-$metric spaces.

Theorem 2.1. *Let $(X, S, \preceq)$ be a partially ordered $S-$metric space. Let $f : X \to X$ and $g : X \to X$ be mappings such that f is $g-$non-decreasing and there exists $x_0 \in X$ with $gx_0 \preceq fx_0$. Let there be a constant $k \in (0, 1)$ such that the following holds:*

$$\int_0^{S(fx,fy,fz)} \varphi(t)\,dt \leq k \int_0^{S(gx,gy,gz)} \varphi(t)\,dt \tag{1}$$

for all $x, y, z \in X$ with $gx \preceq gy \preceq gz$ or $gx \succeq gy \succeq gz$ where $\varphi : [0, \infty) \to [0, \infty)$ is Lebesgue integrable, summable on each compact subset of $[0, \infty)$ and $\int_0^{\varepsilon} \varphi(t)\,dt > 0$ for each $\varepsilon > 0$. Assume the following conditions hold:

(i) $f(X) \subset g(X)$,

(ii) f and g are continuous and compatible and (X, S) is a complete, or

(iv) $(X, S, \preceq)$ is regular and one of $f(X)$ or $g(X)$ is complete.

Then f and g have a coincidence point in X, i.e., there exists $u \in X$ such that $fu = gu$.

Proof. We shall prove this Theorem in several steps.

Step I.

Since there exists $x_0 \in X$ with $gx_0 \preceq fx_0$ then by (i) we can define so-called Jungck sequence $y_n = f(x_n) = g(x_{n+1})$ for all $n \in \mathbb{N}$. If $y_k = y_{k+1}$ for some $k \in \mathbb{N}_0$ then $u = x_{k+1}$ is a request coincidence point. So, we will suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}_0$. In this case we shall prove that

$$y_n \prec y_{n+1}. \tag{2}$$

We will use the mathematical induction as well as the $g-$non-decreasing property for the mapping f. Because from $g(x_0) \preceq f(x_0) = g(x_1)$ follows $f(x_0) \prec f(x_1)$, i.e., $y_0 \prec y_1$. Let $y_n \prec y_{n+1}$ for some fixed $n \in \mathbb{N}$, i.e., $gx_{n+1} \prec gx_{n+2}$. As the mapping f is $g-$non-decreasing we get that $f(x_{n+1}) \prec f(x_{n+2})$, i.e., $y_{n+1} \prec y_{n+2}$. The proof for (2) is complete.

Step II.

The sequence $\{y_n\}$ is Cauchy. Firstly, we have

$$\begin{aligned}
&\int_0^{S(y_{n+1},y_{n+1},y_n)} \varphi(t)\,dt \\
= &\int_0^{S(fx_{n+1},fx_{n+1},fx_n)} \varphi(t)\,dt \\
\leq\ & k \int_0^{S(gx_{n+1},gx_{n+1},gx_n)} \varphi(t)\,dt = k \int_0^{S(y_n,y_n,y_{n-1})} \varphi(t)\,dt \\
\leq\ & k^2 \int_0^{S(y_{n-1},y_{n-1},y_{n-2})} \varphi(t)\,dt \leq \ldots \leq k^n \int_0^{S(y_1,y_1,y_0)} \varphi(t)\,dt \to 0, \text{ as } k \to \infty.
\end{aligned} \tag{3}$$

Hence according to Lemma 1.4 we get that $S(y_{n+1}, y_{n+1}, y_n) \to 0$ as $k \to \infty$. If $\{y_n\}$ is not Cauchy sequence in $S-$metric space (X, S), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$S(y_{m_k+1}, y_{m_k+1}, y_{n_k+1}) \text{ and } S(y_{m_k}, y_{m_k}, y_{n_k}), \tag{4}$$

where $y_n = f(x_n) = g(x_{n+1})$ (for all details see Lemma 2.1). Putting now in (1) $x = y = x_{m_k}, z = x_{n_k}$ (it is possible, because it is by (2) $gx \preceq gy \preceq gz$) we obtain

$$\int_0^{S\left(y_{m_k+1},y_{m_k+1},y_{n_k+1}\right)} \varphi(t)\,dt \leq k \int_0^{S\left(y_{m_k},y_{m_k},y_{n_k}\right)} \varphi(t)\,dt, \tag{5}$$

from which, letting the limit as $k \to \infty$, follows

$$\int_0^{\varepsilon} \varphi(t)\,dt \leq k \int_0^{\varepsilon} \varphi(t)\,dt < \int_0^{\varepsilon} \varphi(t)\,dt, \tag{6}$$

whih is a contradiction. Hence, $\{y_n\}$ is a Cauchy sequence.

Step III.

Let now (ii) holds. Then we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = u, \tag{7}$$

for some $u \in X$. Further, since f and g are continuous and compatible we get that

$$\lim_{n\to\infty} f(g(x_n)) = f(u), \lim_{n\to\infty} g(f(x_n)) = g(u) \tag{8}$$

$$\text{and } \lim_{n\to\infty} S(f(g(x_n)), f(g(x_n)), g(f(x_n))) = 0.$$

We shall show that $f(u) = g(u)$. Indeed, on the other hand we have that

$$\lim_{n\to\infty} S(f(g(x_n)), f(g(x_n)), g(f(x_n))) = S(f(u), f(u), g(u)). \tag{9}$$

Now, (8) and (9) imply $S(f(u), f(u), g(u)) = 0$, i.e., $f(u) = g(u)$.

If (iv) holds we obtain that $y_n = f(x_n) = g(x_{n+1}) \to g(v)$ for some $v \in X$. Also, because $(X, S, \preceq)$ is regular we have that $g(x_n) \preceq g(v)$ for all $n \in \mathbb{N}$. Now, putting in (1) $x = y = x_n, z = v$ we get

$$\int_0^{S(fx_n,fx_n,fv)} \varphi(t)\,dt \leq k \int_0^{S(gx_n,gx_n,gv)} \varphi(t)\,dt \to 0, \text{ as } k \to \infty.$$

Since, on the other hand $S(fx_n, fx_n, fv) \to S(g(v), g(v), f(v))$ we again obtained that f and g have a coincidence point. Here this is v, i.e., $f(v) = g(v)$. Theorem 2.1 is proved.

□

Putting $g = i_X$-identity mapping on X, we get Branciari's theorem in the framework of partially ordered $S-$metric spaces.

Corollary 2.1. *Let $(X, S, \preceq)$ be a partially ordered S-metric space and let $f : X \times X \to X$ be mapping such that f is non-decreasing and there be $x_0 \in X$ with $x_0 \preceq fx_0$. Let there exist a constant $k \in (0, 1)$ such that the following holds:*

$$\int_0^{S(fx,fy,fz)} \varphi(t)dt \leq k \int_0^{S(x,y,z)} \varphi(t)dt, \tag{10}$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$ or $x \succeq y \succeq z$, where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$ and $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$. Assume the following conditions:
(a) *f is continuous and (X, S) is complete, or*
(b) *$(X, S, \preceq)$ is regular and $f(X)$ is complete.*
Then f has a fixed point in X, i.e., there exists $u \in X$ such that $fu = u$.

Example 2.2. *Let $X = [0, 1]$ and let $S : X^3 \to [0, \infty)$ be the $S-$metric defined as follows:*

$$S(x, y, z) = |x - z| + |y - z|,$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$ or $x \succeq y \succeq z$ where $a \preceq b$ if and only if $b \leq a$ for all $a, b \in X$. It is clear that $(X, S, \preceq)$ is a partially ordered complete $S-$metric space. Define the self mappings f and g on X by $f(x) = \frac{1}{8}x$ and $g(x) = \frac{1}{4}x$. Further, consider $\varphi(t) = 2t$ and $k = \frac{1}{2}$. It is not hard to see that all the conditions of Theorem 2.1 are satisfied ((i), (ii)) and 0 is the coincidence point.

Our second result is a coupled coincidence point theorem in partially ordered $S-$metric spaces.

Theorem 2.2. *Let $(X, S, \preceq)$ be a partially ordered $S-$metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed $g-$monotone property on X and there exist two elements $a_0, b_0 \in X$ with $g(a_0) \preceq F(a_0, b_0)$ and $g(b_0) \succeq F(b_0, a_0)$. Let there be a constant $k \in (0, 1)$ such that the following holds:*

$$\int_0^{S(F(a,b),F(p,q),F(c,r))} \varphi(t)dt \leq k \int_0^{\max\{S(ga,gp,gc),S(gb,gq,gr)\}} \varphi(t)dt, \tag{11}$$

for all $a, b, c, p, q, r \in X$ with $ga \succeq gp \succeq gc$ and $gb \preceq gq \preceq gr$ or $ga \preceq gp \preceq gc$ and $gb \succeq gq \succeq gr$ where $\varphi : [0, +\infty) \to [0, +\infty]$ is Lebesgue integrable, summable on each compact subset of $[0, +\infty)$ and $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$. Assume the following conditions hold:

(a) $F(X \times X) \subset g(X)$,
(b) $g(X)$ *is complete,*
(c) g *is continuous and commutes with* F.

Then F *and* g *have a coupled coincidence point. Moreover, then there exists* $a \in X$ *such that* $g(a) = F(a, a) = a$.

Proof. Let a_0, b_0 be such that $g(a_0) \preceq F(a_0, b_0)$ and $g(b_0) \succeq F(b_0, a_0)$. As $F(X\times X) \subset g(X)$, we may choose a_1, b_1 in a way that $g(a_1) = F(a_0, b_0)$ and $g(b_1) = F(b_0, a_0)$.
Again since $F(X \times X) \subset g(X)$, we may choose a_2, $b_2 \in X$ such that $g(a_2) = F(a_1, b_1)$ and $g(b_2) = F(b_1, a_1)$. Repeating this process, we can build two sequences $\{a_n\}$ and $\{b_n\}$ in X such that,

$$g(a_{n+1}) = F(a_n, b_n) \text{ and } g(b_{n+1}) = F(b_n, a_n), \text{ for all } n \geq 0. \tag{12}$$

Now, we claim that for all $n \geq 0$,

$$g(a_n) \preceq g(a_{n+1}), \tag{13}$$

and

$$g(b_n) \succeq g(b_{n+1}). \tag{14}$$

Now we will use mathematical induction. Suppose $n = 0$. Since $g(a_0) \preceq F(a_0, b_0)$ and $g(b_0) \succeq F(b_0, a_0)$, we see that $g(a_1) = F(a_0, b_0)$ and $g(b_1) = F(b_0, a_0)$, and so $g(a_0) \preceq g(a_1)$ and $g(b_0) \succeq g(b_1)$, i.e., (13) and (14) holds for $n = 0$. We now suppose that (13) and (14) are valid for some $n > 0$. As we know that F has the mixed g-monotone property and also $g(a_n) \preceq g(a_{n+1})$, $g(b_n) \succeq g(b_{n+1})$, then from (12), we have

$$g(a_{n+1}) = F(a_n, b_n) \preceq F(a_{n+1}, b_n)$$

and

$$F(b_{n+1}, a_n) \preceq F(b_n, a_n) = g(b_{n+1}).$$

Also we have,

$$g(a_{n+2}) = F(a_{n+1}, b_{n+1}) \succeq F(a_{n+1}, b_n)$$

and

$$F(b_{n+1}, a_n) \succeq F(b_{n+1}, a_{n+1}) = g(b_{n+2}).$$

Then from (12) and (13), we get

$$g(a_{n+1}) \preceq g(a_{n+2}) \text{ and } g(b_{n+1}) \succeq g(b_{n+2}).$$

We conclude by mathematical induction that (13) and (14) hold for all $n \geq 0$. Continuing this process, we see clearly that

$$g(a_0) \preceq g(a_1) \preceq g(a_2) \preceq ... \preceq g(a_{n+1})...$$

and

$$g(b_0) \succeq g(b_1) \succeq g(b_2) \succeq ... \succeq g(b_{n+1})....$$

If $(a_{n+1}, b_{n+1}) = (a_n, b_n)$, then F and g have a coupled coincidence point. So we suppose that $(a_{n+1}, b_{n+1}) \neq (a_n, b_n)$ for all $n \geq 0$, i.e., we suppose that either $g(a_{n+1}) = F(a_n, b_n) \neq g(a_n)$

or $g(b_{n+1}) = F(b_n, a_n) \neq g(b_n)$.

Now, using (11), we have,

$$\int_0^{S(ga_{n+1},ga_{n+1},ga_n)} \varphi(t)dt \tag{15}$$

$$= \int_0^{S(F(a_n,b_n),F(a_n,b_n),F(a_{n-1},b_{n-1}))} \varphi(t)dt$$
$$\leq k\int_0^{\max\{S(ga_n,ga_n,ga_{n-1}),S(gb_n,gb_n,gb_{n-1})\}} \varphi(t)dt.$$

Since

$$\int_0^{S(ga_n,ga_n,ga_{n-1})} \varphi(t)dt = \int_0^{S(F(a_{n-1},b_{n-1}),F(a_{n-1},b_{n-1}),F(a_{n-2},b_{n-2}))} \varphi(t)dt \leq \tag{16}$$
$$k\int_0^{\max\{S(ga_{n-1},ga_{n-1},ga_{n-2}),S(gb_{n-1},gb_{n-1},gb_{n-2})\}} \varphi(t)dt,$$

and

$$\int_0^{S(gb_n,gb_n,gb_{n-1})} \varphi(t)dt = \int_0^{S(F(b_{n-1},a_{n-1}),F(b_{n-1},a_{n-1}),F(b_{n-2},a_{n-2}))} \varphi(t)dt \leq \tag{17}$$
$$k\int_0^{\max\{S(gb_{n-1},gb_{n-1},gb_{n-2}),S(ga_{n-1},ga_{n-1},ga_{n-2})\}} \varphi(t)dt,$$

using (15) we have

$$\int_0^{S(ga_{n+1},ga_{n+1},ga_n)} \varphi(t)dt = \int_0^{S(F(a_n,b_n),F(a_n,b_n),F(a_{n-1},b_{n-1}))} \varphi(t)dt \leq \tag{18}$$
$$k^2\int_0^{\max\{S(ga_{n-1},ga_{n-1},ga_{n-2}),S(gb_{n-1},gb_{n-1},gb_{n-2})\}} \varphi(t)dt \leq \cdots \leq$$
$$k^n\int_0^{\max\{S(ga_1,ga_1,ga_0),S(gb_1,gb_1,gb_0)\}} \varphi(t)dt.$$

Analogous, we have

$$\int_0^{S(gb_{n+1},gb_{n+1},gb_n)} \varphi(t)dt = \int_0^{S(F(b_n,a_n),F(b_n,a_n),F(b_{n-1},a_{n-1}))} \varphi(t)dt \leq \tag{19}$$
$$k^n\int_0^{\max\{S(gb_1,gb_1,gb_0),S(ga_1,ga_1,ga_0)\}} \varphi(t)dt.$$

Letting $n \to \infty$ in (18) and (19) and using Lemma 1.4 we conclude that

$$\lim_{n\to\infty} S(ga_{n+1}, ga_{n+1}, ga_n) = 0 \text{ and } \lim_{n\to\infty} S(gb_{n+1}, gb_{n+1}, gb_n) = 0.$$

Now we have to prove that the sequences $\{ga_n\}$ and $\{gb_n\}$ are Cauchy sequences.

Since $g(a_{n+1}) = F(a_n, b_n)$, $g(b_{n+1}) = F(b_n, a_n)$ and $ga_{n+1} \neq gb_{n+1}$ for all $n \in \mathbb{N}_0$ $\{ga_{n+1}\}, \{gb_{n+1}\}$ are Cauchy sequences if and only if $\{(ga_{n+1}, gb_{n+1})\}$ is Cauchy sequence in $\left(X^2, S_+\right)$ that is., in $\left(X^2, S_{\max}\right)$, where

$$S_+((a,b),(p,q),(c,r)) = S(a,p,c) + S(b,q,r)$$

that is.,

$$S_{\max}((a,b),(p,q),(c,r)) = \max\{S(a,p,c), S(b,q,r)\}.$$

Putting $G_n = (ga_n, gb_n) \in X^2$. If $\{G_n\}$ is not Cauchy sequence, then according to Lemma 2.1 the following subsequences tend ε when $k \to \infty$:

$$S(ga_{m_k+1}, ga_{m_k+1}, ga_{n_k+1}) \text{ and } S_{\max}(G_{m_k}, G_{m_k}, G_{n_k}),$$

where

$$G_{m_k} = (ga_{m_k}, gb_{m_k}), G_{n_k} = (ga_{n_k}, gb_{n_k}),$$

Taking in (11) $a = p = a_{m_k}, b = q = b_{m_k}$ and $c = a_{n_k}, r = b_{n_k}$ we get

$$\int_0^{S\left(ga_{m_k+1}, ga_{m_k+1}, ga_{n_k+1}\right)} \varphi(t)\,dt \leq k \int_0^{S_{max}(G_{m_k}, G_{m_k}, G_{n_k})} \varphi(t)\,dt. \tag{20}$$

Now, letting $k \to \infty$ in (20) we obtain

$$\int_0^{\varepsilon} \varphi(t)\,dt \leq k \int_0^{\varepsilon} \varphi(t)\,dt < \int_0^{\varepsilon} \varphi(t)\,dt.$$

A contradiction. Hence $\{ga_{n+1}\}$ and $\{gb_{n+1}\}$ are a Cauchy sequences in $g(X)$. Since $g(X)$ is complete, we have $\{ga_n\}$ and $\{gb_n\}$ are convergent to some $a \in X$ and $b \in X$ respectively. Since g is continuous, we have $\{g(ga_n)\}$ is convergent to ga and $\{g(gb_n)\}$ is convergent to gb, that is,

$$\lim_{n\to\infty} g(g(a_n)) = g(a) \text{ and } \lim_{n\to\infty} g(g(b_n)) = g(b).$$

Since, F and g are commutative, we have

$$F(g(a_n), g(b_n)) = g(F(a_n, b_n)) = g(g(a_{n+1}))$$

and

$$F(g(b_n), g(a_n)) = g(F(b_n, a_n)) = g(g(b_{n+1})).$$

Next, we have prove that (a, b) is coupled coincidence point of F and g.
From (11) we obtain

$$\begin{aligned} \int_0^{S(F(a,b),F(a,b),gga_{n+1})} \varphi(t)dt &= \int_0^{S(F(a,b),F(a,b),F(ga_n,gb_n))} \varphi(t)dt \\ &\leq k \int_0^{\max\{S(ga,ga,gga_n),S(gb,gb,ggb_n)\}} . \end{aligned}$$

Letting $n \to \infty$ we have $F(a, b) = ga$. Similarly we have $F(b, a) = gb$. Next, we claim that a is a common fixed point of the mappings F and g. Suppose now that $ga \neq gb$. Then from (11) we have

$$\begin{aligned}\int_0^{S(gb,gb,ga)}\varphi(t)dt &= \int_0^{S(F(b,a),F(b,a),F(a,b))}\varphi(t)dt\\ &\leq k\int_0^{\max\{S(gb,gb,ga),S(ga,ga,gb)\}}\varphi(t)dt\\ &= k\int_0^{\max\{S(gb,gb,ga),S(gb,gb,ga)\}}\varphi(t)dt\\ &< \int_0^{S(gb,gb,ga)}\varphi(t)dt.\end{aligned}$$

Contradiction. Therefore $ga = gb$ i.e. $F(a,b) = ga = gb = F(b,a)$. It remains to be prove that $ga = a$ and $gb = b$. Using (11) we have

$$\begin{aligned}\int_0^{S(ga,ga,ga_{n+1})}\varphi(t)dt &= \int_0^{S(F(a,b),F(a,b),F(a_n,b_n))}\varphi(t)dt\\ &\leq k\int_0^{\max\{S(ga,ga,ga_n),S(gb,gb,gb_n)\}}\varphi(t)dt.\end{aligned}$$

Letting $n\to\infty$ we get

$$\int_0^{S(ga,ga,a)}\varphi(t)dt \leq k\int_0^{\max\{S(ga,ga,a),S(gb,gb,b)\}}\varphi(t)dt. \tag{21}$$

Similarly

$$\int_0^{S(gb,gb,b)}\varphi(t)dt \leq k\int_0^{\max\{S(gb,gb,b),S(ga,ga,a)\}}\varphi(t)dt. \tag{22}$$

From (21) and (22) we have

$$\begin{aligned}\int_0^{\max\{S(ga,ga,a),S(gb,gb,b)\}}\varphi(t)dt &\leq k\int_0^{\max\{S(gb,gb,b),S(ga,ga,a)\}}\varphi(t)dt\\ &< \int_0^{\max\{S(gb,gb,b),S(ga,ga,a)\}}\varphi(t)dt.\end{aligned}$$

Contradiction. So $a = ga = F(a,b) = F(b,a) = gb = b$.

□

The next is our third new result in the framework of partially ordered $S-$metric spaces with the coupled question. It is not comparable with previously.

Theorem 2.3. *Let $(X,S,\preceq)$ be a partially ordered S-metric space. Let $F: X\times X\to X$ and $g: X\to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $a_0,b_0\in X$ with $g(a_0)\preceq F(a_0,b_0)$ and $g(b_0)\succeq F(b_0,a_0)$. Let there be a constant $k\in(0,1)$ such that the following holds:*

$$\int_0^{S(F(b,a),F(q,p),F(r,c))+S(F(a,b),F(p,q),F(c,r))}\varphi(t)dt \leq k\int_0^{[S(ga,gp,gc)+S(gb,gq,gr)]}\varphi(t)dt, \tag{23}$$

for all $a,b,c,p,q,r\in X$ with $ga\succeq gp\succeq gc$ and $gb\preceq gq\preceq gr$ or $ga\preceq gp\preceq gc$ and $gb\succeq gq\succeq gr$, where $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue integrable mapping which is summable, *non-negative, integrable function and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. *Assume the following conditions:*

(a) $F(X \times X) \subset g(X)$,

(b) $g(X)$ *is complete,*

(c) g *is continuous and commutes with* F.

Then F *and* g *have a coupled coincidence point. Moreover, then there exists* $a \in X$ *such that* $g(a) = F(a, a) = a$.

Proof. Let $T_g : X \times X \to X \times X$ and $T_F : X \times X \to X \times X$ be defined as $T_g(a, b) = (ga, gb)$, $T_F(a, b) = (F(a, b), F(b, a))$. Let $S_+ : \left(X^2\right)^3 \to [0, \infty)$ be defined as $S_+((a, b), (p, q), (r, c)) = S(a, p, r) + S(b, q, c)$. If $(X, S, \preceq)$ is partially ordered S metric space, then $(X^2, S_+, \sqsubseteq)$ is also partially ordered S metric space.

It is obvious that (23) becomes

$$\int_0^{S_+(T_F(Y),T_F(V),T_F(Z))} \varphi(t)\, dt \leq k \int_0^{S_+(T_g(Y),T_g(V),T_g(Z))} \varphi(t)\, dt, \tag{24}$$

where $Y = (a, b), V = (p, q), Z = (c, r)$. Also the conditions in Theorem 2.3 imply that $T_g(a_0, b_0) \sqsubseteq T_F(a_0, b_0)$, $T_g(Y) \sqsubseteq T_g(V) \sqsubseteq T_g(Z)$ or $T_g(Y) \sqsupseteq T_g(V) \sqsupseteq T_g(Z)$ as well as $T_F(X \times X) \sqsubseteq T_g(X \times X)$. Further we obtain that $T_g(X \times X)$ is complete and T_g is continuous and commutes with T_F.

The proof further follows according to Theorem 2.5. □

Let $A, B \subseteq \mathbb{R}$. The function $\varphi : A \to B$ is called sub-additive integrable function if and only if for $x, y \in A$

$$\int_0^{x+y} \varphi(t)\, dt \leq \int_0^{x} \varphi(t)\, dt + \int_0^{y} \varphi(t)\, dt.$$

For such functions we have the following result:

Corollary 2.2. *Let* $(X, S, \preceq)$ *be a partially ordered* S*-metric space. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be mappings such that* F *has the mixed* g*-monotone property on* X *and there exist two elements* $a_0, b_0 \in X$ *with* $g(a_0) \preceq F(a_0, b_0)$ *and* $g(b_0) \succeq F(b_0, a_0)$. *Let there be a constant* $k \in (0, 1)$ *such that the following holds:*

$$\int_0^{S(F(b,a),F(q,p),F(r,c))} \varphi(t)dt + \int_0^{S(F(a,b),F(p,q),F(c,r))} \varphi(t)dt \leq \tag{25}$$

$$k \int_0^{[S(ga,gp,gc)+S(gb,gq,gr)]} \varphi(t)dt,$$

for all $a, b, c, p, q, r \in X$ *with* $ga \succeq gp \succeq gc$ *and* $gb \preceq gq \preceq gr$ *or* $ga \preceq gp \preceq gc$ *and* $gb \succeq gq \succeq gr$, *where* $\varphi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable mapping which is summable, non-negative, sub-additive integrable function and such that for each* $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. *Assume the following conditions:*

(a) $F(X \times X) \subset g(X)$,

(b) $g(X)$ *is complete,*

(c) g *is continuous and commutes with* F.

Then F *and* g *have a coupled coincidence point. Moreover, there exists* $a \in X$ *such that* $g(a) = F(a, a) = a$.

Proof. Since the function φ is sub-additive we have that from (25) follows

$$\int_0^{S(F(b,a),F(q,p),F(r,c))+S(F(a,b),F(p,q),F(c,r))} \varphi(t)dt \leq k \int_0^{[S(ga,gp,gc)+S(gb,gq,gr)]} \varphi(t)dt, \quad (26)$$

and using Theorem 2.3 we conclude that the result follows. □

Corollary 2.3. *Let $(X, S, \preceq)$ be a partially ordered S-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $a_0, b_0 \in X$ with $g(a_0) \preceq F(a_0, b_0)$ and $g(b_0) \succeq F(b_0, a_0)$. Let there be a constant $k \in (0, \frac{1}{2})$ such that the following holds:*

$$\int_0^{S(F(b,a),F(q,p),F(r,c))+S(F(a,b),F(p,q),F(c,r))} \varphi(t)dt \leq k \int_0^{\max\{S(ga,gp,gc),S(gb,gq,gr)\}} \varphi(t)dt,$$

for all $a, b, c, p, q, r \in X$ with $ga \succeq gp \succeq gc$ and $gb \preceq gq \preceq gr$ or $ga \preceq gp \preceq gc$ and $gb \succeq gq \succeq gr$, where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable, non-negative, integrable function and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Assume the following conditions:

(a) *$F(X \times X) \subset g(X)$,*

(b) *$g(X)$ is complete,*

(c) *g is continuous and commutes with F.*

Then F and g have a coupled coincidence point. Moreover, there exists $a \in X$ such that $g(a) = F(a, a) = a$.

Proof. Since

$$\begin{aligned} & \int_0^{S(F(b,a),F(q,p),F(r,c))+S(F(a,b),F(p,q),F(c,r))} \varphi(t)dt \\ \leq \;& k \int_0^{\max\{S(ga,gp,gc),S(gb,gq,gr)\}} \varphi(t)dt \\ \leq \;& k \int_0^{S(ga,gp,gc)+S(gb,gq,gr)} \varphi(t)dt, \end{aligned}$$

results follows from Theorem 2.3. □

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REFERENCES

[1] R. P. Agarwal, M. A. El-Gebeily, D. Oregan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87**(1) (2008), 109-116.

[2] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generallized weak contractive mappings in partially ordered G_b-metric spaces*, Filomat, **28**(6), 1087-1101, (2014).

[3] I. Altun, H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory App. , Article ID 621492, 17 pages, (2010).

[4] H. Aydi, S. Chauhan and S. Radenović, *Fixed points of weakly compatible mappings in $G-$metric spaces satisfying common limit range property,* Facta Universitatis (Niš), **28**(3), 197-210, (2013).

[5] H. Aydi, W. Shatanawi, M. Postolache, Z. Mustafa, N. Tahat, *Theorems for Boyd-Wong type contractions in ordered metric spaces,* Abstr. Appl. Anal., (2012), ID: 359054.

[6] H. Aydi, E. Karapinar, M. Postolache, *Tripled coincidence point theorems for weak phi$-$contractions in partially ordered metric spaces,* Fixed Point Theory Appl., **2012**, Art No. 44, (2012).

[7] H. Aydi, M. Postolache, W. Shatanawi, W, *Coupled fixed point results for $(\psi, \varphi)-$weakly contractive mappings in ordered G-metric spaces,* Comput. Math. Appl., **63**(1), 298-309, (2012).

[8] S. Banach, *Sur les operations dans les enseembles abstraits et leur application aux equations integrals,* Fundam. Math., **3**, 133-181, (1922).
[9] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis **65**, 1379-1393, (2006).
[10] A. Branciari, *A fixed point theorem for mappings satisfying a gemneral contractive condition of integrasl type*, Int. J. Math.Sci. **29**(9), 531-536, (2002).
[11] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, *Partical metric spaces*, Am. Math. Mon. **116**, 708-718, (2009).
[12] S. Chandok, Z. Mustafa, M. Postolache, *Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces,* U. Politeh. Buch. Ser. A, **75**(4), 13-2,6 (2013).
[13] Lj. Ćirić, V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Stoch. Anal. Appl., **27**(6), 1246-1259, (2009).
[14] Lj. Ćirić, D. Mihet, R. Saadati, *Monotone generalized contractiobs in partiality ordered probabilitic metric spacea*, Topology and its Appl., **156**(17), 2838-2844, (2009).
[15] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti det Semmario Matematico e Fisico dell universita dr Modena, **46**(2), 263-276, (1998).
[16] N. V. Dung, *On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces*, Fixed Point Theory Appl., Article ID 48 (2013).
[17] S. Gahler, *2-metrische Raume und ther topoloische Struktur*, Math. Nachr. **26**, 115-148, (1963).
[18] O. Hadžić, *On common fixed point theorems in 2-metric spaces*, Review of Research faculty of Science-University of Novi Sad, **12** 7-18, (1982).
[19] J. Harjani, K. Sadarani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal., **71**(7-8), 3403-3410, (2008).
[20] H. K. Nashine, *Coupled common fixed point results ordered G-metric space*, J. Nonlinear Sci. Appl., **1**, 1-13, (2012).
[21] L. G. Huang, X. Zhng, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. App., **332**, 1468-1476, (2007).
[22] V. Lakshmikantham, Lj.B. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis **70**, 4341-4349, (2009).
[23] Z. Liu, J. Li, SM. Kang, *Fixed point theorems of contractive mappings of integral type*, Fixed Point Theory Appl., Article ID 300 (2013).
[24] N. V. Luong, N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal., **74**(3), 983-992, (2011).
[25] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7**, 289-297, (2006).
[26] S. Radenović, *Remarks on some coupled coincidence point results in partially ordered metric spaces,* Arab J Math. Sci **20** (1) (**2014**), 29-39.
[27] S. Radenović, *Bhaskar-Lakshmikantham type-results for monotone mappings in partially ordered metric spaces,* Int. J. Nonlinear Anal. Appl. **5** (2014) No. 2, 37-49.
[28] S. Radenović, *Coupled fixed point theorems for monotone mappings in partially ordered metric spaces,* Kragujevac Journal of Mathematics; 38 (2) (2014), 249-257
[29] S. Sedghi, I. Altun, N. Shobe, M. Salahshour, *Some properties of S-metric space and fixed point results*, Kyungpook Math. J., **54**, 113-122, (2014).
[30] S. Sedghi, N. Shobe, Aliouche, *A generalization of fixed point theorem in S-metric spaces*, Mat. Vesnik **64**, 258-266, (2012).
[31] S. Sedghi, N. Shobe, T. Došenović, *Fixed point results in S-metric spaces*, Nonlinear Functional Analysis and Applications, **20** (1), 55-67, (2015).
[32] S. Sedghi, N. Shobe, H. Zhou, *A common fixed point theorem in D^*-metric spaces*, Fixed Point Theory Appl., Article ID 27906, 13 pages, (2007).
[33] W. Shatanawi, M. Postolache, *Some fixed point results for a G-weak contraction in G-metric spaces,* Abstr. Appl. Anal., Vol. 2012, ID: 815870.
[34] W. Shatanawi, M. Postolache, *Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces,* Fixed Point Theory Appl., **2013**, Art. No. 271 (2013).
[35] W. Shatanawi, M. Postolache, Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces, Fixed Point Theory Appl., **2013**, Art. No. 60 (2013).
[36] W. Shatanawi, M. Postolache, Z. Mustafa, *Tripled and coincidence fixed point theorems for contractive mappings satisfying Φ-maps in partially ordered metric spaces,* An. Sti. U. Ovid. Co-Mat., **22**(3), 179-203, (2014).