U.P.B. Sci. Bull., Series A, Vol. 84, Iss. 1, 2022 ISSN 1223-7027

FRACTIONAL DIFFERENTIAL EQUATION IN PARTIALLY ORDERED CONTROLLED METRIC SPACES

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In this paper a fractional equation (with derivatives of order α less than 1) is considered. We study the existence of solutions of boundary value problems for system of fractional equations. Our approach is based on Banach fixed point theorem.

Keywords: fractional equations, fixed point, controlled metric type space.
MSC2020: 47H10, 34A08.

1. Introduction

Solving fractional problems continues to motivate many researchers especially in last decade. Indeed, the fractional differential equations are used to modelling many phenomena in biology, physics,... [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]. Consequently, this topic is gaining much importance. Many papers dealing with the existence of solution to the value problem for some differential equation see [16], [17], [18], [19]. In the literature, many techniques are combined to obtained the desired results. One of the approach used is the fixed point theory which is considered as powerful tool in this area. Indeed, this theory is developing faster during the last years and several results have been reported in many different metric spaces [20], [21], [22], [23], [24], [25], [26], [28].

In this paper, we discuss the existence of a solution to the following problem:

$$(\mathcal{P}) : \left\{ \begin{array}{ccc} D^\alpha x(t) & = & f(t, x(t)) = Fx(t) \text{ if } t \in J_0 = (0, T] \\ x(0) & = & x(T) = r \end{array} \right\}$$

where $T > 0$ and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $J = [0, T]$ and $D^\alpha x$ denotes a Riemann-Liouville fractional derivative of x with $\alpha \in (0, 1)$.

The paper is organized as follows. In Section 2, we start by recalling the partially ordered controlled metric space then we prove a version of fixed point theorem on this context under some hypothesis. In Section 3, we discuss the existence of solution for the value problem $(\mathcal{P})$. To achieve this we prove Theorem 3.1 by applying the fixed point result obtained in Section 2 with a corresponding weighted norm.

2. Fixed point results in partially ordered controlled metric spaces

First, we remind the reader of the definition of a controlled metric space.

Definition 2.1. [27] *Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$ be a mapping. The function $\mu : X \times X \to [0, \infty)$ is called a controlled metric type if*
(c_1) $\mu(x, y) = 0$ if and only if $x = y$,

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(c_2) $\mu(x,y) = \mu(y,x)$,
(c_3) $\mu(x,y) \leq \theta(x,z)\mu(x,z) + \theta(z,y)\mu(z,y)$,
for all $x, y, z \in X$. The pair (X, μ) is called a controlled metric type space.

We define Cauchy and convergent sequences in controlled metric type spaces as follows:

Definition 2.2. *Let (X, μ) be a controlled metric type space and $\{t_n\}_{n\geq 0}$ be a sequence in X.*
(1) We say that the sequence $\{t_n\}$ converges to some t in X, if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $\mu(t_n, t) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n\to\infty} t_n = t$.
(2) We say that the sequence $\{t_n\}$ is Cauchy, if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $\mu(t_m, t_n) < \epsilon$ for all $m, n \geq N$.
(3) The controlled metric type space (X, μ) is called complete if every Cauchy sequence is convergent.

Definition 2.3. *Let X be a nonempty set. If (X, μ) is a controlled metric space and $(X, \prec)$ is a partially ordered set, then $(X, \mu, \prec)$ is called a partially ordered controlled metric space. $t_1, t_2 \in X$ are called comparable if $t_1 \prec t_2$ or $t_2 \prec t_1$ holds.*

Theorem 2.1. *Let $(X, \mu, \prec)$ be a complete partially ordered controlled metric space. Let $h\colon X \to X$ be a increasing mapping. Assume that there exists $t_0 \prec h(t_0)$ and define the sequence $\{t_n\}$ by $t_1 = h(t_0)$, $t_2 = h(t_1)$,...$t_n = h(t_{n-1})$.*
Suppose that there exists a function $\sigma : [0, \infty) \to [0, k)$ where $0 < k < 1$ satisfying $\sigma(t_n) \to 1$ implies $t_n \to 0$ such that

$$\mu(h(a), h(b)) \leq \sigma(\mu(a,b))\mu(a,b) \text{ for each } a, b \in X \text{ with } a \prec b. \tag{1}$$

Assume that h is continuous or X is such that :
if an increasing sequence $\{t_n\} \to t$ in X then $t_n \prec t$ $\forall n$.
Moreover, if for each $a, b \in X$ there exists $c \in X$ which is comparable to a and b.
In addition, assume that, for every $a \in X$, we have

$$\lim_{n\to\infty} \theta(t_{n+1}, t) \text{ and } \lim_{n\to\infty} \theta(t, t_{n+1}) \text{ exist and are finite.} \tag{2}$$

Suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\theta(t_{i+1}, t_{i+2})}{\theta(t_i, t_{i+1})} \theta(t_{i+1}, t_m) < \frac{1}{k}. \tag{3}$$

Then, h has a unique fixed point.

Proof. Since $t_0 \prec h(t_0)$ and h is an increasing function, we obtain by induction that:

$$t_0 \prec h(t_0) \prec h^2(t_0) \prec ... \prec h^n(t_0) \prec h^{n+1}(t_0).$$

We denote $t_n = h^n(t_0)$, n=1,2,... . Since $t_n \prec t_{n+1}$ for each $n \in \mathbb{N}$ then by (1) we get

$$\begin{aligned} \mu(t_{n+1}, t_{n+2}) &= \mu(h^{n+1}(t_0), h^{n+2}(t_0)) \\ &\leq \sigma(\mu(t_n, t_{n+1}))\mu(t_n, t_{n+1}) \\ &\leq k\mu(t_n, t_{n+1}) \\ &\leq k^2\mu(t_{n-1}, t_n) \\ &\leq k^n\mu(t_0, t_1) \end{aligned} \tag{4}$$

Therefore, we can conclude from (4) that

$$\lim_{n\to\infty} \mu(t_n, t_{n+1}) = 0.$$

In the next step we show that $\{t_n\}$ is a Cauchy sequence. Using the triangle inequality we obtain:

$$\begin{aligned}
\mu(t_n,t_m) &\le \theta(t_n,t_{n+1})\mu(t_n,t_{n+1})+\theta(t_{n+1},t_m)\mu(t_{n+1},t_m)\\
&\le \theta(t_n,t_{n+1})\mu(t_n,t_{n+1})+\theta(t_{n+1},t_m)\theta(t_{n+1},t_{n+2})\mu(t_{n+1},t_{n+2})\\
&+ \theta(t_{n+1},t_m)\theta(t_{n+2},t_m)\mu(t_{n+2},t_m)\\
&\vdots\\
&\le \theta(t_n,t_{n+1})\mu(t_n,t_{n+1})+\sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\theta(t_j,t_m)\Big)\theta(t_i,t_{i+1})\mu(t_i,t_{i+1})\\
&+ \prod_{s=n+1}^{m-1}\theta(t_s,t_m)\mu(t_{m-1},t_m)\\
&\le \theta(t_n,t_{n+1})k^n\mu(t_0,t_1)+\sum_{i=n+1}^{m-2}\Big(\prod_{j=n+1}^{i}\theta(t_j,t_m)\Big)\theta(t_i,t_{i+1})k^i\mu(t_0,t_1)\\
&+ \prod_{s=n+1}^{m-1}\theta(t_s,t_m)k^m\mu(t_0,t_1)\\
&\le \theta(t_n,t_{n+1})k^n\mu(t_0,t_1)+\sum_{i=n+1}^{m-1}\Big(\prod_{j=n+1}^{i}\theta(t_j,t_m)\Big)\theta(t_i,t_{i+1})k^i\mu(t_0,t_1)\\
&\le \theta(t_n,t_{n+1})k^n\mu(t_0,t_1)+\sum_{i=n+1}^{m-1}\Big(\prod_{j=0}^{i}\theta(t_j,t_m)\Big)\theta(t_i,t_{i+1})k^i\mu(t_0,t_1) \qquad (5)
\end{aligned}$$

We denote $\psi_s=\sum_{i=0}^{s}\Big(\prod_{j=0}^{i}\theta(t_j,t_m)\Big)\theta(t_i,t_{i+1})k^i$. Then we have from (5)

$$\mu(t_n,t_m)\le\mu(t_0,t_1)[k^n\theta(t_n,t_{n+1})+(\psi_{m-1}-\psi_n)]. \qquad (6)$$

Using (6) and by taking into account (2) and (3) we deduce that $\lim_{n\to\infty}\psi_n$ exists and the sequence $\{\psi_n\}$ is Cauchy. Hence, if we take the limit in the inequality (6) as $n,m\leftarrow\infty$, we conclude that

$$\lim_{n,m\to\infty}\mu(t_n,t_m)=0,$$

which affirms that $\{t_n\}$ is a Cauchy sequence in the complete partially ordered controlled metric space $(X,\mu,\prec)$, then $\{t_n\}$ converges to some $t\in X$.
Let prove that t is a fixed point of h. Since h is continuous we have

$$t=\lim_{n\to\infty}t_n=\lim_{n\to\infty}h^n(t_0)=\lim_{n\to\infty}h^{n+1}(t_0)=h(\lim_{n\to\infty}h^n(t_0))=h(t).$$

Then t is a fixed point of h.
To prove the uniqueness of the fixed point, let u be another fixed point of h then

$$\mu(t,u)=\mu(h(t),h(u))\le\sigma(\mu(t,u))\mu(t,u)$$

which holds unless $\mu(t,u)=0$ then $t=u$ and h has a unique fixed point. □

In the next section, we prove the existence of a solution for a fractional differential equation using our result, we also, prove the same result using monotone iterative method.

3. Fractional differential equation

3.1. Fixed point method

Let $C_{1-\alpha}(J,\mathbb{R}) = \{f \in C((0,T],\mathbb{R}) : t^{1-\alpha} \in C(J,\mathbb{R})\}$. We define the following weighted norm

$$||f|| = \max_{[0,T]} t^{1-\alpha}|x(t)|.$$

Theorem 3.1. *Let $\alpha \in (0,1)$, $f \in C(J\times\mathbb{R},\mathbb{R})$ increasing and $\sigma : [0,\infty) \longrightarrow [0,k)$ where $0 < k < 1$. In addition, we assume the following hypothesis:*

(1) $|f(u_1(t),v_1(t)) - f(u_2(t),v_2(t))| \leq \frac{\Gamma(2\alpha)}{T^\alpha}\sigma\Big(t^{1-\alpha}(v_1-v_2)\Big)|v_1 - v2|$

(2) $\frac{\Gamma(2\alpha)}{T^\alpha} \leq \frac{1}{k}$.

Then the problem $(\mathcal{P})$ has a unique solution.

Proof. Problem $(\mathcal{P})$ is equivalent to the problem $\mathcal{M}x = \mathcal{M}$ where

$$\mathcal{M}x(t) = rt^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Fx(s)ds.$$

In fact, proving that the operator $\mathcal{M}$ has a fixed point is sufficient to say that problem $\mathcal{M}$ has a unique solution. We use Banach fixed point theorem. Therefore we need to check that hypothesis in Theorem 2.1 are satisfied.
Indeed, $A = C_{1-\alpha}(J,\mathbb{R})$ is a partially ordered set if we define the following order relation in A:

$$U,V \in C_{1-\alpha}(J,\mathbb{R}), U \leq V \text{ if and only if } U(t) \leq V(t)\forall t \in J.$$

Also, (A,μ) is a complete controlled metric space if we choose:

$$\mu(x,y) = \max_{[0,T]} t^{1-\alpha}|x(t) - y(t)|, x,y \in C_{1-\alpha}(J,\mathbb{R}).$$

The mapping $\mathcal{M}$ is increasing since f is increasing.
Now, we must prove that $\mathcal{M}$ is a contraction map. Let $x,y \in C_{1-\alpha}(J,\mathbb{R})$, $0 < \alpha < 1$.

$$||\mathcal{M}x - \mathcal{M}y|| \leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,T]} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}|f(t,x(s)) - f(t,y(s))|ds$$

Since $||\mathcal{M}x - \mathcal{M}y|| = \max t^{1-\alpha}|x(s) - y(s)|$ then

$$|x(s) - y(s)| = ||x(s) - y(s)|| \max s^{\alpha-1}.$$

Subsequently, using the first hypothesis of the theorem we get

$$\begin{aligned}
||\mathcal{M}x - \mathcal{M}y|| &\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,T]} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\frac{\Gamma(2\alpha)}{T^\alpha}\sigma\big(s^{1-\alpha}|x(s)-y(s)|\big)|x(s)-y(s)| \\
&= \frac{1}{\Gamma(\alpha)}\max_{t\in[0,T]} t^{1-\alpha}\int_0^t \Big[(t-s)^{\alpha-1}\frac{\Gamma(2\alpha)}{T^\alpha}\sigma\big(s^{1-\alpha}||x(s)-y(s)||\max s^{\alpha-1}\big) \\
&\quad ||x(s)-y(s)||\max s^{\alpha-1}\Big]ds \\
&\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,T]} t^{1-\alpha}||x(s)-y(s)||\sigma\big(||x(s)-y(s)||\big)\frac{\Gamma(2\alpha)}{T^\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds.
\end{aligned}$$

From the Riemann-Liouville fractional integral we have

$$\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{2\alpha-1}.$$

Therefore, we have

$$||\mathcal{M}x - \mathcal{M}y|| \leq \sigma\big(||x(s) - y(s)||\big)||x(s) - y(s)||$$

□

3.2. Monotone iterative method

First, we present the following hypothesis;

Hypothesis 3.2 (H_1)**.** *(1) $L(t) = L, t \in J$ or,*
(2) the function L is non constant on J and

$$\frac{a^\alpha}{\Gamma(2\alpha)} \max_{t\in J} |L(t)| < 1, \quad \textit{only if} \quad \alpha \in (0, \frac{1}{2})$$

Next, we present the following consequence of Theorem 3.1.

Lemma 3.1. *If $\alpha \in (0,1), L \in C(J,\mathbb{R}), z \in C_{1-\alpha}(J,\mathbb{R})$,and hypothesis (H_1) holds, then the problem $(\mathcal{P})$ has a unique solution.*

Hypothesis 3.3 (H_2)**.** *(1) $L(t) = L, t \in J$ or,*
(2) the function L is non constant and if $L(t)$ is negative, then there exists $\bar{L}$ non decreasing where $-L(t) \leq \bar{L}(t)$ on J and for every $x \in J$ we have

$$\frac{1}{\Gamma(\alpha)} \int_0^a (a-\tau)^{\alpha-1} \bar{L}(\tau) d\tau < 1.$$

Now, for our purpose we prove the following useful lemma.

Lemma 3.2. *Let $\alpha \in (0,1)$ and $L \in C(J,[0,\infty))$ or $L \in C(J,(-\infty,0])$. Assume that $q \in C_{1-\alpha}(J,\mathbb{R})$ is a solution to the following problem:*

$$D^\alpha q(t) \leq -L(t)q(t), \quad t \in J_0 \tag{7}$$
$$\tilde{q}(0) < 0.$$

If (H_2) holds, then $q(t) \leq 0$ for all $t \in J$.

Proof. Assume that our lemma is false, that is there exists $x, y \in (0,a]$ such that $q(x) = 0, q(y) > 0$ and $q(t) \leq 0$ for $t \in (0,x]$; $q(t) > 0$ for $t \in (x,y]$. Let x_0 be the first maximal point of q on $[x,y]$

Case1:

Assume that $L(t) \geq 0$ for all $t \in J$. Thus, $D^{\alpha,\rho} q(t) \leq 0$ for $t \in [x,y]$. Hence,

$$\int_x^{x_0} D^\alpha q(t) \leq 0.$$

Therefore, $B \equiv I^{1-\alpha}q(x_0) - I^{1-\alpha}q(x) \leq 0$. But,

$$\begin{aligned}
B &= \frac{1}{\Gamma(1-\alpha)}[\int_0^{x_0}(x_0-\tau)^{-\alpha}q(\tau)d\tau - \int_0^x (x-\tau)^{-\alpha}q(\tau)d\tau] \\
&= \frac{1}{\Gamma(1-\alpha)}\{\int_0^x [(x_0-\tau)^{-\alpha} - (x-\tau)^{-\alpha}]q(\tau)d\tau \\
&+ \int_x^{x_0}(x_0-\tau)^{-\alpha}q(\tau)d\tau\} \\
&> \frac{1}{\Gamma(1-\alpha)}\int_x^{x_0}(x_0-\tau)^{-\alpha}q(\tau)d\tau > 0.
\end{aligned}$$

Which leads us to a contradiction given the fact that $B \leq 0$

Case2:

Assume that $L(t) \leq 0$ for all $t \in J$, and consider $\tilde{L}$ to be nondecreasing on J. Now, if we apply I^α on problem (7) we obtain;

$$q(t) - \tilde{q}(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} \leq -I^\alpha[L(t)q(t)] \quad \text{for} \quad t \in [x, x_0].$$

Notice that, $\tilde{q}(0)\frac{t^{\alpha-1}}{\Gamma(\alpha)} \leq 0$ and that is due to the fact that $\tilde{q}(0) \leq 0$. Thus,

$$\begin{aligned}
q(x_0) &\leq -\frac{1}{\Gamma(\alpha)}\int_0^{x_0}(x_0-\tau)^{\alpha-1}L(\tau)q(\tau)d\tau \\
&= -\frac{1}{\Gamma(\alpha)}[\int_0^{x}(x_0-\tau)^{\alpha-1}L(\tau)q(\tau)d\tau + \int_x^{x_0}(x_0-\tau)^{\alpha-1}L(\tau)q(\tau)d\tau] \\
&< -\frac{q(x_0)}{\Gamma(\alpha)}\int_0^{x_0}(x_0-\tau)^{\alpha-1}L(\tau)d\tau, \quad \text{let} \quad \sigma = \frac{\tau}{x_0} \\
&= -\frac{q(x_0)x_0^\alpha}{\Gamma(\alpha)}\int_0^{1}(1-\sigma)^{\alpha-1}L(\sigma x_0)d\sigma \\
&\leq \frac{q(x_0)x_0^\alpha}{\Gamma(\alpha)}\int_0^{1}(1-\sigma)^{\alpha-1}\tilde{L}(\sigma a)d\sigma \\
&= \frac{q(x_0)x_0^\alpha}{\Gamma(\alpha)a^\alpha}\int_0^{a}(a-\tau)^{\alpha-1}\tilde{L}(\tau)d\tau \\
&\leq \frac{q(x_0)}{\Gamma(\alpha)}\int_0^{a}(a-\tau)^{\alpha-1}\tilde{L}(\tau)d\tau.
\end{aligned}$$

Hence, $q(x_0)[1 - \frac{1}{\Gamma(\alpha)}\int_0^a (a-\tau)^{\alpha-1}\tilde{L}(\tau)d\tau] \leq 0$. Using hypothesis (H_2) this implies that $q(x_0) \leq 0$, which leads us to a contradiction, and concludes our proof. □

We say that y is a lower solution of problem $(\mathcal{P})$, if

$$D^\alpha y(t) \leq \mathcal{F}y(t), \quad t \in J_0; \quad \tilde{y}(0) \leq 0,$$

and we say that y is an upper solution of problem $(\mathcal{P})$, if

$$D^\alpha y(t) \geq \mathcal{F}y(t), \quad t \in J_0; \quad \tilde{y}(0) \leq 0$$

Next, define the following hypothesis;

Hypothesis 3.4 (H_3)**.** *There exists a function $L \in C(J, \mathbb{R})$ where*

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \leq L(t)|v_1 - u_1| \quad \textit{whenever} \quad x_0 \leq u_1 \leq v_1 \leq y_0, \quad u_2 \leq v_2$$

Theorem 3.5. *Assume that x_0 is a lower solution of problem $(\mathcal{P})$, and y_0 is an upper solution of problem $(\mathcal{P})$, where $x_0, y_0 \in C_{1-\alpha}(J,\mathbb{R})$. Moreover, assume that hypothesis H_1, H_2 and H_3 hold, the problem $(\mathcal{P})$ has solutions in $[x_0, y_0] = \{y \in C_{1-\alpha}(J,\mathbb{R}) \mid x_0(t) \leq y(t) \leq y_0(t), t \in J_0, \tilde{x}_0(0) \leq \tilde{y}(0) \leq \tilde{y}_0(0)\}$.*

Proof. Using Lemma 3.1, and Lemma 3.2, the proof is similar to the proof of Theorem 2 in [28]. □

Now, we present the following example.

Example I

Let $0 < \alpha < 1$, and $\mathbb{A}, \mathbb{B} \in C([0,1],(0,\infty))$ such that $\mathbb{A}(t) \leq \mathbb{B}(t)$ for $t \in [0,1]$. Now consider the following problem;

$$\begin{aligned}
&D^\alpha \xi(t) \equiv \mathcal{F}\xi(t); t \in J_0 = (0,1]. \qquad (8)\\
&\tilde{\xi}(0) = 0
\end{aligned}$$

where

$$\mathcal{F}\xi(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \mathbb{A}(t)[t-1-\xi(t)]^3 + \mathbb{B}(t)\int_0^t [\cos(t\tau)]^4 \xi(\tau)d\tau$$

Now, let $x_0(t) = 0$ and $y_0(t) = t$, first note that $x_0(t)$ is a lower solution of the problem (8). Next, we show that $y_0(t)$ is an upper solution of problem (8);

$$\begin{aligned}
\mathcal{F}y_0(t) &= \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \mathbb{B}(t)\int_0^t [\cos(t\tau)]^4 \tau d\tau \\
&\leq \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \mathbb{B}(t)\int_0^t d\tau \\
&= \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \mathbb{A}(t) + \mathbb{B}(t) \\
&< \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \\
&= D^\alpha y_0(t).
\end{aligned}$$

Thus, $y_0(t)$ is an upper solution of the problem (8). Now, it is not difficult to see that all the hypothesis of Theorem 3.5, are satisfied. Therefore, problem (8) has solutions in $[x_0, y_0]$ if $\alpha \in (\frac{1}{2}, 1)$ and for $\alpha \in (0, \frac{1}{2}]$ we need to assume that $\frac{1}{\Gamma(2\alpha)} \max\limits_{t\in[0,1]} |\mathbb{A}(t)| < 1$.

4. **Conclusion**

We proved the existence and uniqueness of fixed point for a contractive mapping in partially ordered controlled metric type spaces, we were able to use our result to show that the Fractional differential equation ($\mathcal{P}$) has a solution. Moreover, we used the monotone iterative method to show that ($\mathcal{P}$) has a solution.

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