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SOME INTEGRAL INEQUALITIES FOR PRODUCT OF HARMONIC log-CONVEX FUNCTIONS

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In this paper, we establish some new Hermite-Hadamard like integral inequalities involving harmonic log-*convex functions.*

Keywords: Harmonic log-convex functions, Hermite-Hadamard inequality.

1. Introduction

It is well known that convexity had played an important and significant part in the development of several branches of pure and applied sciences, see [1, 2, 6, 7, 9, 10, 15] and the references therein.

A function $f\colon [a,b] \to \mathbb{R}$ is said to be a convex function, if and only if, f satisfies the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x \in [a,b]. \tag{1}$$

The inequality (1) is known as Hermite-Hadamard inequality, see [4, 5]. For applications and other aspects of Hermite-Hadamard inequality and their variant forms, see [2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17].

In recent years, convex functions and convex sets have been generalized in many directions using some innovative ideas A significant generalization of convex functions is harmonic convex functions and harmonic log-convex functions introduced by Iscan [6] and Noor *et al.* [9] respectively. The main purpose of this paper is to establish some new Hermite-Hadamard integral inequalities for harmonic-log-convex functions. Our results represent significant refinement of the known results.

2. Preliminaries

In this section, we recall some basic concepts and results.

Definition 2.1 ([16]). *A set $I \subseteq \mathbb{R}\setminus\{0\}$ is said to be a harmonic convex set, if*

$$\frac{xy}{tx+(1-t)y} \in I, \qquad \forall x,y \in I, t \in [0,1].$$

Definition 2.2 ([6]). *Let I be a harmonic convex set. A function $f\colon I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be a harmonic convex function, if and only if,*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x,y \in I, t \in [0,1].$$

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A function f is a harmonic convex function, if and only if, $-f$ is a harmonic concave function.

If $t = \frac{1}{2}$, then we have

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2}, \qquad \forall x, y \in I,$$

which is called Jensen harmonic convex function.

Definition 2.3 ([9]). *A function $f\colon I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$ is said to be a harmonic* log-*convex function on I, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \big(f(x)\big)^{1-t}\big(f(y)\big)^{t}, \qquad \forall x, y \in I, t \in [0,1]. \tag{2}$$

If the inequality (2) is reversed, then f is called harmonic log-concave function.

If $t = \frac{1}{2}$, then we have Jensen harmonic log-convex function, that is,

$$f\left(\frac{2xy}{x+y}\right) \leq \sqrt{f(x)f(y)}, \qquad \forall x, y \in I.$$

It follows that

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)\log f(x) + t\log f(y), \qquad \forall x, y \in I, t \in [0,1].$$

Using the arithmetic-geometric inequality, (2) can be written as

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &\leq \big(f(x)\big)^{1-t}\big(f(y)\big)^{t} \\ &\leq (1-t)\log f(x) + t\log f(y), \qquad \forall x, y \in I, t \in [0,1]. \end{aligned}$$

This shows that harmonic log-convex functions are harmonic convex functions, but the converse is not true, see [1].

Theorem 2.1 ([6]). *Let $f\colon I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic convex function. If $f \in L[a,b]$, then*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x \in [a,b]. \tag{3}$$

Theorem 2.2 ([9]). *Let $f\colon I = [a,b] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonic* log-*convex function. If $f \in L[a,b]$, then*

$$f\left(\frac{2ab}{a+b}\right) \leq \exp\left[\frac{ab}{b-a}\int_a^b \log\left(\frac{f(x)}{x^2}\right)\mathrm{d}x\right] \leq \sqrt{f(a)f(b)}, \qquad x \in [a,b].$$

Recall the following special means.

(1) The arithmetic mean:

$$A(a,b) = \frac{a+b}{2} \qquad \forall a, b, a \neq b \in \mathbb{R}_+$$

(2) The geometric mean:

$$G(a,b) = \sqrt{ab} \qquad \forall a, b, a \neq b \in \mathbb{R}_+$$

(3) The harmonic mean:

$$H(a,b) = \frac{2ab}{a+b} \qquad \forall a, b, a \neq b \in \mathbb{R}_+$$

(4) The quadratic mean:

$$K(a,b)=\sqrt{\frac{a^2+b^2}{2}} \qquad \forall a,b,a\neq b\in\mathbb{R}_+$$

(5) The logarithmic mean:

$$L(a,b)=\begin{cases}\frac{a-b}{\ln a-\ln b}, & a\neq b\\ a, & a=b.\end{cases} \qquad \forall a,b\in\mathbb{R}_+$$

(6) The generalized logarithmic mean:

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & a\neq b,p\neq -1,0\\ \frac{a-b}{\ln a-\ln b}, & a\neq b,p=-1\\ a, & a=b.\end{cases} \qquad \forall a,b\in\mathbb{R}_+$$

In particular, we have the following inequality

$$H\leq G\leq L\leq A.$$

3. Main results

In this section, we derive Hermite-Hadamard inequalities for harmonic log-convex functions

Theorem 3.1. *Let $f\colon I_h\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be an increasing and a harmonic* log-*convex function. If $f\in L[a,b]$, then*

$$\begin{aligned}&f\left(\frac{2ab}{a+b}\right)L(f(a),f(b))\\ \leq&\frac{ab}{8(b-a)}\int_a^b\frac{f^4(x)}{x^2}\mathrm{d}x+\frac{1}{8}\frac{f(a)^2+f(b)^2}{2}\frac{f(a)+f(b)}{2}\frac{f(b)-f(a)}{\log f(b)-\log f(a)}+1.\end{aligned}$$

Proof. Let f be harmonic log-convex function. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right)\leq[f(a)]^{1-t}[f(b)]^t.$$

Now using the inequality

$$8xy\leq x^4+y^4+8, \qquad x,y\in\mathbb{R},$$

we have

$$\begin{aligned}&8\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)[f(a)]^{1-t}[f(b)]^t\mathrm{d}t\\ \leq&\int_0^1 f^4\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(a)]^{4(1-t)}[f(b)]^{4t}\mathrm{d}t+8.\end{aligned}$$

Since f is an increasing function, we have

$$\begin{aligned}&8\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\int_0^1[f(a)]^{1-t}[f(b)]^t\mathrm{d}t\\ \leq&\int_0^1 f^4\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(a)]^{4(1-t)}[f(b)]^{4t}\mathrm{d}t+8.\end{aligned}$$

From the above inequality it is easy to observe that

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)L(f(a),f(b))\\
\leq&\frac{ab}{8(b-a)}\int_a^b \frac{f^4(x)}{x^2}\mathrm{d}x+\frac{1}{8}K^2(f(a),f(b))A(f(a),f(b))L(f(a),f(b))+1\\
=&\frac{ab}{8(b-a)}\int_a^b \frac{f^4(x)}{x^2}\mathrm{d}x+\frac{1}{8}\frac{f(a)^2+f(b)^2}{2}\frac{f(a)+f(b)}{2}\frac{f(b)-f(a)}{\log f(b)-\log f(a)}+1,
\end{aligned}$$

which is the required result. □

Theorem 3.2. *Let* $f,g:I=[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be an increasing and harmonic* log-*convex functions. If* $f\in L[a,b]$, *then*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)L(f(a),f(b))+2A(f(a),f(b))L(f(a),f(b))-L^2(f(a),f(b))\\
&+\frac{a^2b}{(b-a)^2}f(a)\int_a^b \frac{(b-x)f(x)}{x^3}\mathrm{d}x+\frac{ab^2}{(b-a)^2}f(b)\int_a^b \frac{(x-a)f(x)}{x^3}\mathrm{d}x\\
&\leq\frac{ab}{b-a}\int_a^b \frac{f^2(x)}{x^2}\mathrm{d}x+A(f(a),f(b))L(f(a),f(b))+\psi(a,b),
\end{aligned}$$

where

$$\psi(a,b)=\frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.$$

Proof. Let f,g be harmonic log-convex functions on I. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right)\leq[f(a)]^{1-t}[f(b)]^t\leq(1-t)f(a)+tf(b)\quad\forall a,b\in I,t\in[0,1].$$

Using the elementary inequality

$$xy+yz+zx\leq x^2+y^2+z^2,\quad x,y,z\in\mathbb{R},$$

we have

$$\begin{aligned}
&\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)[f(a)]^{1-t}[f(b)]^t\mathrm{d}t+\int_0^1(1-t)[f(a)]^{2-t}[f(b)]^t\mathrm{d}t+\int_0^1 t[f(a)]^{1-t}[f(b)]^{1+t}\mathrm{d}t\\
&+f(a)\int_0^1(1-t)f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+f(b)\int_0^1 tf\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\\
&\leq\int_0^1 f^2\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t+\int_0^1[f(a)]^{2(1-t)}[f(b)]^{2t}\mathrm{d}t\\
&+f^2(a)\int_0^1(1-t)^2+f^2(b)\int_0^1 t^2\mathrm{d}t+f(a)f(b)\int_0^1 2t(1-t)\mathrm{d}t \qquad (4)
\end{aligned}$$

As it is easy to see that

$$\int_0^1 f^2\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t=\frac{ab}{b-a}\int_a^b \frac{f^2(x)}{x^2}\mathrm{d}x. \qquad (5)$$

By substituting $2t=u$, we obtain

$$\begin{aligned}
\int_0^1[f(a)]^{2(1-t)}[f(b)]^{2t}\mathrm{d}t=\frac{1}{2}f^2(a)\int_0^2\left(\frac{f(b)}{f(a)}\right)^u\mathrm{d}u=&\frac{1}{2}\frac{f^2(b)-f^2(a)}{\log f(b)-\log f(a)}\\
=&\frac{(f(a)+f(b))(f(b)-f(a))}{2\log f(b)-\log f(a)}\\
=&A(f(a),f(b))L(f(a),f(b)). \qquad (6)
\end{aligned}$$

Using the left half of the Hermite-Hadamard inequality (3), one can obtain

$$\begin{aligned}\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)[f(a)]^{1-t}[f(b)]^t\mathrm{d}t &\geq \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\int_0^1[f(a)]^{1-t}[f(b)]^t\mathrm{d}t\\ &= \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\frac{f(b)-f(a)}{\log f(b)-\log f(a)}\\ &\geq f\left(\frac{2ab}{a+b}\right)L(f(a),f(b)). \end{aligned}\tag{7}$$

Also

$$\begin{aligned}\int_0^1(1-t)[f(a)]^{2-t}[f(b)]^t\mathrm{d}t &= f^2(a)\int_0^1(1-t)\left(\frac{f(b)}{f(a)}\right)^t\mathrm{d}t\\ &= \frac{-f^2(a)}{\log\frac{f(b)}{f(a)}}+\frac{f(a)f(b)-f^2(a)}{\left(\log\frac{f(b)}{f(a)}\right)^2}\end{aligned}\tag{8}$$

and

$$\int_0^1 t[f(a)]^{1-t}[f(b)]^{1+t}\mathrm{d}t = f(a)f(b)\int_0^1 t\left(\frac{f(b)}{f(a)}\right)^t\mathrm{d}t \tag{9}$$

$$= \frac{f^2(b)}{\log\frac{f(b)}{f(a)}}-\frac{f^2(b)-f(a)f(b)}{\left(\log\frac{f(b)}{f(a)}\right)^2} \tag{10}$$

Take $\frac{ab}{ta+(1-t)b}=x$, to have

$$f(a)\int_0^1(1-t)f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t = \frac{a^2b}{(b-a)^2}f(a)\int_a^b\frac{(b-x)f(x)}{x^3}\mathrm{d}x \tag{11}$$

$$f(b)\int_0^1 tf\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t = \frac{ab^2}{(b-a)^2}f(b)\int_a^b\frac{(x-a)f(x)}{x^3}\mathrm{d}x \tag{12}$$

From (5)-(12), (4) reduces to

$$\begin{aligned}&f\left(\frac{2ab}{a+b}\right)L(f(a),f(b))+2A(f(a),f(b))L(f(a),f(b))-L^2(f(a),f(b))\\ &+\frac{a^2b}{(b-a)^2}f(a)\int_a^b\frac{(b-x)f(x)}{x^3}\mathrm{d}x+\frac{ab^2}{(b-a)^2}f(b)\int_a^b\frac{(x-a)f(x)}{x^3}\mathrm{d}x\\ &\leq\frac{ab}{b-a}\int_a^b\frac{f^2(x)}{x^2}\mathrm{d}x+A(f(a),f(b))L(f(a),f(b))+\psi(a,b)\end{aligned}$$

which is the required result. □

Theorem 3.3. *Let $f,g\colon I_h\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be harmonic $\log$-convex functions. If $fg\in L[a,b]$, then*

$$\begin{aligned}&\frac{ab}{b-a}\int_a^b\left(\frac{f(x)}{x^2}\right)[g(a)]^{\frac{a(x-b)}{x(a-b)}}[g(b)]^{\frac{b(a-x)}{x(a-b)}}\mathrm{d}x+\frac{ab}{b-a}\int_a^b\left(\frac{g(x)}{x^2}\right)[f(a)]^{\frac{a(x-b)}{x(a-b)}}[f(b)]^{\frac{b(a-x)}{x(a-b)}}\mathrm{d}x\\ &\leq\frac{ab}{b-a}\int_a^b\left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x+\frac{1}{2}M(a,b).\end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$.

Proof. Let f and g be harmonic log-convex functions. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq [f(a)]^{1-t}[f(b)]^t,$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \leq [g(a)]^{1-t}[g(b)]^t.$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned} &f\left(\frac{ab}{ta+(1-t)b}\right)[g(a)]^{1-t}[g(b)]^t + g\left(\frac{ab}{ta+(1-t)b}\right)[f(a)]^{1-t}[f(b)]^t \\ \leq &f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + [f(a)]^{1-t}[f(b)]^t[g(a)]^{1-t}[g(b)]^t. \end{aligned}$$

Integrating over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)[g(a)]^{\frac{a(x-b)}{x(a-b)}}[g(b)]^{\frac{b(a-x)}{x(a-b)}}\mathrm{d}t + \int_0^1 g\left(\frac{ab}{ta+(1-t)b}\right)[f(a)]^{\frac{a(x-b)}{x(a-b)}}[f(b)]^{\frac{b(a-x)}{x(a-b)}}\mathrm{d}t \\ &\leq \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t + \int_0^1 [f(a)]^{1-t}[f(b)]^t[g(a)]^{1-t}[g(b)]^t\mathrm{d}t. \end{aligned}$$

From the above inequality it follows that

$$\begin{aligned} &\frac{ab}{b-a}\int_a^b \left(\frac{f(x)}{x^2}\right)[g(a)]^{\frac{a(x-b)}{x(a-b)}}[g(b)]^{\frac{b(a-x)}{x(a-b)}}\mathrm{d}x + \frac{ab}{b-a}\int_a^b \left(\frac{g(x)}{x^2}\right)[f(a)]^{\frac{a(x-b)}{x(a-b)}}[f(b)]^{\frac{b(a-x)}{x(a-b)}}\mathrm{d}x \\ &\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + L(G^2(f(a),g(a)), G^2(f(b),g(b))) \\ &\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + A(G^2(f(a),g(a)), G^2(f(b),g(b))) \\ &= \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + \frac{1}{2}M(a,b), \end{aligned}$$

which is the required result. □

Theorem 3.4. *Let $f, g\colon I_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be an increasing and harmonic* log-*convex functions. If $fg \in L[a,b]$, then*

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right)\mathbb{L}[g(a),g(b)] + g\left(\frac{2ab}{a+b}\right)\mathbb{L}[f(a),f(b)] \\ &\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x}\right)\mathrm{d}x + \mathbb{L}[f(a)g(a), f(b)g(b)]. \\ &\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + \frac{f(a)g(a)+f(b)g(b)}{2}. \end{aligned}$$

Proof. Let f, g be harmonic log-convex functions. Then

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq [f(a)]^{1-t}[f(b)]^t.$$
$$g\left(\frac{ab}{ta+(1-t)b}\right) \leq [g(a)]^{1-t}[g(b)]^t.$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we get

$$\begin{aligned}
&f\left(\frac{ab}{ta+(1-t)b}\right)[g(a)]^{1-t}[g(b)]^t + g\left(\frac{ab}{ta+(1-t)b}\right)[f(a)]^{1-t}[f(b)]^t\\
\leq &f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right) + [f(a)]^{1-t}[f(b)]^t[g(a)]^{1-t}[g(b)]^t.
\end{aligned}$$

Integrating over $[0, 1]$, we have

$$\begin{aligned}
&\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)[g(a)]^{1-t}[g(b)]^t \mathrm{d}t + \int_0^1 g\left(\frac{ab}{ta+(1-t)b}\right)[f(a)]^{1-t}[f(b)]^t \mathrm{d}t\\
&\leq \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t + \int_0^1 [f(a)]^{1-t}[f(b)]^t[g(a)]^{1-t}[g(b)]^t \mathrm{d}t.
\end{aligned}$$

Now, since f, g are increasing functions, we have

$$\begin{aligned}
&\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \int_0^1 [g(a)]^{1-t}[g(b)]^t \mathrm{d}t + \int_0^1 g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t \int_0^1 [f(a)]^{1-t}[f(b)]^t \mathrm{d}t\\
&\leq \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t + \int_0^1 [f(a)]^{1-t}[f(b)]^t[g(a)]^{1-t}[g(b)]^t \mathrm{d}t.
\end{aligned}$$

From the above inequality it follows that

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\mathbb{L}[g(a), g(b)] + g\left(\frac{2ab}{a+b}\right)\mathbb{L}[f(a), f(b)]\\
&\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + \mathbb{L}[f(a)g(a), f(b)g(b)]\\
&\leq \frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)\mathrm{d}x + \frac{f(a)g(a)+f(b)g(b)}{2},
\end{aligned}$$

which is the required result. □

Theorem 3.5. *Let $f, g\colon I_h \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be harmonic* log*-convex functions. If $fg \in L[a, b]$, then*

$$\begin{aligned}
&\frac{a^2b}{(b-a)^2}\int_a^b \frac{(b-x)}{x^3}[\log f(a)\log g(x) + \log g(a)\log f(x)]\mathrm{d}x\\
&+ \frac{ab^2}{(b-a)^2}\int_a^b \frac{(x-a)}{x^3}[\log f(b)\log g(x) + \log g(b)\log f(x)]\mathrm{d}x\\
\leq&\frac{1}{3}[\log f(a)\log g(a) + \log f(b)\log g(b)] + \frac{1}{6}[\log f(a)g(b) + \log f(b)g(a)]\\
&+ \frac{ab}{b-a}\int_a^b \frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x.
\end{aligned}$$

Proof. Let f, g be harmonic log-convex functions. Then

$$\log f\left(\frac{ab}{ta+(1-t)b}\right) \leq (1-t)\log f(a) + t\log f(b).$$
$$\log g\left(\frac{ab}{ta+(1-t)b}\right) \leq (1-t)\log g(a) + t\log g(b).$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned}
&\log g(a)(1-t)\log f\left(\frac{ab}{ta+(1-t)b}\right) + \log g(b)t\log f\left(\frac{ab}{ta+(1-t)b}\right)\\
&+ \log f(a)(1-t)\log g\left(\frac{ab}{(1-t)a+tb}\right) + \log f(b)t\log g\left(\frac{ab}{ta+(1-t)b}\right)\\
\leq &\log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{ta+(1-t)b}\right)\\
&+ [(1-t)\log f(a) + t\log f(b)][(1-t)\log g(a) + t\log g(b)].
\end{aligned}$$

Integrating over $[0,1]$, we get

$$\begin{aligned}
&\log g(a)\int_0^1 (1-t)\log f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t + \log g(b)\int_0^1 t\log f\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\\
&+ \log f(a)\int_0^1 (1-t)\log g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t + \log f(b)\int_0^1 t\log g\left(\frac{ab}{(1-t)a+tb}\right)\mathrm{d}t\\
\leq &\int_0^1 \log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\\
&+ \log f(a)\log g(a)\int_0^1 (1-t)^2 + \log f(b)\log g(b)\int_0^1 t^2\mathrm{d}t\\
&+ [\log f(a)g(b) + \log f(b)g(a)]\int_0^1 t(1-t)\mathrm{d}t.
\end{aligned}$$

Now after simple integration, we have

$$\begin{aligned}
&\frac{a^2b}{(b-a)^2}\int_a^b \frac{(b-x)}{x^3}[\log f(a)\log g(x) + \log g(a)\log f(x)]\mathrm{d}x\\
&+ \frac{ab^2}{(b-a)^2}\int_a^b \frac{(x-a)}{x^3}[\log f(b)\log g(x) + \log g(b)\log f(x)]\mathrm{d}x\\
\leq &\frac{1}{3}[\log f(a)\log g(a) + \log f(b)\log g(b)] + \frac{1}{6}[\log f(a)g(b) + \log f(b)g(a)]\\
&+ \frac{ab}{b-a}\int_a^b \frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

which is the required result. □

Theorem 3.6. *Let $f, g\colon I = [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic convex functions. If $fg \in L[a,b]$, then*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\frac{a^2b^2}{(b-a)^2}\int_a^b \frac{\log g(x)}{x^2}\mathrm{d}x\int_a^b \frac{\log f(x)}{x^2}\mathrm{d}x\\
\leq &\exp\left[\frac{ab}{2(b-a)}\int_a^b \frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x + \frac{1}{12}M(a,b) + \frac{1}{6}N(a,b) + \log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)\right].
\end{aligned}$$

Proof. Let f, g be harmonic log-convex functions with $t = \frac{1}{2}$, then

$$\log f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[\log f\left(\frac{ab}{ta+(1-t)b}\right) + \log f\left(\frac{ab}{(1-t)a+tb}\right)\right].$$

$$\log g\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[\log g\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{(1-t)a+tb}\right)\right].$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we get

$$\begin{aligned}
&\frac{1}{2}\log f\left(\frac{2ab}{a+b}\right)\left[\log g\left(\frac{ab}{ta+(1-t)b}\right)+\log g\left(\frac{ab}{(1-t)a+tb}\right)\right]\\
&+\frac{1}{2}\log g\left(\frac{2ab}{a+b}\right)\left[\log f\left(\frac{ab}{ta+(1-t)b}\right)+\log f\left(\frac{ab}{(1-t)a+tb}\right)\right]\\
\leq&\frac{1}{4}\left[\log f\left(\frac{ab}{ta+(1-t)b}\right)+\log f\left(\frac{ab}{(1-t)a+tb}\right)\right]\left[\log g\left(\frac{ab}{ta+(1-t)b}\right)\right.\\
&\left.+\log g\left(\frac{ab}{(1-t)a+tb}\right)\right]+\log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)\\
=&\frac{1}{4}\left[\log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{ta+(1-t)b}\right)+\log f\left(\frac{ab}{(1-t)a+tb}\right)\log g\left(\frac{ab}{(1-t)a+b}\right)\right.\\
&+\log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{(1-t)a+tb}\right)+\log f\left(\frac{ab}{(1-t)a+tb}\right)\\
&\left.\log g\left(\frac{ab}{ta+(1-t)b}\right)\right]+\log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)\\
\leq&\frac{1}{4}\left[\log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{ta+(1-t)b}\right)+\log f\left(\frac{ab}{(1-t)a+tb}\right)\log g\left(\frac{ab}{(1-t)a+b}\right)\right.\\
&+[(1-t)\log f(a)+t\log f(b)][t\log g(a)+(1-t)\log g(b)]\\
&\left.+[t\log f(a)+(1-t)\log f(b)][(1-t)\log g(a)+t\log g(b)]\right]+\log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)\\
\leq&\frac{1}{4}\left[\log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{ta+(1-t)b}\right)+\log f\left(\frac{ab}{(1-t)a+tb}\right)\log g\left(\frac{ab}{(1-t)a+b}\right)\right.\\
&+2t(1-t)[\log f(a)\log g(a)+\log f(b)\log g(b)]\\
&\left.+[\log f(a)\log g(b)+\log f(b)\log g(a)][t^2+(1-t)^2]\right]+\log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)
\end{aligned}$$

Integrating over [0, 1], we have

$$\begin{aligned}
&\frac{1}{2}\log f\left(\frac{2ab}{a+b}\right)\int_0^1\left[\log g\left(\frac{ab}{ta+(1-t)b}\right)+\log g\left(\frac{ab}{(1-t)a+tb}\right)\right]\mathrm{d}t\\
&+\frac{1}{2}\log g\left(\frac{2ab}{a+b}\right)\int_0^1\left[\log f\left(\frac{ab}{ta+(1-t)b}\right)+\log f\left(\frac{ab}{(1-t)a+tb}\right)\right]\mathrm{d}t\\
\leq&\frac{1}{4}\left[\int_0^1\log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\right.\\
&+\int_0^1\log f\left(\frac{ab}{(1-t)a+tb}\right)\log g\left(\frac{ab}{(1-t)a+b}\right)\mathrm{d}t\\
&+2[\log f(a)\log g(a)+\log f(b)\log g(b)]\int_0^1 t(1-t)\mathrm{d}t\\
&\left.+[\log f(a)\log g(b)+\log f(b)\log g(a)]\int_0^1[t^2+(1-t)^2]\mathrm{d}t\right]\\
&+\log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)\int_0^1\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{4}\Bigg[\int_0^1 \log f\left(\frac{ab}{ta+(1-t)b}\right)\log g\left(\frac{ab}{ta+(1-t)b}\right)\mathrm{d}t\\
&+\int_0^1 \log f\left(\frac{ab}{(1-t)a+tb}\right)\log g\left(\frac{ab}{(1-t)a+b}\right)\mathrm{d}t\\
&+\frac{1}{3}[\log f(a)\log g(a)+\log f(b)\log g(b)]\\
&+\frac{2}{3}\big[\log f(a)\log g(b)+\log f(b)\log g(a)\big]\Bigg]+\log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)
\end{aligned}
$$

From the above inequality it follows that

$$
\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right)\frac{a^2b^2}{(b-a)^2}\int_a^b\frac{\log g(x)}{x^2}\mathrm{d}x\int_a^b\frac{\log f(x)}{x^2}\mathrm{d}x\\
\le&\exp\left[\frac{ab}{2(b-a)}\int_a^b\frac{\log f(x)\log g(x)}{x^2}\mathrm{d}x+\frac{1}{12}M(a,b)+\frac{1}{6}N(a,b)+\log f\left(\frac{2ab}{a+b}\right)\log g\left(\frac{2ab}{a+b}\right)\right],
\end{aligned}
$$

which is the required result. □

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