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A NEW SCALARIZATION FUNCTION AND WELL-POSEDNESS OF VECTOR OPTIMIZATION PROBLEM

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In this paper, a new scalarization function which is Gerstewitz type and non-linear is introduced. This function can be used to scalarize not only solid but also non-solid optimization problems. A few properties of this newly defined function are established. Two types of well-posedness are considered for a vector optimization problem (V, f) in terms of its weak effi-cient solutions. Using the above function, a scalar problem SOP (V, f) cor-responding to (V, f) is considered. Few characterizations of weak minimal solutions of (V, f) in terms of solutions of SOP (V, f) are obtained. Equiv-alence of well-posedness of (V, f) with that of SOP (V, f) is established. At the end, a characterization of well-posedness of (V, f) with respect to level set is given.

Keywords: Vector optimization, Well-posedness, Non-linear scalarization function, Weak efficiency.

MSC2000: 49K40, 90C29

1. **Introduction**

Most of the optimization problems concerned with real-life require the optimization of several functions simultaneously which are popularly called vector optimization problems. Consider the following vector optimization problem

$$(V, f) \qquad \min f(x)$$
$$\text{subject to} \qquad x \in V$$

where $f : X \to Y$ and X, Y are real normed linear spaces and V is a non-empty subset of X.

As all the functions of a vector optimization problem cannot be optimized simultaneously, there are several concepts of optimality which are available in literature with the help of partial ordering induced by the ordering cones

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in terms of efficient, weak efficient, properly efficient solutions (see [1, 2, 15]). From a mathematical point of view, the concept of weak efficient solution is more tractable than efficient solution; whereas efficiency is more important than weak efficiency for concrete applications.

In literature, different methods have been used to obtain the solutions of (V, f). One of the significant approaches used by many authors is to study a corresponding dual problem and by establishing duality between these two problems and concluding about the optimal efficient solution of the vector optimization problem (V, f), for reference (see [9, 19]). In literature, we also find that many references are available where the (V, f) is studied for its various aspects through establishing a corresponding variational inequality (in short (VI)) for example (see [12, 14]) where the solutions of (V, f) and (VI) have been connected under certain conditions. Another well-known method is a scalarization technique in which an optimization problem is converted into a suitable scalar optimization problem and the solution of the problem under consideration is obtained through its corresponding scalar optimization problem, for reference (see [20, 28]). In this paper we use this technique of scalarization for the problem (V, f). In case of a vector optimization problem (V, f) different types of linear and non-linear scalarization functions have been used for converting it into a scalarization problem. It has been found that, linear scalarization functions are not of much use as they can only be applied to convex problems having convex separation theorems (see [21] and the refrences therein). So, to deal with non-convex optimization problems, various non-linear scalarization functions have been used by many authors (see [4, 7, 11, 16, 22, 23, 29]).

Gerstewitz scalarization function was introduced by Gerstewitz in 1985 to obtain the solutions of a vector optimization problem by converting it to a scalar optimization problem using this function. This non-linear scalarization function is now being frequently used (see [3, 8, 10]). Basic properties of this function have been investigated and can be found in [25, 26, 27].

In literature, it is observed that the scalarization done by using Gerstewitz function can only be applied to solid optimization problems where the interior of the partial ordering cone is non-empty. There also exist problems where this interior could be empty. For example, the ordering cone $S = \{(x, 0) : x \geq 0\}$ has an empty interior. To overcome this shortcoming, we have considered a scalarization function which is Gerstewitz type, non-linear and is also valid for both solid as well as non-solid optimization problems where the interior of the ordering cone could be empty.

The concept of well-posedness was first introduced by Tykhonov ([24]) for an unconstrained scalar minimization problems which says that the problem is Tykhonov well posed if every minimizing sequence converges to the unique solution of the problem. This notion and its generalizations to vector optimization problems have been discussed thoroughly in [5, 6] and the references therein. The concept of well-posedness plays an important role in establishing the stability of an optimization problem.

In [21, 29] the relationship between the well-posedness of a vector optimization problem and the well-posedness of the associated scalar problem is investigated. While converting the vector optimization problem to scalar optimization problem using the appropriate scalarization technique, it has been observed that the properties of the well-posedness remain preserved.

In [4, 18, 29] it is observed that the definitions of well-posedness are applicable only for the vector optimization problems having ordering cone with non-empty interior and hence are valid only for solid optimization problems.

Motivated by the above approaches, in this paper we have relaxed this condition that the interior of the ordering cone is non-empty by defining a Gerstewitz type non-linear scalarization function and used it to scalarize the considered (V, f). Here, the (V, f) can be both solid as well as non-solid optimization problem. Few properties of this newly defined scalarization function have been established which ensures that a solution of the associated scalar optimization problem is a solution to the corresponding vector optimization problem. Two types of well-posedness have been introduced for (V, f) and one of them is applicable to both solid and non-solid optimization problems.

This paper, in four sections is organised as follows: In Section 2 we have given the preliminaries and the basic results required in the sequel. In Section 3 a non-linear scalarization function has been defined and some of its properties have been investigated like convexity, monotonicity and subadditivity. In Section 4 two kinds of well-posedness for vector optimization problem (V, f) are introduced and equivalence between well-posedness of (V, f) and the corresponding scalar optimization problem is obtained. Also, a characterization of well-posedness of (V, f) with respect to level sets is given. Finally, some conclusions are drawn.

2. Preliminaries

Let K be a closed, convex and pointed cone in Y which introduces a partial order on Y as follows: Let $y_1, y_2 \in Y$; we have

$$y_1 \leq_K y_2 \quad \Leftrightarrow \quad y_2 - y_1 \in K;$$
$$y_1 <_K y_2 \quad \Leftrightarrow \quad y_2 - y_1 \in \text{int}K.$$

where int$K$ stands for interior of cone K. Recall that a cone K is solid when interior of cone is non-empty.

Let $\text{int}K \neq \emptyset$. In [15], an element $t \in T$ is a weak minimal element of $T \subseteq Y$ if

$$(T - t) \cap (-\text{int}K) = \emptyset.$$

The set of weakly minimal elements of T is denoted by $\text{WMin}_K T$.

Using the above minimality notion of a set, we define the corresponding minimality notion of problem (V, f) with respect to the set $f(V)$ in Y.

A point $y \in f(V)$ is called a weak minimal solution of (V, f) if $y \in \text{WMin}_K f(V)$.

A point $x \in V$ is called an weak efficient solution of (V, f) if $f(x) \in \mathrm{WMin}_K f(V)$. The set of all the weak efficient solutions of problem (V, f) is given by

$$\mathrm{WEff}_K(f, V) := \{x \in V : (f(V) - f(x)) \cap (-\mathrm{int}K) = \emptyset\}.$$

Definition 2.1. *A vector-valued function $f : A \subseteq X \to Y$ where, A is a non-empty convex subset of X is said to be*
(i) *K-convex, if for all $x, y \in A,\ t \in [0, 1]$*

$$f(tx + (1-t)y) \leq_K tf(x) + (1-t)f(y).$$

(ii) *strictly K-convex, if int$K \neq \emptyset$ and for all $x, y \in A, x \neq y,\ t \in]0, 1[$*

$$f(tx + (1-t)y) <_K tf(x) + (1-t)f(y).$$

Definition 2.2. *The level set of f at a point $y \in Y$ is denoted by $\mathcal{L}(y)$ and is given by $\mathcal{L}(y) = \{x \in V : f(x) \leq_K y\}$.*

Remark 2.1. [8] For a function $g : X \to \mathbb{R} \cup \{\pm\infty\}$ the epigraph and domain are defined as

$$\mathrm{epi}\, g := \{(x, p) \in X \times \mathbb{R} : g(x) \leq p\} \text{ and } \mathrm{dom}\, g := \{x \in X : g(x) < +\infty\}.$$

The function g is said to be proper if $\mathrm{dom}\, g \neq \emptyset$ and $g > -\infty$ and convex if $\mathrm{epi}\, g$ is a convex set.

Consider a scalar problem

$$\begin{array}{cc} (S, h) & \min h(x) \\ \text{subject to} & x \in S, \end{array}$$

where $h : X \to \mathbb{R}$ and S is a non-empty subset of X. The set of all solutions of (S, h) is denoted by $\mathrm{argmin}(S, h)$.

Definition 2.3. [13] *The function h is said to be*
(i) *monotonically increasing on S, if for every $\overline{a} \in S$*

$$a \in S,\ \ \overline{a} \leq_K a \ \Rightarrow\ h(\overline{a}) \leq h(a).$$

(ii) *strictly monotonically increasing on S, if for every $\overline{a} \in S$*

$$a \in S \setminus \{\overline{a}\},\ \ \overline{a} <_K a \ \Rightarrow\ h(\overline{a}) < h(a).$$

(iii) *strongly monotonically increasing on S, if for every $\overline{a} \in S$*

$$a \in S,\ \ a \neq \overline{a},\ \ \overline{a} \leq_K a \ \Rightarrow\ h(\overline{a}) < h(a).$$

Recall that a sequence $\{x_k\}$ in S is called minimizing sequence for the problem (S, h) if $h(x_k) \to h(x_0)$ where $h(x_0) = \inf_{x \in S} h(x)$.

Definition 2.4. *The problem (S, h) is said to be Tykhonov well-posed if and only if*
(i) *it has a unique solution,*
(ii) *each minimizing sequence converges to argmin(S, h).*

Proposition 2.1. [4] *Let S be a convex set and $h : S \subseteq X \to \mathbb{R}$ be a convex function. If f has a unique global minimum point over S, then (S, h) is Tykhonov well-posed.*

Definition 2.5. *The problem (S, h) is said to be well-posed in the generalized sense if and only if*
(i) *argmin$(S, h) \neq \emptyset$,*
(ii) *for each minimizing sequence $\{x_k\}$, there exists a subsequence $\{x_{k_n}\}$ converging to some point of argmin(S, h).*

Remark 2.2. *If argmin(S, h) is singleton then (S, h) is Tykhonov well-posed if and only if it is well-posed in the generalised sense.*

For vector optimization problems several techniques can be used to check the well-posedness. One of such technique is scalarization. Many authors used Gerstewitz function to get scalarization in terms of well-posedness. In 1985, Gerstewitz introduced the scalarization function $\psi_M := \psi_{M,l^0} : Y \to \overline{\mathbb{R}}$,

$$\psi_{M,l^0}(y) := \inf\{t \in \mathbb{R} : y \in tl^0 + M\} \tag{1}$$

where M is a non-empty subset and $l^0 \neq \emptyset$ is an element of a real linear space for vector optimization. In [7] the properties of the function (1) are derived. Luc [17] had given the following scalarizing function for vectors, $y \in Y$

$$\chi(y) := \inf\{t \in \mathbb{R} : y \in tk^0 + N - \text{int}C\} \tag{2}$$

where C is a solid closed convex cone and N is a subset of a topological vector space and $k^0 \in C$. The scalarizing function in (2) is not applicable for the non-solid problems. To overcome this, we modified this function for both solid and non-solid problems.

3. Gerstewitz's Type Scalarization Function

In this section, we introduce a non-linear Gerstewitz type scalarizing function for (V, f) and establish certain properties of this function like convexity and monotonicity. Throughout this paper, $\overline{B}_0$ denotes a closed unit ball with unit radius and center at origin and $\mathbb{R}^+$ represents the set of non-negative real numbers. Let A be any nonempty set in Y.

Definition 3.1. *Let $a \in A$ be a fixed point. We define the non-linear scalar function $\varphi_{\overline{B}_0,a} : Y \to \mathbb{R}^+$ for every $y \in Y$ as*

$$\varphi_{\overline{B}_0,a}(y) = \inf\{t \in \mathbb{R}^+ : y \in t\overline{B}_0 + a - K\}.$$

The function $\varphi_{\overline{B}_0,a}$ is proper and well-defined.

Now, we give an example to illustrate Definition 3.1.

Example 3.1. *Let $Y = \mathbb{R}^2, K = \{(x, y) : x \leq 0, y \leq 0\}$ and $A = \{(0, y) : y \in \mathbb{R}^+\}$. Take $a = (0, 1) \in A$. Then, for $y = (1, \frac{1}{2})$, $\varphi_{\overline{B}_0,(0,1)}((1, \frac{1}{2})) = \inf\{t \in \mathbb{R}^+ : (1, \frac{1}{2}) \in t\overline{B}_0 + (0, 1) - K\} = \inf\{t \in \mathbb{R}^+ : (1, \frac{1}{2}) \in \overline{B}((0, 1), t) - K\}$.*

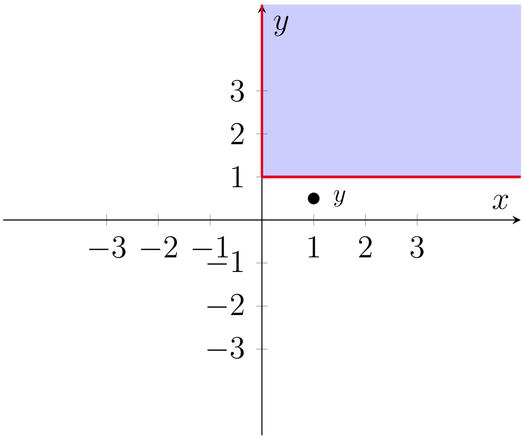


Figure 1: $\varphi_{\overline{B_0},(0,1)}((1,\frac{1}{2}))$ for $t=0$

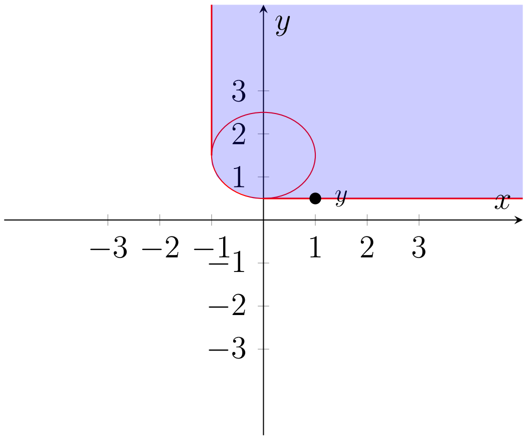


Figure 2: $\varphi_{\overline{B_0},(0,1)}((1,\frac{1}{2}))$ for $t=\frac{1}{2}$

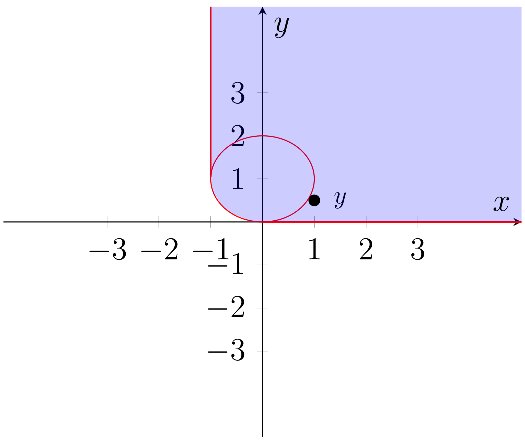


Figure 3: $\varphi_{\overline{B_0},(0,1)}((1,\frac{1}{2}))$ for $t=1$

As illustrated in Figures, clearly for $t<\frac{1}{2}$ *the point* $y=(1,\frac{1}{2})$ *does not belong to the set* $\overline{B}((0,1),t)-K$ *but for* $t\geq\frac{1}{2}$ *the point* $y=(1,\frac{1}{2})$ *belongs to the set* $\overline{B}((0,1),t)-K$*. Therefore,* $\varphi_{\overline{B}_0,(0,1)}((1,\frac{1}{2}))=\frac{1}{2}$*. Similarly, it can be checked that* $\varphi_{\overline{B}_0,(0,1)}((1,1))=0$.

Proposition 3.1. *Let* $\epsilon>0$*. For every* $a\in A, y\in Y$ *and* $\lambda\in\mathbb{R}^+$*, we have the following*

(i) $\varphi_{\overline{B}_0,a}(y)\leq\lambda\Leftrightarrow y\in(\lambda+\epsilon)\overline{B}_0+a-K$;

(ii) $\varphi_{\overline{B}_0,a}(y)=\lambda\Rightarrow y\in(\lambda+\epsilon)\overline{B}_0+a-K$;

(iii) *For* $\lambda\neq 0$, $\varphi_{\overline{B}_0,a}(y)<\lambda\Leftarrow y\in\lambda\overline{B}_0+a-intK$;

(iv) $\varphi_{\overline{B}_0,a}(y)>\lambda\Leftrightarrow y\notin(\lambda+\epsilon)\overline{B}_0+a-K$;

(v) $y\notin(\lambda+\epsilon)\overline{B}_0+a-K\Rightarrow\varphi_{\overline{B}_0,a}(y)\neq\lambda$.

Proof. Let $S=\{t\in\mathbb{R}^+ : y\in t\overline{B}_0+a-K\}$. So, $\varphi_{\overline{B}_0,a}(y)=\inf S=\mu$ (say).

(i) Firstly, let $\varphi_{\overline{B}_0,a}(y) \leq \lambda$ i.e. $\mu \leq \lambda$.
Then, two cases arise:
Case (i) $\mu < \lambda$.
Then, there exists $t \in S$ such that $t < \lambda$ which gives

$$y \in t\overline{B}_0 + a - K \subset \lambda\overline{B}_0 + a - K \subseteq (\lambda + \epsilon)\overline{B}_0 + a - K \quad \text{for every} \quad \epsilon > 0.$$

Case (ii) $\mu = \lambda$.
Then, for every $\epsilon > 0$ there exists $t \in S$ such that $t < (\lambda + \epsilon)$ which gives

$$y \in t\overline{B}_0 + a - K \subseteq (\lambda + \epsilon)\overline{B}_0 + a - K \quad \text{for every} \quad \epsilon > 0.$$

Conversly, let $y \in (\lambda + \epsilon)\overline{B}_0 + a - K$ for every $\epsilon > 0$.
This implies, $\inf S \leq (\lambda + \epsilon)$ for every $\epsilon > 0$. Hence, $\varphi_{\overline{B}_0,a}(y) \leq \lambda$.

(ii) Firstly, let $\varphi_{\overline{B}_0,a}(y) = \lambda$ which gives $\mu = \lambda$. Then, it follows from (i).

(iii) For $\lambda \neq 0$. Let $y \in \lambda\overline{B}_0 + a - \text{int}K$.
Then there exists some $\epsilon > 0$ such that

$$y \in (\lambda - \epsilon)\overline{B}_0 + a - \text{int}K \subset (\lambda - \epsilon)\overline{B}_0 + a - K,$$

which gives $\varphi_{\overline{B}_0,a}(y) \leq \lambda - \epsilon < \lambda$.

(iv) The assertion follows from (i).

(v) The assertion follows from (ii).

□

Remark 3.1. The converse of the part *(ii)* of Proposition 3.1 need not be true. This remark is illustrated in the following example.

Example 3.2. *Let* $Y = \mathbb{R}^2, a = (1, 1), y = (-1, -1)$ *and* $K = \{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0\}$. *If we choose* $\lambda = 4$ *then clearly* $y \in (\lambda + \epsilon)\overline{B}_0 + a - K$ *for any* $\epsilon > 0$. *But it can be seen easily that* $\varphi_{\overline{B}_0,a}(y) = 2\sqrt{2} \neq 4 = \lambda$.

Remark 3.2. In particular, for $\lambda = 0$ and for every $\epsilon > 0$ in Proposition 3.1 we get,

$$\varphi_{\overline{B}_0,a}(y) = 0 \Leftrightarrow y \in \epsilon\overline{B}_0 + a - K.$$

Proposition 3.2. *For any* $a \in A$ *and* $y_1, y_2 \in Y$ *the function* $\varphi_{\overline{B}_0,a}$ *has the following properties:*

(i) $\varphi_{\overline{B}_0,a}$ *is monotonically increasing.*

$$\textit{i.e.,} \quad y_1 \leq_K y_2 \ \textit{ implies } \ \varphi_{\overline{B}_0,a}(y_1) \leq \varphi_{\overline{B}_0,a}(y_2).$$

(ii) $\varphi_{\overline{B}_0,a}$ *is convex.*

$$\textit{i.e., for} \ 0 \leq \lambda \leq 1, \ \varphi_{\overline{B}_0,a}(\lambda y_1 + (1 - \lambda)y_2) \leq \lambda\varphi_{\overline{B}_0,a}(y_1) + (1 - \lambda)\varphi_{\overline{B}_0,a}(y_2).$$

(iii) *If* $\varphi_{\overline{B}_0,0}(y_1) \leq t_1$ *and* $\varphi_{\overline{B}_0,0}(y_2) \leq t_2$. *Then,* $\varphi_{\overline{B}_0,0}(y_1 + y_2) \leq t_1 + t_2$.

(iv) $\varphi_{\overline{B}_0,0}$ *is positively homogeneous.*

$$\textit{i.e., for} \ \alpha > 0, \ \varphi_{\overline{B}_0,0}(\alpha y) = \alpha\varphi_{\overline{B}_0,0}(y).$$

(v) $\varphi_{\overline{B}_0,0}$ *is subadditive.*

$$i.e., \quad \varphi_{\overline{B}_0,0}(y_1+y_2) \leq \varphi_{\overline{B}_0,0}(y_1) + \varphi_{\overline{B}_0,0}(y_2).$$

(vi) *For* $intK \neq \emptyset$. $\varphi_{\overline{B}_0,a}$ *is strictly monotone.*

$$i.e., \quad y_1 \geq_{intK} y_2 \quad implies \quad \varphi_{\overline{B}_0,a}(y_1) > \varphi_{\overline{B}_0,a}(y_2).$$

Proof. (i) Let $y_1 \in y_2 - K$ for any $y_1, y_2 \in Y$.
Let $S_1 = \{t_1 \in \mathbb{R}^+ : y_1 \in t_1\overline{B}_0 + a - K\}$ and $S_2 = \{t_2 \in \mathbb{R}^+ : y_2 \in t_2\overline{B}_0 + a - K\}$.
For $t \in S_2$, we have $y_2 \in t\overline{B}_0 + a - K$. Thus, $y_1 \in t\overline{B}_0 + a - K$ gives $t \in S_1$.
Therefore, $\inf S_1 \leq \inf S_2$. Hence, $\varphi_{\overline{B}_0,a}(y_1) \leq \varphi_{\overline{B}_0,a}(y_2)$.

(ii) To prove the convexity of $\varphi_{\overline{B}_0,a}$, we will show that epi$\varphi_{\overline{B}_0,a}$ is convex.
Let $(y_1,t_1),(y_2,t_2) \in$ epi$\varphi_{\overline{B}_0,a}(y)$ and let $0 \leq \lambda \leq 1$.
$\varphi_{\overline{B}_0,a}(y_1) \leq t_1$ and $\varphi_{\overline{B}_0,a}(y_2) \leq t_2$ gives $y_1 \in (t_1+\epsilon_1)\overline{B}_0 + a - K$ and $y_2 \in (t_2+\epsilon_2)\overline{B}_0 + a - K$, for $\epsilon_1, \epsilon_2 > 0$. Choose $\epsilon = \max\{\epsilon_1, \epsilon_2\}$.
Therefore, $y_1 \in (t_1+\epsilon)\overline{B}_0 + a - K$ and $y_2 \in (t_2+\epsilon)\overline{B}_0 + a - K$, for $\epsilon > 0$.
Thus, $\lambda y_1 + (1-\lambda)y_2 \in (\lambda t_1 + (1-\lambda)t_2 + \epsilon)\overline{B}_0 + a - K$, for $\epsilon > 0$.
Since above equation is true for every $\epsilon > 0$. In Particular, choose $\epsilon = \frac{1}{n}$.
Thus, we get $\lambda y_1 + (1-\lambda)y_2 \in (\lambda t_1 + (1-\lambda)t_2 + \frac{1}{n})\overline{B}_0 + a - K$.
We can always find a sequence $\{x_n\} \in \frac{1}{n}\overline{B}_0$, for every n such that $\{x_n\} \to 0$ as $n \to \infty$.
Therefore, $\lambda y_1 + (1-\lambda)y_2 \in (\lambda t_1 + (1-\lambda)t_2)\overline{B}_0 + a - K$.
Hence, $\varphi_{\overline{B}_0,a}(\lambda y_1 + (1-\lambda)y_2) \leq \lambda t_1 + (1-\lambda)t_2$.

(iii) $\varphi_{\overline{B}_0,0}(y_1) \leq t_1$ and $\varphi_{\overline{B}_0,0}(y_2) \leq t_2$ gives $y_1 \in (t_1+\epsilon_1)\overline{B}_0 - K$ and $y_2 \in (t_2+\epsilon_2)\overline{B}_0 - K$, for $\epsilon_1, \epsilon_2 > 0$. Choose $\epsilon = \max\{\epsilon_1, \epsilon_2\}$.
Therefore, $y_1 \in (t_1+\epsilon)\overline{B}_0 - K$ and $y_2 \in (t_2+\epsilon)\overline{B}_0 - K$, for $\epsilon > 0$.
On adding, we get $y_1 + y_2 \in (t_1+t_2+2\epsilon)\overline{B}_0 - K$, for $\epsilon > 0$, which implies $y_1 + y_2 \in (t_1+t_2)\overline{B}_0 - K$.
Thus, $\inf\{t \in \mathbb{R}^+ : y_1 + y_2 \in t\overline{B}_0 - K\} \leq t_1 + t_2$. Hence, $\varphi_{\overline{B}_0,0}(y_1+y_2) \leq t_1 + t_2$.

(iv) For $\alpha > 0$. Consider,

$$\begin{aligned}\varphi_{\overline{B}_0,0}(\alpha y) &= \inf\{t \in \mathbb{R}^+ : \alpha y \in t\overline{B}_0 - K\} \\ &= \alpha \inf\{\frac{t}{\alpha} \in \mathbb{R}^+ : y \in \frac{t}{\alpha}\overline{B}_0 - K\} \\ &= \alpha \inf\{\lambda \in \mathbb{R}^+ : y \in \lambda\overline{B}_0 - K\} \\ &= \alpha\, \varphi_{\overline{B}_0,0}(y).\end{aligned}$$

(v) Let $\varphi_{\overline{B}_0,0}(y_1) = t_1$ and $\varphi_{\overline{B}_0,0}(y_2) = t_2$ gives $y_1 \in (t_1+\epsilon_1)\overline{B}_0 - K$ and $y_2 \in (t_2+\epsilon_2)\overline{B}_0 - K$ for $\epsilon_1, \epsilon_2 > 0$. Choose $\epsilon = \max\{\epsilon_1, \epsilon_2\}$.
Therefore, $y_1 \in (t_1+\epsilon)\overline{B}_0 - K$ and $y_2 \in (t_2+\epsilon)\overline{B}_0 - K$ for $\epsilon > 0$.
Thus, $y_1 + y_2 \in (t_1+t_2+2\epsilon)\overline{B}_0 - K$ for $\epsilon > 0$.

Hence, $\inf\{t \in \mathbb{R}^+ : y_1 + y_2 \in t\overline{B}_0 - K\} \leq t_1 + t_2$ gives $\varphi_{\overline{B}_0,0}(y_1 + y_2) \leq t_1 + t_2 = \varphi_{\overline{B}_0,0}(y_1) + \varphi_{\overline{B}_0,0}(y_2)$.

(vi) Let $y_1 \in y_2 + \text{int}K$. Set $r = \varphi_{\overline{B}_0,a}(y_1)$. As

$$\begin{aligned} y_2 &\in y_1 - \text{int}K \\ &\subseteq r\overline{B}_0 + a - K - \text{int}K \\ &\subseteq r\overline{B}_0 + a - \text{int}K. \end{aligned}$$

Therefore, by Proposition 3.1 (v), we have $\varphi_{\overline{B}_0,a}(y_2) < r = \varphi_{\overline{B}_0,a}(y_1)$.

□

Note 1. *We note that the function $\varphi_{\overline{B}_0,a}(y)$ is not strongly monotone. It is for this reason that the function $\varphi_{\overline{B}_0,a}(y)$ is more useful in dealing with weak minimal points.*

Example 3.3. *Let $Y = \mathbb{R}^2, K = \{(x, y) : x \leq 0, y \leq 0\}$ and $A = \{(0, y) : y \in \mathbb{R}^+\}$. Take $a = (0, 0) \in A$. Therefore, for $y_1 = (0, 4)$ and $y_2 = (0, 6)$ we have $y_1 - y_2 \in K \setminus \{(0, 0)\}$ but $\varphi_{\overline{B}_0,(0,0)}(y_1) = 0 = \varphi_{\overline{B}_0,(0,0)}(y_2)$. Hence, the function $\varphi_{\overline{B}_0,a}(y)$ is not strongly monotone.*

4. Well-posedness of (V, f)

In this section, we introduce two types of well-posedness for vector optimization problem (V, f) and establish an equivalence between well-posedness of vector optimization problem (V, f) and well-posedness of scalar optimization problem which is given using Gerstewitz's type scalarization function defined in previous section. Throughout this section, we assume that $\text{WEff}_K(f, V)$ is non-empty set.

Consider the following scalar optimization problem

$$SOP\,(V, f) \qquad \min(\varphi_{\overline{B}_0,f(x_0)} \circ f)(x) \qquad \text{subject to} \qquad x \in V,$$

where $(\varphi_{\overline{B}_0,f(x_0)} \circ f)(x) = \inf\{t \in \mathbb{R}^+ : f(x) \in t\overline{B}_0 + f(x_0) - K\}$, for all $x \in V$ and $x_0 \in \text{WEff}_K(f, V)$. For the sake of convenience, throughout this section we denote $(\varphi_{\overline{B}_0,f(x_0)} \circ f)(x)$ by $(\varphi_{f(x_0)} \circ f)(x)$ and the solutions of $SOP(V, f)$ is denoted by $\text{argmin}(V, \varphi_{f(x_0)} \circ f)$.

Remark 4.1. One may observe that, $(\varphi_{f(x_0)} \circ f)(x) \geq 0$ for all $x \in V$ and the optimal value is always zero. Also, $x_0 \in \text{argmin}(V, \varphi_{f(x_0)} \circ f)$.

Theorem 4.1. *For any $x_0 \in WEff_K(f, V)$.*

$$argmin(V, \varphi_{f(x_0)} \circ f) \subseteq WEff_K(f, V).$$

Proof. Let $\overline{x} \in \text{argmin}(V, \varphi_{f(x_0)} \circ f)$, then

$$(\varphi_{f(x_0)} \circ f)(\overline{x}) \leq (\varphi_{f(x_0)} \circ f)(x), \quad \forall x \in V.$$

Since, $\varphi_{f(x_0)}$ is a strictly monotone function, we deduce that

$$f(x) - f(\overline{x}) \notin -\mathrm{int}K, \quad \forall x \in V.$$

This implies that $\overline{x} \in \mathrm{WEff}_K(f, V)$. So,

$$\mathrm{argmin}(V, \varphi_{f(x_0)} \circ f) \subseteq \mathrm{WEff}_K(f, V).$$

□

It can be easily varified that every weak efficient solution of (V, f) is an optimal solution of $SOP(V, f)$.

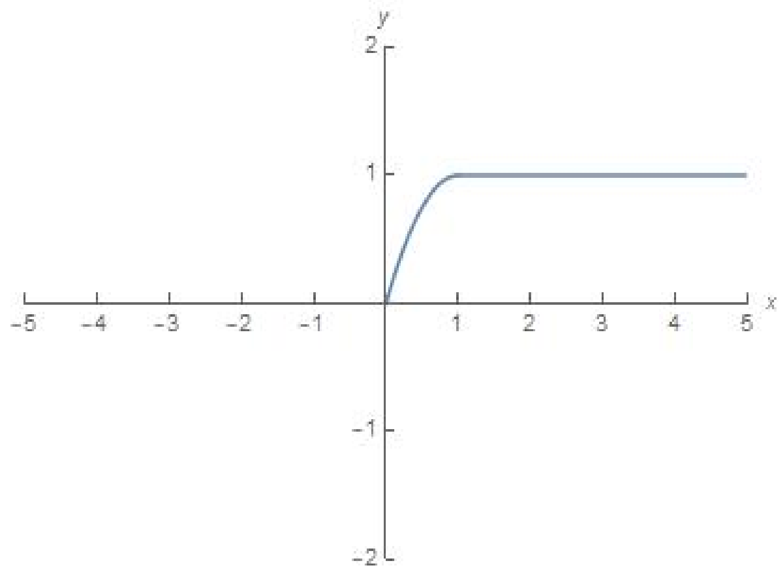
Theorem 4.2. $\bigcup_{x_0 \in WEff_K(f,V)}$ *argmin*$(V, \varphi_{f(x_0)} \circ f) =$ *WEff*$_K(f, V)$.

Theorem 4.2 is illustrated with the following example.

Example 4.1. *Let* $X = \mathbb{R}, Y = \mathbb{R}^2, V = [0, \infty)$ *and* $K = \{(x, y) : x \geq 0, y \geq 0\}$. *Define* $f : X \to Y$ *by*

$$f(x) = \begin{cases} (x, x(2-x)), & \text{if } 0 \leq x \leq 1, \\ (x, 1) & \text{if } x \geq 1 \end{cases}$$

It can be easily seen from the figure that for $x = 0, (f(V) - f(0)) \cap (-intK) = \emptyset$ *and for any* $x \neq 0, (f(V) - f(x)) \cap (-intK) \neq \emptyset$. *Therefore,* *WEff*$_K(f, V) = \{0\}$.



Graphical representation of $f(x)$

Now,

$$
\begin{aligned}
(\varphi_{\overline{B_0},f(x_0)} \circ f)(x) &= (\varphi_{\overline{B_0},(0,0)} \circ f)(x) \\
&= \inf\{t \in \mathbb{R}^+ : f(x) \in t\overline{B}_0 + (0,0) - K\} \\
&= \inf\{t \in \mathbb{R}^+ : f(x) \in t\overline{B}_0 - K\}.
\end{aligned}
$$

We observe that the function $(\varphi_{\overline{B_0},(0,0)} \circ f)(x)$ is nothing but the distance of each point of V from the origin. So,

$$
(\varphi_{\overline{B_0},(0,0)} \circ f)(x) = \begin{cases} x\sqrt{x^2 - 4x + 5}, & \text{if } 0 \le x \le 1, \\ \sqrt{x^2 + 1}, & \text{if } x \ge 1. \end{cases}
$$

Also, argmin$(V, \varphi_{(0,0)} \circ f) = \{0\}$. Hence, WEff$_K(f, V)$ = argmin$(V, \varphi_{(0,0)} \circ f)$.

In the following proposition, we show that some properties of function f are inherited to the function $\varphi_{f(x_0)} \circ f$.

Proposition 4.1. *Let $V \subseteq X$ be a convex set. The following holds:*

(i) *If f is K-convex on V then $\varphi_{f(x_0)} \circ f$ is convex for all $x_0 \in$ WEff$_K(f, V)$.*

(ii) *If f is strictly K-convex on V then $\varphi_{f(x_0)} \circ f$ is strictly convex for all $x_0 \in$ WEff$_K(f, V)$.*

Proof. (i) Let $x_1, x_2 \in V$ and $0 \le \mu \le 1$ be given. Since, f is K-convex, we have

$$
f(\mu x_1 + (1-\mu)x_2) \le_K \mu f(x_1) + (1-\mu)f(x_2).
$$

By the monotonicity and convexity of $\varphi_{f(x_0)}$, we have

$$
\begin{aligned}
\varphi_{f(x_0)} \circ f(\mu x_1 + (1-\mu)x_2) &\le_K \varphi_{f(x_0)}(\mu f(x_1) + (1-\mu)f(x_2)) \\
&\le_K \mu\, \varphi_{f(x_0)} \circ f(x_1) + (1-\mu)\varphi_{f(x_0)} \circ f(x_2).
\end{aligned}
$$

Hence, $\varphi_{f(x_0)} \circ f$ is convex for all $x_0 \in \mathrm{WEff}_K(f, V)$.

(ii) It follows by using the same reasoning as part (i).

□

We now give the definitions of well-posedness for (V, f).

Definition 4.1. *Let $x_0 \in$ WEff$_K(f, V)$.*
A sequence $\{x_n\}$ in V is called $\overline{B}_0$-minimizing sequence at x_0 to the problem (V, f) if for every positive integer n

(i) *there exists a sequence $\{t_n\}$, $t_n \ge 0$ and $t_n \to 0$,*

(ii) *$f(x_n) \in t_n\overline{B_0} + f(x_0) - K$.*

Definition 4.2. *The problem (V, f) is said to be $\overline{B}_0$-well-posed at $x_0 \in$ WEff$_K(f, V)$ if for each $\overline{B}_0$-minimizing sequence $\{x_n\}$ at x_0, we have $x_n \to x_0$.*

Definition 4.3. *The problem (V, f) is said to be L-$\overline{B}_0$-well-posed at $x_0 \in$ WEff$_K(f, V)$ if for each $\overline{B}_0$-minimizing sequence $\{x_n\}$ at x_0 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, \mathrm{WEff}_K(f, V)) \to 0$.*

The following example gives illustration of the above definitions.

Example 4.2. *Let $X = \mathbb{R}, Y = \mathbb{R}^2, V = [0, \infty)$ and $K = \{(x, y) : x \geq 0, y \geq 0\}$. Define $f : X \to Y$ by*

$$f(x) = \begin{cases} (x, 0), & \text{if } 0 \leq x \leq 1, \\ (x, x-1), & \text{if } x > 1. \end{cases}$$

Here, $WMin_K f(V) = \{(x, 0) : 0 \leq x \leq 1\}$ and $WEff_K(f, V) = [0, 1]$. It can be easily checked that the (V, f) is $\overline{B}_0$-well-posed as well as L-$\overline{B}_0$-well-posed at $x_0 = 0$.

Example 4.3. *Let $X = \mathbb{R}, Y = \mathbb{R}^2$ and $K = \{(x, y) : x \geq 0, y \geq 0\}$.*

(i) *Let $V = [0, 2]$ and define $f : X \to Y$ by*

$$f(x) = \begin{cases} (x, x^3), & \text{if } 0 \leq x \leq 1, \\ (x, x), & \text{if } 1 \leq x \leq 2. \end{cases}$$

Here, $WMin_K f(V) = \{(0, 0)\}$ and $WEff_K(f, V) = \{0\}$.

(ii) *Let $V = [-1, 2]$ and define $f : X \to Y$ by*

$$f(x) = \begin{cases} (x, x+1), & \text{if } -1 \leq x \leq 1, \\ (x, 2), & \text{if } 1 \leq x \leq 2. \end{cases}$$

Here, $WMin_K f(V) = \{(-1, 0)\}$ and $WEff_K(f, V) = \{-1\}$.
It can be observed that the (V, f) is neither $\overline{B}_0$-well-posed nor L-$\overline{B}_0$-well-posed.

In next theorem, the relation between $\overline{B}_0$-well-posedness and L-$\overline{B}_0$-well-posedness is given.

Theorem 4.3. *Let $x_0 \in WEff_K(f, V)$. If the problem (V, f) is $\overline{B}_0$-well-posed at x_0 then it is L-$\overline{B}_0$-well-posed at x_0.*

Proof. The proof follows from the definition.

Remark 4.2. The converse of above theorem need not be always true.

The following example shows that the converse of Theorem 4.3 is not always true.

Example 4.4. *Let $V = [-1, 1]$ and define $f : \mathbb{R} \to \mathbb{R}^2$ by*

$$f(x) = \begin{cases} (x, -(x^2 - 1)), & \text{if } -1 \leq x \leq 0, \\ (x, 1), & \text{if } 0 \leq x \leq 1. \end{cases}$$

We observe that when $K = \{(x, y) : x \leq 0, y \leq 0\}$.

$$WMin_K f(V) = \{(x, 1) : 0 \leq x \leq 1\}$$

and

$$WEff_K(f, V) = [0, 1].$$

It can be seen that the (V, f) *is* L-$\overline{B}_0$*-well-posed but not* $\overline{B}_0$*-well-posed at* $x_0 = 1$.

The following theorems characterizes $\overline{B}_0$ and L-$\overline{B}_0$-well-posedness of (V, f) in terms of well-posedness of $SOP(V, f)$.

Theorem 4.4. *Suppose that* $x_0 \in WEff_K(f, V)$*. The problem* (V, f) *is* $\overline{B}_0$*-well-posed at* x_0 *if and only if the problem* $SOP(V, f)$ *is Tykhonov well-posed.*

Proof. Suppose that, (V, f) is $\overline{B}_0$-well-posed at x_0 and $\{x_n\}$ be a sequence in V such that $(\varphi_{f(x_0)} \circ f)(x_n) \to \inf_{x \in V} (\varphi_{f(x_0)} \circ f)(x) = 0$.
Let $t_n = (\varphi_{f(x_0)} \circ f)(x_n) \geq 0, t_n \to 0$. Define $\epsilon_n = t_n + 1/n$, then $\epsilon_n \geq 0, \epsilon_n \to 0$ and $f(x_n) \in \epsilon_n \overline{B_0} + f(x_0) - K$.
This implies that $\{x_n\}$ is a $\overline{B}_0$-minimizing sequence at x_0 and hence $x_n \to x_0 \in \mathrm{WEff}_K(f, V) \subseteq \bigcup_{x_0 \in \mathrm{WEff}_K(f,V)} \mathrm{argmin}(V, \varphi_{f(x_0)} \circ f)$.
To verify that x_0 is the unique minimizer of the scalar problem $SOP(V, f)$. Let us assume that y_0 is the minimizer of $SOP(V, f)$ such that $(\varphi_{f(x_0)} \circ f)(y_0) = 0$.
This implies $f(y_0) \in \epsilon_n \overline{B_0} + f(x_0) - K$, for every $\epsilon_n > 0$.
Take $y_0 = x_n$ and $\epsilon_n = 1/n \geq 0, \epsilon_n \to 0$. Therefore, $f(x_n) \in \epsilon_n \overline{B_0} + f(y_0) - K$.
Thus, $\{x_n\}$ is a $\overline{B}_0$-minimizing sequence and therefore $\{x_n\} \to y_0$ but $\{x_n\} \to x_0$, which is a contradiction.
Conversly, let $SOP(V, f)$ is Tykhonov well-posed. Therefore, $SOP(V, f)$ has unique global minimum. Let $\mathrm{argmin}(V, \varphi_{f(x_0)} \circ f) = \{x_0\}$.
Let $\{x_n\} \subseteq V$ be a $\overline{B}_0$-minimizing sequence of (V, f) at x_0. Then for any positive integer there exists a sequence $\{t_n\}$, $t_n \geq 0, t_n \to 0$ and $f(x_n) \in t_n \overline{B_0} + f(x_0) - K$.
Therefore, we get $0 \leq (\varphi_{f(x_0)} \circ f)(x_n) \leq t_n$ which gives $(\varphi_{f(x_0)} \circ f)(x_n) \to 0 = \inf_{x \in V} (\varphi_{f(x_0)} \circ f)(x)$. Thus, $x_n \to \mathrm{argmin}(V, \varphi_{f(x_0)} \circ f) = x_0$. Hence, (V, f) is $\overline{B}_0$-well-posed.

Theorem 4.5. *Suppose that* $x_0 \in WEff_K(f, V)$*. The following statements hold:*

(i) *If* (V, f) *is* L-$\overline{B}_0$*-well-posed at* x_0 *then* $SOP(V, f)$ *is generalised well-posed provided* $WEff_K(f, V)$ *is a compact set.*
(ii) *If* $SOP(V, f)$ *is generalised well-posed then* (V, f) *is* L-$\overline{B}_0$*-well-posed at* x_0*.*

Proof. (i) Suppose that the problem (V, f) is L-$\overline{B}_0$-well-posed at x_0 and $\{x_n\}$ is a sequence in V such that $(\varphi_{f(x_0)} \circ f)(x_n) \to \inf_{x \in V} (\varphi_{f(x_0)} \circ f)(x) = 0$. Thus, $\{x_n\}$ is a $\overline{B}_0$-minimizing sequence at x_0. So, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, \mathrm{WEff}_K(f, V)) \to 0$. Using the compactness of $\mathrm{WEff}_K(f, V)$ there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\} \to \overline{x} \in \mathrm{WEff}_K(f, V) \subseteq \bigcup_{x_0 \in \mathrm{WEff}_K(f,V)} \mathrm{argmin}(V, \varphi_{f(x_0)} \circ f)$. Hence, $SOP(V, f)$ is generalized well-posed.

(ii) Suppose that the problem $SOP(V, f)$ is generalized well-posed. Let $\{x_n\} \subseteq V$ be a sequence such that there exists $\{t_n\}, t_n \geq 0, t_n \to 0$ and $f(x_n) \in t_n\overline{B_0} + f(x_0) - K$.
Therefore we get, $0 \leq (\varphi_{f(x_0)} \circ f)(x_n) \leq t_n$. This implies $\varphi_{f(x_0)}(f(x_n)) \to 0 = \inf_{x\in V} \varphi_{f(x_0)}(f(x_n))$ which gives that $\{x_n\}$ is a minimizing sequence of $SOP(V, f)$. So, there exists a subsequence $\{x_{n_k}\} \to \overline{x} \in \operatorname{argmin}(V, \varphi_{f(x_0)} \circ f) \subseteq \mathrm{WEff}_K(f, V)$. Therefore, $d(x_{n_k}, \mathrm{WEff}_K(f, V)) \to 0$. Hence, (V, f) is L-$\overline{B}_0$-well-posed at x_0.

□

Let $\mathcal{L}(x_0, \beta)$ be the level set at $x_0 \in \mathrm{WEff}_K(f, V)$ with level $\beta \in \mathbb{R}$ where

$$\mathcal{L}(x_0, \beta) = \{x \in V : f(x) \leq_K \beta\overline{B}_0 + f(x_0)\}.$$

We now give the characterization of $\overline{B}_0$-well-posedness of (V, f) in terms of level sets $\mathcal{L}(x_0, \beta)$.

Theorem 4.6. *Suppose that $x_0 \in WEff_K(f, V)$. The problem (V, f) is $\overline{B}_0$-well-posed at x_0 if and only if $\inf_{\beta\geq 0}$ diam $\mathcal{L}(x_0, \beta) = 0$.*

Proof. Suppose that the problem (V, f) is $\overline{B}_0$-well-posed at x_0 and $\inf_{\beta\geq 0} \operatorname{diam} \mathcal{L}(x_0, \beta) \neq 0$. Then there exists $\alpha > 0$ such that $\mathcal{L}(x_0, \beta_n) \geq \alpha$ for all n where $\beta_n = \frac{1}{2n}$. Then there exists $x_n, y_n \in \mathcal{L}(x_0, \beta_n)$ such that $d(x_n, y_n) > \frac{\alpha}{2}$ for all n. It implies that $\{x_n\}, \{y_n\}$ are $\overline{B}_0$-minimizing sequence at x_0. Thus, $d(x_n, x_0) \to 0$ and $d(y_n, x_0) \to 0$ as $n \to \infty$ which implies $d(x_n, y_n) \to 0$ as $n \to \infty$, which is a contradiction. Hence, $\inf_{\beta\geq 0} \operatorname{diam} \mathcal{L}(x_0, \beta) = 0$.
Conversly, suppose that $\inf_{\beta\geq 0} \operatorname{diam} \mathcal{L}(x_0, \beta) = 0$ and let $\{x_n\}$ be a $\overline{B}_0$-minimizing sequence at x_0. Then, there exists $\{\beta_n\}, \beta_n \geq 0, \beta_n \to 0$ such that $f(x_n) \leq_K \beta_n\overline{B}_0 + f(x_0)$ which follows that $x_n \in \mathcal{L}(x_0, \beta_n)$ for all n. Also, $x_0 \in \mathcal{L}(x_0, \beta_n)$ for all n. If x_n does not converges to x_0, then there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k}, x_0) \geq \epsilon$ for all k which implies that $\inf_{\beta\geq 0} \operatorname{diam} \mathcal{L}(x_0, \beta) > 0$, which is a contradiction. Thus, $x_n \to x_0$ and hence (V, f) is $\overline{B}_0$-well-posed at x_0. □

Theorem 4.7. *If the following conditions holds:*
(i) *V is a convex set;*
(ii) *$\varphi_{f(x_0)} \circ f$ is a convex function;*
(iii) *argmin$(\varphi_{f(x_0)} \circ f)$ is a singleton set.*
Then the problem (V, f) is $\overline{B}_0$-well-posed.

Proof. The proof follows from Proposition 2.1 and Theorem 4.4. □

We illustrate with an example that the above theorem does not hold if $\operatorname{argmin}(\varphi_{f(x_0)} \circ f)$ is not a singleton set.

From Example 4.4 we get,

$$(\varphi_{f(0)} \circ f)(x) = \begin{cases} |x\sqrt{x^2+1}|, & \text{if } -1 \leq x \leq 0, \\ 0, & \text{if } 0 \leq x \leq 1. \end{cases}$$

Here, $\operatorname{argmin}(V, \varphi_{f(0)} \circ f) = [0, 1]$, which is not a singleton set. Hence, it can be verified that (V, f) is not B_0-well-posed at $x_0 = 0$.

5. Conclusions

In this paper, we introduce a scalar function which is non-linear and is valid for both solid and non-solid optimization problems. We established that this function is convex, monotonic and sub-linear. Two types of well-posedness for (V, f) namely, $\overline{B}_0$-well-posedness and L-$\overline{B}_0$-well-posedness have been introduced. The equivalence of these well-posedness with the well-posedness of corresponding scalar problem have been established via this newly defined scalarization function. It is important to note that the results in our paper are more general as found in literature in terms of that they are valid not only for solid optimization problems but also for non-solid optimization problems.

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