ISOTIMIC CURVES FOR DESCRIPTION AND CONTROL OF INTENSITY PROFILE DYNAMICS OF SOLUTIONS TO THE PARAXIAL WAVE EQUATION

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We propose a method of computing the initial condition for the paraxial wave equation using the desired isotimic curves of the intensity profile.

Keywords: wave equation, non-diffractive beams, self-accelerating beams, isotimic curves

1. Introduction

Laser beam dynamics relies on descriptive methods to characterize the properties of the beam profile during propagation, most of which being based on computing the solution to some partial differential equation (PDE) for some given border and initial conditions. This approach, along with some further introduced parameters, is considered to be the standard method for controlling the propagation [1][2][3]. Although in principle the computation, be it analytic or numeric, can help characterize the beam profile, it is not as useful when a different approach is needed. Such different approaches have appeared in the study of non-diffractive and self-accelerating beam, where the emphasis lays on the choice for the initial condition, rather than on the solution of the PDE. This approach is heavily related to soliton theory [4][5], while in the linear regime it relies on the theoretical discovery of Airy function's non-diffractive and self-accelerating properties by Berry et. al. [6], and on the experimental implementation and validation of the result from Siviloglou et. al. [7][8]. Because of this shift in perspective, other methods of describing the propagation are needed. A different approach is presented by Kaganovsky et. al.[9] where the emphasis is on the trajectory of the global intensity peak, the rest of the profile being considered only as a by-product of their procedure. Although the trajectory of the main intensity peak is well controlled by their approach, the non-diffractive property emerges without having it emposed. This motivates us to present a framework in which both trajectory and non-diffractive properties

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can be controlled. Thus we propose a method that allows for characterizing the potentially non-diffractive and self-accelerating properties that arise from an initial condition, properties which can also be used, at least in principle, to generate an initial condition with a desired behaviour during propagation.

2. Theoretical aspects

In the paraxial approximation, the equation for the evolution of a slowly-varying envelope[1] ψ of the beam's electric field has the form

$$\partial_z \psi(x, z) = \frac{i}{2k} \partial_x^2 \psi(x, z) \tag{1}$$

where x and z are the transverse coordinate and the propagation distance respectively, ∂_α^β is the partial derivative of order β with respect to α, and k is the wavenumber. In order to solve the propagation equation an analytic initial condition ψ_0 is considered.

Starting with (1), by computing $(1)\cdot\psi^* + (1)^* \cdot \psi$ the propagation equation for the intensity profile is

$$\partial_z |\psi|^2 = \frac{i}{2k} \left((\partial_x^2 \psi)\psi^* - \psi(\partial_x^2 \psi^*) \right) \tag{2}$$

For an extreme case of non-diffractive solutions of the propagation equation (1) the profile $|\psi|^2$ only shifts along the spatial axis during propagation. This is equivalent to having a condition

$$|\psi|^2(x, z) = |\psi|^2(x - f(z)) \tag{3}$$

$\forall z \in \mathbb{R}$, where $f : \mathbb{R} \to \mathbb{R}$. Incidentally for $f(z) = z^2$ a self-accelerating solution as the Airy beam[6] is emposed by the same condition.

This approach is limited because the condition is applied $\forall z \in \mathbb{R}$, and does not allow for solutions that are nondiffractive and self-accelerating on finite temporal domains. Because of this limitation we have considered revising this approach starting from the condition (3).

Let ψ be a solution of PDE (1), then there exists at least one isotimic curve $\Gamma(s) = (g(s), h(s))$ for which $|\psi|^2\,(x = g(s), z = h(s)) = a, \forall s \in \mathbb{R}$, where $a \in [0, \max(|\psi|^2)]$ is a constant. The isotimic curves are considered to be parametric of parameter s. In a general 1+1 dimensional scenario they can be defined using two functions $g(s)$ and $h(s)$. Based on the choices of $g(s)$ and $h(s)$ the trajectory of the intensity profile $|\psi|^2$ is approximated in the following.

Because the amplitude is constant on an isotimic curve by definition, then it is true that

$$\nabla_{x,z} |\psi|^2 \cdot T_\Gamma = 0 \tag{4}$$

where $\nabla_{x,z}$ is the gradient in the (x, z) space, " $\cdot$ " is the scalar product and $T_\Gamma(s) = (\partial_s g(s), \partial_s h(s))$ is the tangent of the isotimic curve. Written explicitly, condition (4) becomes

$$\partial_s g \cdot \partial_x |\psi|^2 + \partial_s h \cdot \partial_z |\psi|^2 = 0. \tag{5}$$

Introducing equation (2) in (5) gives

$$\partial_s g \cdot \partial_x |\psi|^2 + \frac{i}{2k}\partial_s h \cdot \left((\partial_x^2\psi)\psi^* - \psi(\partial_x^2\psi^*)\right) = 0 \tag{6}$$

which is the condition that relates the propagation equation (1) with the isotimic curves introduced by condition (4). Using this equation, for a set of isotimic curves, the solution ψ can be derived. We present in the following section how (6) can be used on some simple scenarios.

Unlike condition (3), equation (6) allows for a more general description of the isotimic curves. In principle, if the equations for the isotimic curves are known, equation (6) can be used in order to generate both diffractive and non-diffractive scenarios, which is not as straightforward from (3).

We have to remark that by $\Gamma(s) = (g(s), h(s))$ we refer to parametric curves that have the following properties:

(1) Any 2 isotimic curves Γ_1 and Γ_2 are non-intersecting unless $\Gamma_1 \equiv \Gamma_2$.
(2) Let Γ be an isotimic curve, it can either be a point, a closed curve, or an open curve that extends from $-\infty$ to ∞ on the z-axis.
(3) We consider curves Γ that do not self-intersect.

The analysis that follows focuses on the non-diffractive scenario of (3) where all the isotimic curves are open and are given by $\Gamma(z) = (x_0 + g(z), z)$ where x_0 is used to define the intersection of the curve Γ with the x-axis. Based on the choice of x_0 different curves are defined, so it is conveninent to label din dependency as Γ_{x_0} since we are interested in the entire set of isotimic curves that help to characterize our solution. For this particular case equations (5) and (6) become

$$\partial_z g \cdot \partial_x |\psi|^2 + \partial_z |\psi|^2 = 0 \tag{7}$$

and

$$\partial_z g \cdot \partial_x |\psi|^2 + \frac{i}{2k}\left((\partial_x^2\psi)\psi^* - \psi(\partial_x^2\psi^*)\right) = 0 \tag{8}$$

respectively.

We are interested in computing an initial condition that will generate at least an approximate solution to the one emposed by the isotimic curves. For this we write $g(z)$ as a Taylor series around $z = 0$. This is introduced in (7) which gives

$$\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\partial_z^{n+1} g(z=0)\right) \cdot \partial_x |\psi|^2 + \partial_z |\psi|^2 = 0 \tag{9}$$

Next we evaluate equation (9) at $z = 0$ which gives condition

$$\left(\partial_z g \cdot \partial_x |\psi|^2 + \partial_z |\psi|^2\right)\Big|_{z=0} = 0 \tag{10}$$

or

$$\partial_z g(0) \cdot \partial_x |\psi_0|^2 + \frac{i}{2k}\left((\partial_x^2\psi_0)\psi_0^* - \psi_0(\partial_x^2\psi_0^*)\right) = 0. \tag{11}$$

Next we apply ∂_z on (9) and evaluate at $z = 0$ with gives a second condition that involves $\partial_z^2 g(0)$. By repeating this process an infinite number of conditions can be derived. If they are satisfied by the initial condition ψ_0, the solution should propagate according to the choice of $g(z)$, which defines the isotimic curves.

3. Applying the method

This section is meant to showcase two situations on which the method is applied in order to offer a better understanding of it and how we consider that it can be used. First we describe how using the isotimic curves to manipulate a Gaussian beam's direction of propagation by enforcing an amplitude profile for which the phase is computed, then we generate the amplitude for an initial condition that generates a non-diffractive self-accelerating profile similar to the Airy beam[1] by assuming a constant phase.

3.1. Linear curves

In this example we apply the method on an initial condition for which $|\psi_0(x)| = e^{-x^2/2\sigma^2}$ in order to change the direction the beam propagates. If the phase is left constant, then tre result is the well known Gaussian beam[10]. This change of direction easily translates to the isotimic curves

$$\Gamma_{x_0}(z) = (x_0 + z \cdot \partial_z g(0), z) \tag{12}$$

Based on the choice of $\partial_z g(0)$ the slope of the isotimic curve in the (x, z) plane is defined, also changing the direction the gaussian beam propagate.

Since we enforce the amplitude of the beam, the control will be made on the phase $\phi(x)$ at $z = 0$ by considering the ansatz

$$\psi_0(x) = A(x)e^{i\phi(x)}. \tag{13}$$

The condition resulting from enforcing the isotimic curves arises from evaluation (11) by introducing the ansatz (13), which gives

$$2k \cdot \partial_z g(0)\partial_x A = 2\partial_x\phi\partial_x A + A\partial_x^2\phi \tag{14}$$

where we assume that $A(x) \neq 0$.

The naive finite difference implementation of (14) is

$$\phi_{i+2} = \phi_{i+1}\frac{4A_i}{A_{i+2} + 2A_{i+1} - A_i} + \phi_i\frac{A_{i+2} - 2A_{i+1} - A_i}{A_{i+2} + 2A_{i+1} - A_i} + \\ + k\partial_z g(0)\Delta x\frac{A_{i+2} - A_i}{A_{i+2} + 2A_{i+1} - A_i} \tag{15}$$

where Δx is the step for the transverse grid, $\phi_i = \phi(i\Delta x)$, $A_i = A(i\Delta x)$, and the derivatives have been approximated using

$$\partial_x A(i\Delta x) \approx \frac{A_{i+1} - A_{i-1}}{2\Delta x},$$

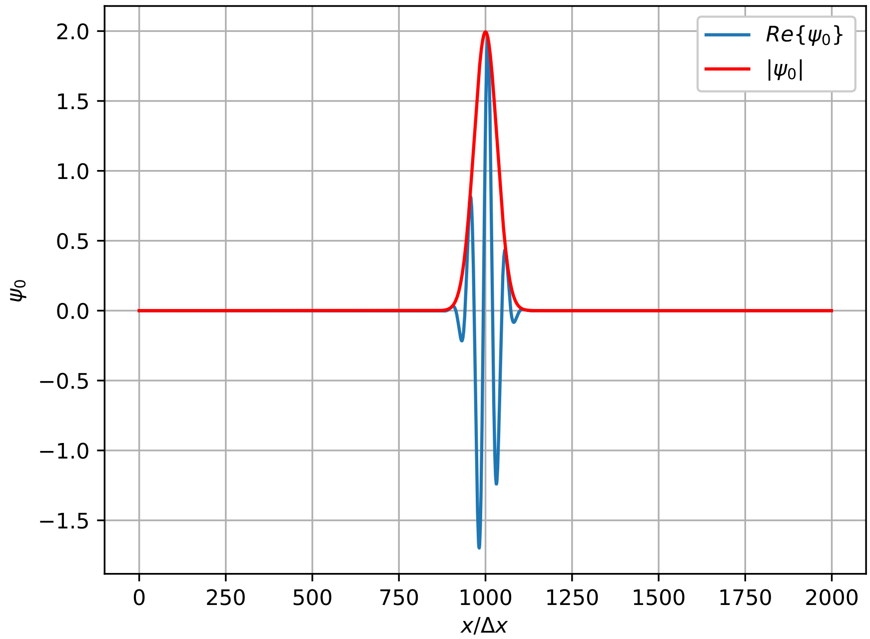

FIG. 1. The real part and amplitude of the initial condition for a tilted Gaussian beam after the phase ϕ is applied.

$$\partial_x \phi(i\Delta x) \approx \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x},$$

and

$$\partial_x^2 \phi(i\Delta x) \approx \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}.$$

The parameters used for computing the phase are $\phi_{i=0} = \phi_{i=1} = 0$, $\Delta x = 6 \cdot 10^{-3} \mu m$, $\lambda = 0.635 \mu m$ since $k = 2\pi/\lambda$ and $\partial_z g(0) = 2$. The resulting initial condition is given in figure 1.

The solution is computed using the Crank-Nicolson scheme on (1) with $\Delta z = 4 \cdot 10^{-4} \mu m$ for the propagation axis. The intensity profile of the solution is given in artibrary units in figure 2. Qualitatively it can be seen that the trajectory of the intensity maximum is tilted.

For a quantitative analysis we fit the trajectory of the maximum with a linear function $f(z) = az + b$. Our result for the solution in figure 2 gives $a = 1.99$ which is comparable with the input value of $\partial_z g(0) = 2$.

Although diffraction is still present during propagation, it is important to mention that the control was applied only by using eq. (11). All the other conditions that appear from eq. (9) by applying ∂_z and evaluating at $z = 0$ (and repeating this process) are not considered. Thus, even if the result is a very rough approximation, it can still easily showcase the tilt of the trajectory.

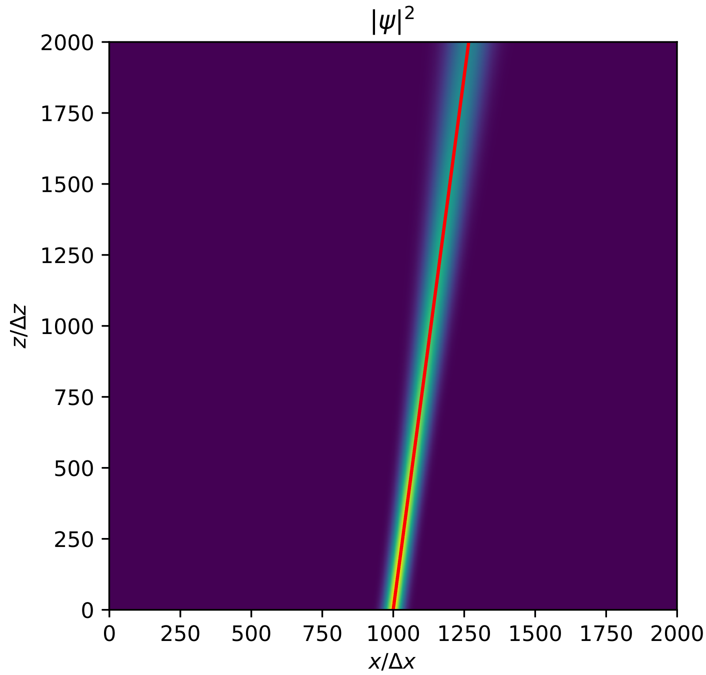

FIG. 2. The intensity plot of the numerical solution for a tilted Gaussian beam. The red line gives the trajectory of the intensity maximum.

3.2. Parabolic curves

In this example we apply the method in order to generate an initial condition that corresponds to a finite non-diffractive and self-accelerated profile using parabolic isotimic curves.

Using our approach, any curve Γ has to be of the form

$$\Gamma_{x_0}(z) = \left(x_0 + \frac{z^2}{2}\partial_z^2 g(0), z\right) \tag{16}$$

which makes $\partial_z^2 g(0)$ a parameter that can be chosen by the user. Using these conditions we derive the restrictions for the initial condition.

Next we use $\partial_z g(0) = 0$ based on the choice for the isotimic curves of (16) and equation (11) from which the first restriction $\partial_z|\psi|^2(x_0, 0) = 0, \forall x_0 \in \mathbb{R}$ is derived. One class of functions that satisfy this restriction is the class of real functions so $\psi_0(x) = \psi_0^*(x), \forall x \in \mathbb{R}$. Applying ∂_z on equation (8) gives

$$\partial_z^2 g \cdot \partial_x|\psi|^2 + \partial_z g \cdot \partial_z\partial_x|\psi|^2 + \partial_z^2|\psi|^2 = 0. \tag{17}$$

By evaluating equation (17) at $z = 0$ and considering $\partial_z g(0) = 0$, equation (17) becomes

$$\partial_z^2 g(0)\partial_x|\psi|^2(x_0, 0) + \partial_z^2|\psi|^2(x_0, 0) = 0 \tag{18}$$

Next we use (2) in (18) and that $\psi_0 \in \mathbb{R}$ which gives

$$\partial_x^4 \psi_0(x_0) = \frac{(\partial_x^2 \psi_0(x_0))^2}{\psi_0(x_0)} + 4k^2 \partial_z^2 g(0) \partial_x \psi_0(x_0). \tag{19}$$

To test this result we have used the naive finite difference implentation of equation (19),

$$\psi_i = 4\psi_{i-1} - 6\psi_{i-2} + 4\psi_{i-3} - \psi_{i-4} + \frac{(\psi_{i-1} - 2\psi_{i-2} + \psi_{i-3})^2}{\psi_{i-2}} + \\ + 4k^2 \partial_t^2 g(0) \frac{\psi_{i-1} - \psi_{i-3}}{2} \Delta x^3 \tag{20}$$

where Δx is the step for the transverse grid, $\psi_i = \psi_0(i\Delta x)$, and the derivatives have been approximated using

$$\partial_x^4 \psi_0(i\Delta x) \approx \frac{\psi_{i+2} - 4\psi_{i+1} + 6\psi_i - 4\psi_{i-1} + \psi_{i-2}}{\Delta x^4},$$

$$\partial_x^2 \psi_0(i\Delta x) \approx \frac{\psi_{i+1} - 2\psi_i + \psi_{i-1}}{\Delta x^2}$$

and

$$\partial_x \psi_0(i\Delta x) \approx \frac{\psi_{i+1} - \psi_{i-1}}{2\Delta x}.$$

The computation of the numerical initial condition requires some initial data for equation (20) to be used, which we have considered to be $\psi_{i=0} = 1.8$, $\psi_{i=1} = 2$, $\psi_{i=2} = 2$, $\psi_{i=3} = 1.8$ in artibrary units, $\Delta x = 6 \cdot 10^{-3} \mu m$, $\lambda = 0.635 \mu m$ since $k = 2\pi/\lambda$ and $\partial_z^2 g(0) = 2\mu m^{-1}$. The resulting initial condition is given in figure 3. The decay regions have been introduces to compensate the absence of a perfectly matched layer on the boundaries of the transversal spatial domain. The central region is computed using equation (20) with the above mentioned parameters.

The computation of the numerical solution is made using the Crank-Nicolson scheme on (1) with the additional parameter for the discrete space along the propagation axis $\Delta z = 4 \cdot 10^{-4} \mu m$. The intensity profile in arbitrary units of the solution is given in figure 4. Qualitatively it can be seen that the trajectory of the intensity maxima are curved and preserve their profile during propagation for a finite distance.

For a quantitative analysis we compute the trajectories for all the local intensity maxima and fit the resulting trajectories with a second order polinomial $f(z) = az^2 + bz + c$. Our result for the solution in figure 4 gives $a = 0.96 \mu m^{-1}$ with a sample standard deviation $\sigma_a = 9.51 \cdot 10^{-16} \mu m^{-1}$, which corresponds to $\partial_z^2 g(0) = 1.92 \mu m^{-1}$ by taking in consideration the factorial from the Taylor series. By direct comparison with the input value $\partial_z^2 g(0) = 2\mu m^{-1}$, we conclude that the values are comparable, which confirms our theoretical approach.

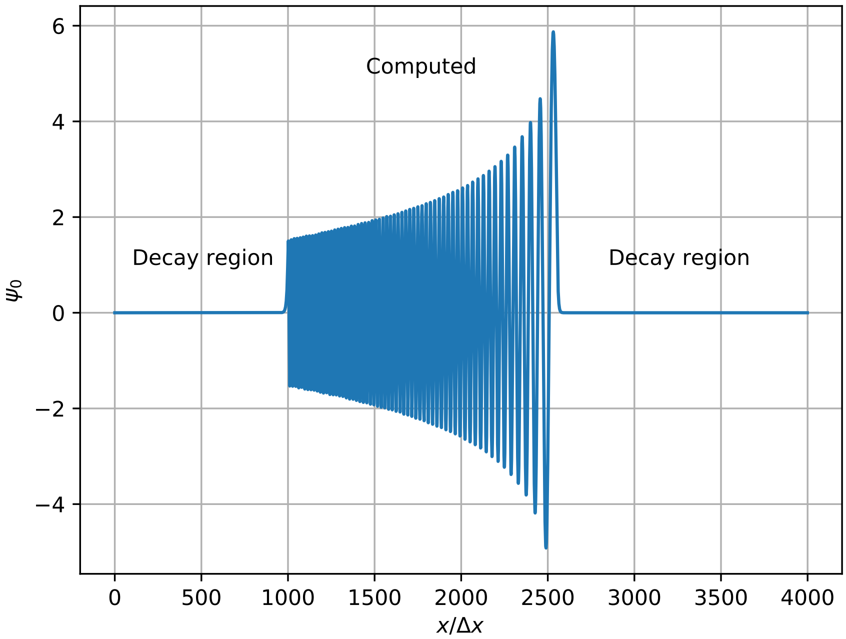

FIG. 3. The initial condition used for the computation of the numerical solution.

As it can be seen in figure 4 the intensity profile preserves its shape on a finite propagation distance. This is due to the limited transversal spatial domain not because of the method. If we consider equation (19) to be in artibrary units and that that

$$4k\partial_z^2 g(0) = 2$$

then the Airy function[6] satisfies the condition. This implies that an initial condition that generates an infinite nondiffractive self-accelerating could be retrieved, at least theoretically, by evaluating (19).

4. **Conclusions**

In this paper we have introduced the isotimic curves as a method to both describe the dynamics of the intensity profile in terms of shape preserving and peak trajectories, and help generate initial conditions that satisfy some desired dynamics. In section 3.1 we have used our method to computed the phase for a Gaussian beam in order to tilt its propagation trajectory in a controlled manner. The result from the numerical simulation recovered with a good approximation the input parameter that was used to define the tilt. In section 3.2 we have used the method to generate an initial condition that satisfies parabolic isotimic curves as a comparison with the known case of the Airy

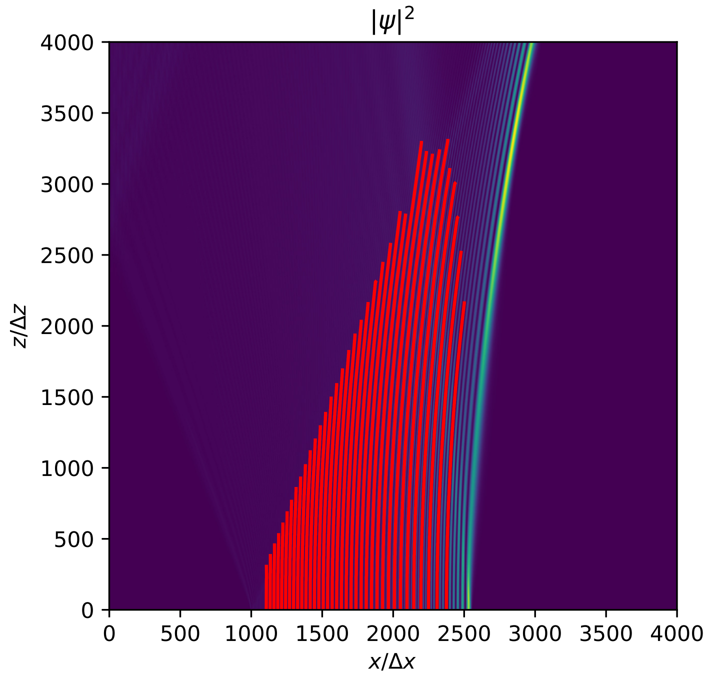

FIG. 4. The intensity plot of the numerical solution. The red lines give the trajectories for some of the local intensity maxima.

beams. The control parameter has been considered the second order coefficient of a second order polynomial. The numerical simulations have retrieved this parameter with a good approximation.

The numerical results have matched the input data in both examples although we have used only a very small number of the potentially infinite number of conditions that can be derived from eq. (9). Considering the finite spatial domain, our results indicate that at least for a relatively small propagation distance, which depends on the choice of parameters in the simulation and the size of the transverse domain, the approximate condition enforced by the isotimic curves is satisfied.

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