U.P.B. Sci. Bull., Series C, Vol. 73, Iss. 1, 2011 ISSN 1454-234x

ENERGY LOCALIZATION IN CHIRP SIGNALS

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În această lucrare este expusă o demonstrație a funcției de localizare a energiei într-un semnal de tip chirp. Această demonstrație se bazează pe o alegere adecvată a unei anumite funcții care are o semnificație fizică. Rezultatul este în conformitate cu distribuția spectrală măsurată experimental pentru semnale de tip chirp modulate exponențial.

In the paper a proof for energy localization in chirp signals is given. It is based on an adequate choice of a certain functional which has a physical significance. The result is in accordance with the experimentally measured spectral distribution for exponentially modulated chirps.

Keywords: energy localization, exponentially sine sweep.

1. Introduction

In the domain of audio measurements for room acoustics, the paper of Farina [1] brought a significant contribution by providing a method of obtaining in the frame of a single measurement the impulse response of the room and the Volterra type products stemming from the loudspeaker nonlinearities. The novelty of this approach termed as Impulse Method Measurement (IMM) lies in:

a) Using an exponentially modulated chirp for sweeping the frequency range

b) The separation, as to the displayed in time, of the linear part of the response and the nonlinear parts.

This last possibility was predicted by Poletti in [2]. Various properties of chirps used in applications are discussed in [3] and [4]. To make clearer the problem arising in IMM, and the origins of our investigation, we will shortly explain its background. Here we will focus our attention on the linear behavior.

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2. The ideas of IMM.

The system, a linear time-invariant one (the room), is actuated by a sound source $x(t)$ (with a fixed location) which produces a signal response $y(t)$ (also in a fixed location). This last is the result of the convolution product [5]
$y(t)=(h*x)(t)$, (1)
where "$*$" denotes the convolution, whereas $h(t)$ is the impulse response or the weighting function of the system.

The operation (1) is performed and the result $y(t)$ is recorded. It is asked to deconvolve (1) in order to get the function $h(t)$. Obviously, here all is described in analog frame; really all is done in the digital frame. If we can construct the inverse of x, let it be denoted by $\tilde{x}$, i.e.

$$x*\tilde{x}=\tilde{x}*x=\delta, \qquad (2)$$

where δ is the Dirac distribution, then we can get to the following result:

$$\tilde{y}(t)=\left[(h*x)*\tilde{x}\right](t)=(h*\delta)(t)=h(t) \qquad (3)$$

Since one really cannot realize δ, one can manage so that the object $x*\tilde{x}$ to meet the following conditions:

a) the modulus of its Fourier transform (FT), be a constant in the frequency range of interest
b) its phase be zero in the frequency range of interest.

Such proprieties describe an equivalent of δ scaled with some constant. To meet the condition b), we use some standard facts of the Fourier transform [5]. We denote by $\bar{x}$ the complex conjugate of x and by $\breve{x}$ the reflected of x, i.e. $\breve{x}(t)=x(-t)$.

We have the correspondences by FT $x\overset{\boldsymbol{F}}{\longleftrightarrow}X$, $\breve{x}\overset{\boldsymbol{F}}{\longleftrightarrow}\breve{X}$, $\bar{x}\overset{\boldsymbol{F}}{\longleftrightarrow}\bar{\breve{X}}$ and if x is a real valued signal $\bar{x}=x\overset{\boldsymbol{F}}{\longleftrightarrow}\bar{\breve{X}}$, $\breve{x}=\breve{\bar{x}}\overset{\boldsymbol{F}}{\longleftrightarrow}\bar{\breve{\breve{X}}}=\bar{X}$, therefore the spectrum of the reverse (reflected of x) is the complex conjugate of X. We have

$$x*\breve{x}\overset{\boldsymbol{F}}{\longleftrightarrow}\boldsymbol{F}[x]\cdot\boldsymbol{F}[\breve{x}]=\boldsymbol{F}[x]\cdot\overline{\boldsymbol{F}[x]}=\left|X(\omega)\right|^2 \qquad (4)$$

We have then obtained a non-negative function. If in addition, $\left|X(\omega)\right|=const$, then the spectrum is a delta-like spectrum scaled by $\left|X(\omega)\right|^2$. There are some cases when this condition is fulfilled: the linear swept used in radiolocation [4]. An example of linear sweep it is shown in Fig. 1. Here the inverse is just $\breve{x}(t)$.

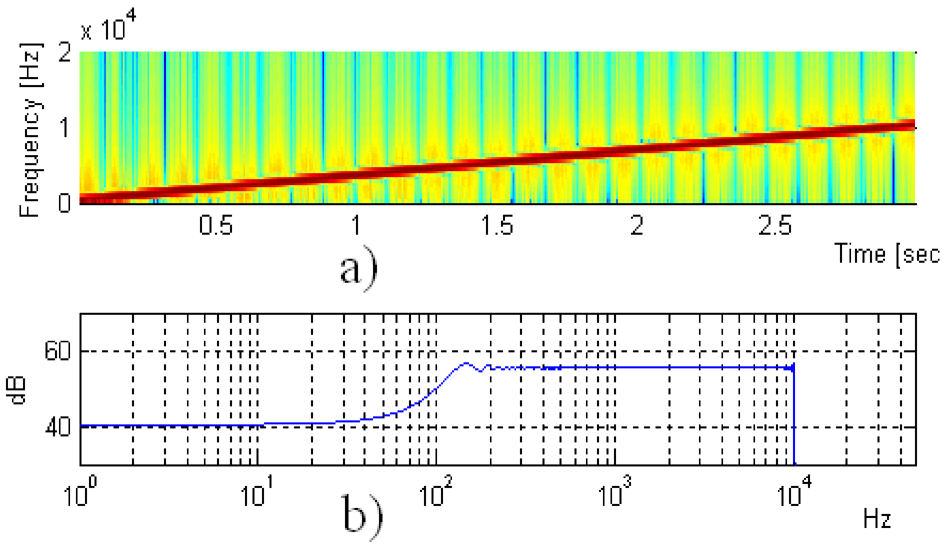

Fig. 1 a) The spectrogram and b) the modulus of the Fourier transform for a chirp signal with linear sweep frequency, $f \in [100, 10000]\, Hz$

3. The structure of the chirps used in IMM

Generally speaking, a chirp is a rapidly varying signal, ex. $\sin(1/t)$. Study of these objects is given in [3]. The used input signal in IMM is termed as a swept sine signal, or more precisely an exponentially modulated one. His mathematical expression is

$$x(t) = \sin[\phi(t)] \tag{5}$$

This is a phase modulated signal. According to the theory of modulated signals, [6], the following quantities are of interest

- the instantaneous phase $\phi(t)$
- the instantaneous frequency

$$\omega(t) = \frac{d\phi}{dt}, \tag{6}$$

The swept sine used in [1] was

$$x(t) = \sin\left[K\left(\exp\left(\frac{t}{L}\right) - 1\right)\right], \tag{7}$$

where L is a time constant and K a non-dimensional constant. If we impose that the swept range to be from ω_1 to ω_2 after T seconds, we need to meet the equalities

$$\left.\frac{d\phi}{dt}\right|_{t=0}=\omega_1, \left.\frac{d\phi}{dt}\right|_{t=T}=\omega_2, \tag{8}$$

from which we get

$$K=\frac{T\cdot\omega_1}{\ln\left(\omega_2/\omega\right)},\ L=\frac{T}{\ln\left(\omega_2/\omega_1\right)} \tag{9}$$

and therefore

$$x(t)=\sin\left\{\frac{\omega_1 T}{\ln\left(\omega_2/\omega_1\right)}\left[\exp\left(\frac{t}{T}\ln\left(\omega_2/\omega_1\right)\right)-1\right]\right\} \tag{10}$$

For example, choosing $\omega_1=2\pi\cdot 20$, $\omega_2=2\pi\cdot 20\cdot 10^3$, $T=5$ seconds we get $x(t)=\sin\left[90.96\left(\exp(1.38t)-1\right)\right]$

For the chirp given by (7) it was found that the spectrum versus frequency is not constant, as it is shown by Fig. 2. The spectrum was obtained by FFT analysis. This fact was pointed out by Farina: a 6dB/oct emphasize was applied in the processing of the inverse signal.

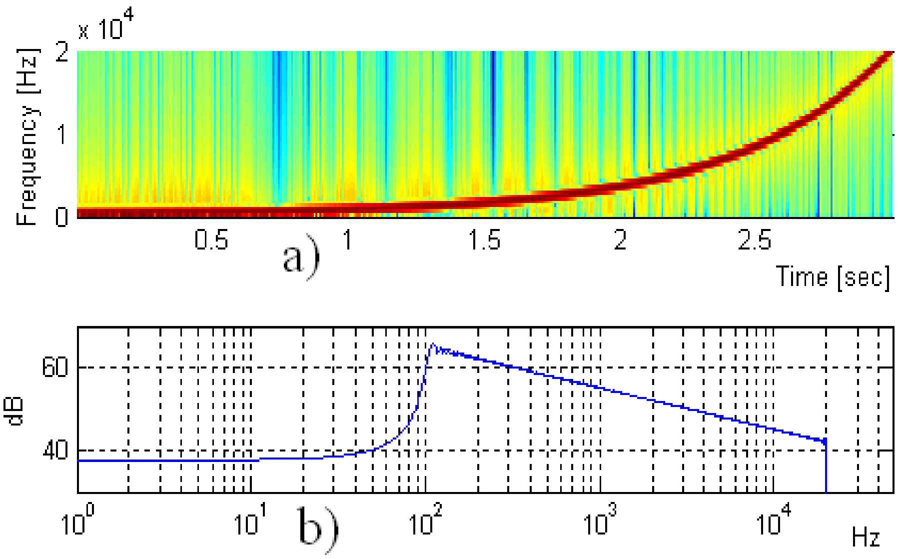

Fig. 2 a) The spectrogram and b) the modulus of the Fourier transform for a chirp signal with exponentially sweep frequency, $f\in[100,20000]\ Hz$

In order to compensate the decreasing of the spectrum as the frequency increases, we must perform a multiplication in the spectral domain with a $A(\omega)$ function so that

$$\left|X(\omega)\right|^2 A(\omega)=1 . \tag{11}$$

The correction $A(\omega)$ can be realized in anyone section of the processing chain. The chosen location is decided by other specific conditions. After the issue of Farina's paper, there appeared some papers related to this approach. We have remarked the paper of Meng et al. [6] where it is asserted that the location of the energy obeys the rule $\left|\omega'(t)\right|^{-1}$, without any reference although this rule is not so obvious.

We have investigated this question (the spectral location of the energy in a chirp) in order to give on rigorous proof for it. The proof is given in the sequel.

4. Properties related to the used chirp

First, we retain two Fresnel-like integrals [7]:

$$\int_{-\infty}^{\infty} \exp\left(\pm \mathrm{j}\frac{u^2}{2}\right) du = \exp\left(\pm \mathrm{j}\frac{\pi}{4}\right)\sqrt{2\pi} \tag{12}$$

$$\int_{-\infty}^{\infty} \exp\left(\pm \mathrm{j}k\frac{u^2}{2}\right) du = \frac{1}{\sqrt{k}}\exp\left(\pm \mathrm{j}\frac{\pi}{4}\right)\sqrt{2\pi} \tag{13}$$

Next, we consider the Taylor expansion of a sufficiently smooth function around a point u_0:

$$\begin{aligned} &\phi(u)=\phi(u_0)+\phi_0^{'}(u-u_0)+\frac{1}{2}\phi_0^{''}(u-u_0)^2+... \\ &\phi_0^{(n)}=\phi^{(n)}(u_0) \end{aligned} \tag{14}$$

The method of stationary phase, [7], states that for an integral containing rapidly varying function (chirp),

$$I=\int_{-\infty}^{\infty} A(u)\exp\left[\mathrm{j}\phi(u)\right] du \tag{15}$$

around a stationary point u_0, with

$$\phi^{'}(u_0)=0 \tag{16}$$

can be calculated as follows

$$I=\int_{-\infty}^{\infty} A(u_0)\exp\left[\mathrm{j}\left(\phi_0+\frac{1}{2}\phi_0^{"}(u-u_0)\right)\right]^2 du=\frac{\sqrt{\mathrm{j}2\pi}}{\sqrt{\left|\phi_0^{"}\right|}}A(u_0)\exp(\mathrm{j}\phi_0) \quad (17)$$

Here it is assumed that A is a slowly varying function

5. Calculation of the spectrum location of a chirp

Firstly we consider a new signal

$$v(t)=\exp\left[\mathrm{j}\phi(t)\right], \quad (18)$$

which obeys $x(t)=\mathrm{Im}\left[v(t)\right]$. It is clear that x and v have similar spectral properties and extremal conditions. It is also obvious that x and v are chirp-like signals. Consider the integral

$$I=\int_{-\infty}^{\infty}\exp\left[\mathrm{j}\phi(u)\right]du \quad (19)$$

and expand ϕ around a fixed point $u=t$. We get:

$$\begin{aligned}\phi(u)&=\phi(t)+\phi^{'}(t)(u-t)+\frac{1}{2}\phi^{"}(t)(u-t)^2+\ldots=\\&=\phi(t)+\omega(t)(u-t)+\frac{1}{2}\omega^{'}(t)(u-t)^2+\ldots\end{aligned} \quad (20)$$

Remark that all the derivatives are evaluated at the point $u=t$. It is seen that I doesn't have a stationary point. Next, we introduce a new functional which depends on "t" and has a stationary point at $u=t$. This new functional looks as follows:

$$J(t)=\int_{-\infty}^{\infty}\exp\left[j\phi(u)-j\omega(t)u\right]du \quad (21)$$

and can be written as

$$J(t)=\int_{-\infty}^{\infty}\exp\left[\mathrm{j}\phi(u)\right]\cdot\exp\left[-\mathrm{j}\omega(t)u\right]du=\boldsymbol{F}\left[v\right]\left[\omega(t)\right], \quad (22)$$

This transform is very difficult to evaluate, but we can do it by using the stationary phase approach. To observe this, we expand the exponent of $J(t)$,

$$\begin{aligned}\phi(u)-\omega(t)u&\cong\phi(t)+\omega(t)(u-t)+\frac{1}{2}\omega'(t)(u-t)^2+\ldots-\omega(t)u=\\&=\phi(t)-t\omega(t)+\frac{1}{2}\omega'(t)(u-t)^2\end{aligned}, \quad (23)$$

and if we express $J(t)$ according to (13)

$$J(t) = \exp\left[\mathrm{j}\phi(t) - \mathrm{j}t\omega(t)\right] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\mathrm{j}\omega'(t)(u-t)\right]^2 dt = \\ = \exp\left[\mathrm{j}\phi(t) - \mathrm{j}t\omega(t)\right] \frac{\sqrt{2\pi\mathrm{j}}}{\sqrt{\omega'(t)}} \quad (24)$$

what is equivalent to

$$\left|J(t)\right| = const \cdot \frac{1}{\sqrt{\omega'(t)}} \quad (25)$$

To get the energy, $E(t)$, located around the point t, we write the squared of (25)

$$E(t) \cong const \frac{1}{\omega'(t)} \quad (26)$$

In case we have a sine sweep with exponential variation, we will obtain

$$E(t) \cong const \cdot \exp\left[-\frac{t}{L}\right]\sigma(t) \quad (27)$$

which has the equivalent in the spectrum domain, the following expression:

$$\boldsymbol{F}\left[E\right](\omega) = const \frac{1}{j\omega + \frac{1}{T}} = const \frac{T}{\omega T + 1} \quad (28)$$

The formula above indicates a 3 dB/oct fall as the frequency increases ($\omega \to \infty$). If we were to check the formula (26) for the linear chirps as well, we would quickly notice that $\omega'(t) = const$ and the expression for the energy localization would be $E(t) = const$. This is in accordance with the calculated module of the Fourier Transform, viewed in Fig. 1 for a particular example of linear chirp.

6. Conclusions

We have presented in this paper the frame in which the energy location appears. Then, we have provided a proof for the energy location expression. This expression can be used to calculate the localization of energy for any kind of chirp signals as long as the frequency varying function is monotonically and known.

A better understanding of the use of chirps has been obtained and perhaps new directions of study will appear.

Acknowledgement

The work has been funded by the Sectoral Operational Programme Human Resources Development 2007-2013 of the Romanian Ministry of Labour, Family and Social Protection through the Financial Agreement POSDRU/6/1.5/S/19.

R E F E R E N C E S

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