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FULLY DEVELOPED TURBULENT PIPE FLOW

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We investigate the problems of boundary layer in a fully developed turbulent pipe flow. In establishing the structure of turbulent boundary layer, we have used the Spalding's (1961) law of the wall in the inner region and Persen's (1974) law in the wake region. The constants, namely the value of intercept and the Von Kármán constant of log-law were chosen to be the same as that used by McKeon et al. (2004) in his turbulent pipe flow analysis. It is found that both Spalding's law and Persen's law nicely described the velocity profile. The computed friction factors are compared with the recent data of Zagarola and Smits (1998), Swanson et al. (2002) and McKeon et al. (2004). The agreement was found highly satisfactory.

Key words: turbulent boundary layer, law of the wall, law of the wake, Reynolds number and friction factor

1. Introduction

Prandtl [6] concluded that time mean velocity, u near the smooth wall must depend upon density ρ and viscosity μ of the fluid, the shear stress at the wall τ_w and on the distance from the wall, y. Thus, near the smooth wall there is a functional relationship

$$u = u(\rho, \mu, \tau_{w,} y) \tag{1}$$

From dimensional analysis, the functional relationship can be written in the from

$$\frac{u}{v_*} = f\left(\frac{y v_*}{\nu}\right) \tag{2}$$

in which shear velocity, $v_*{}^2 = \tau_w / \rho$ and kinemetic viscosity, $\nu = \mu / \rho$. Introducing $u^+ = u / v_*$ and $y^+ = y / v_*$ equation (2) can be written as

$$u^+ = f(y^+) \tag{3}$$

Equation (3) is called law of the wall. Spalding [7] has given a special form of (3)

$$y^+ = f(u^+) = u^+ + A[\exp(\kappa u^+) - 1 - (\kappa u^+)^2/2 - (\kappa u^+)^3/6 - (\kappa u^+)^4/24] \tag{4}$$

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where,

$$A = \exp(-\kappa B) = 0.1108, \kappa = 0.4 \; and B = 5.5. \tag{5}$$

The unique feature of Spalding's equation is that it presents y^+ as a function of u^+ rather than u^+ as a function of y^+.

Spalding [6] used the constants of logarithmic law given in equation (5) and compared his law (4) with the data of Laufer [2] for fully developed pipe flow for Reynolds number 50000 to 500000.

Zagarola and Smith [9] found appropriate values of κ = 0.436 and B = 6.15 for their measurements in fully developed pipe flow for Reynolds numbers in the range 31000 to 35000000. Recently, McKeon et al. [4] reported the values of κ = 0.421 and B = 5.6 for their measurements in fully developed pipe flow for Reynolds numbers in the range 74000 to 35000000.

Spalding [7] pointed out that the constants used in equation (4) must not be regarded as sacrosanct. The constants, κ = 0.421 and B = 5.6 are used to fit the equation (4) with the profiles of McKeon et al. [4] measured for pipe flow (Fig. 1). From Fig. 1 one can see that law of the wall cannot trace the data for the whole boundary layer.

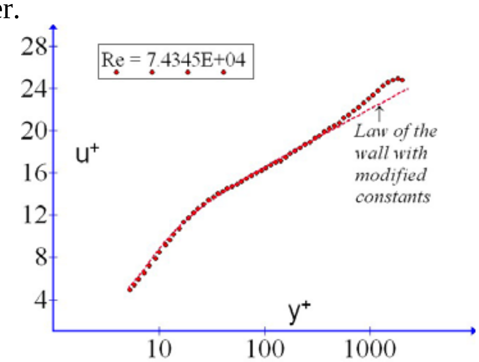

Fig. 1. Law of the wall compared with the experimental data.

2. Coles' wake function [4]

Fig.1 shows a clear picture of existence of wake function and equation (4) is too simple for describing the wake region. Accordingly, based on the idea of Coles [1], equation (4) should be replaced by

$$u^+ = f(y^+) + A(x)w(\eta) \tag{6}$$

where $A(x)$ is the amplitude function, $w(\eta)$ is the wake function and $\eta = y/R$, R being the radius of the pipe.

The wake function is generally defined as the difference between the measured data in the outer region of the boundary layer and extension of logarithmic law in this region. The method to be followed here will be somewhat different. Values of wake are determined as difference between the measured data in the outer region of the boundary layer and values obtained from equations (4) (with κ = 0.421 and B = 5.6) in that region. As $y^+ \to y_o^{\ +} = R^+ = Rv_*/\nu,\ \eta \to 1$ one then obtains from equation (6) the relation

$$\xi = f(y_0^{\ +}) + A(x)w(1) \tag{7}$$

The maximum value of wake, $w_{\max}$ and its position η_m at which it occurs are found by fitting a parabola through three points around the maximum value. Form now the wake function $w(\eta)$ is defined such that

$$w(\eta) = 1 \quad , \quad A(x) = \mathrm{w}_{\max} \tag{8}$$

Position of $w_{\max}$, η_m is plotted against Reynolds number in Fig. 2. It can be seen that the position of η_m varies within a narrow range, the average being value is 0.777.

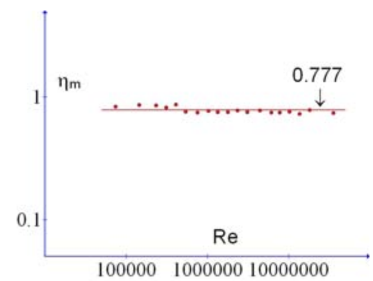

Fig. 2. η_m as function of Re

Variation of maximum value of wake $w_{\max}$ is shown in Fig. 3 with Re. It reveals that $w_{\max}$ is independent of Reynolds number at large Reynolds number.

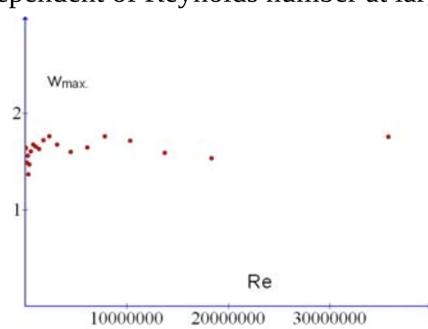

Fig. 3. Plot of $w_{\max}$ as function of Re.

In the present study, it is adopted that the flow field of a two-dimensional turbulent boundary layer may be split up into two regions only: I. near to wall region where Spalding's [7] law with κ = 0.421 and B = 5.6 governs the flow and II. wake region where Persen's [5] wake law governs the flow. The near to wall region is characterized by the fact that it is considered influenced by outside manipulations of the layer while the wake region is characterized by the fact that the influence of outside manipulations.

3. Persen's wake law

Persen [5] presented a formulation for the wake function for representation of the data-points in wake region and that may take care of outside manipulation if any.

Persen [5] proposed the following expression for wake law

$$(u^+ - u_\infty^+)/(\xi - u_\infty^+) = exp[-(y_o^+ - y^+)^2/\alpha^2] \tag{9}$$

where

$$\left.\begin{array}{l} u \to U_c \text{ as } y \to R \\ u^+ \to U_c/v_* = \xi \text{ as } y^+ \to Rv_*/\nu = y_0^+ \end{array}\right\} \tag{10}$$

u_∞^+ = constant, U_c is the centerline velocity and

$$1/\alpha^2 = [\ln(\xi - u_\infty^+) - \ln(u_1^+ - u_\infty^+)]/(y_o^+ - y_1^+)^2 \tag{11}$$

Here (u_1^+ , y_1^+) is the point where law of the wall meets with the law of the wake. The boundary layer ends up at the centerline of the pipe at y^+ = $y_o^+ = R^+ = Rv_*/\nu$, for which centerline velocity $u^+ = \xi$. It is to be mentioned that the Spalding's formulation for the law of the wall (4) with values of constants, A and κ determined by McKeon et al. [4] has an undisputed advantage that it satisfies no-slip condition at the wall and the proposed wake law exhibits a horizontal tangent at the outer edge ($u^+ = \xi$, $y^+ = y_o^+$) of the boundary layer and that applies also for the manipulated boundary layer (adverse pressure gradient etc.). An elaborate discussion on Persen's two dimensional turbulent boundary layer theory has been given in [3].

4. Data: pipe flow [4]

Experiments were made in an aluminium pipe with nominal diameter of 129mm. The pipe was mounted in a closed loop with compressed-air facility. Readers may consult [9] for detailed description of the facility.

We now turn to draw velocity profiles in the light of McKeon et al. data [4]. Experimental data for velocity profiles are fitted with the law of the wall (4) with the value of κ = 0.421 and B = 5.6 and law of the wake (9). The equation (4) is valid for $u^+ \le u_1^+$ while equation (9) is valid for $u_1^+ \le u^+ \le \xi$. Because u_1^+, y_1^+ is a common point on the two curves, the following relation holds

$$u_1^+ = f(y_1^+) \tag{12}$$

The two curves must have a common tangent at that point. This tangent is given by $\tan\alpha$, where

$$\tan\alpha = du^+ / d(\ln(y^+)) = y^+ du^+ / dy^+$$

From equation (3)

$$\tan\alpha = y^+ f'(y^+) \tag{13}$$

and, from equation (9), it follows that

$$\tan\alpha = (2/\alpha^2)(u^+ - u_\infty{}^+)(y_o{}^+ - y^+)y^+ \tag{14}$$

These expressions for $tan\alpha$ must be equal at the point: $u^+ = u_1^+, y^+ = y_1^+$:

Thus,

$$y_1^+ f'(y_1^+) = (2/\alpha^2)(u_1^+ - u_\infty{}^+)(y_o{}^+ - y_1^+)y_1^+ \tag{15}$$

Once the values of ξ and $y_o{}^+$ are known, u_1^+ and y_1^+ can be determined from equations (12), (15). The value of $u_\infty{}^+$ turns out to be 60 for a good fit to the experimental data.

The velocity profiles for various Reynolds number are displayed in figure 4. Experimental data of McKeon et al. [4] are found to be in good agreement both with the law of the wall (4, 5) and law of the wake (9).

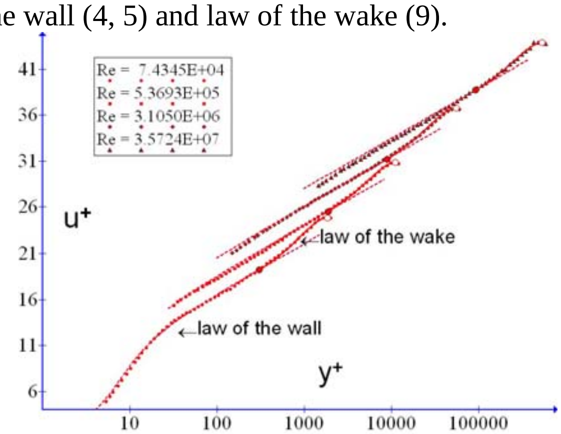

Fig. 4. The law of the wall and law of the wake compared with the McKeon [4] data at four different Reynolds numbers. The points (u_1^+, y_1^+) and (ξ, R^+) are shown as closed circle and open circle, respectively.

Locus of ξ (that is plot of R^+ vs. ξ) is plotted in Fig. 5. A linear relation between ξ and R^+ is found out for the present case as

$$\xi = 2.3450201\ln(R^+) + 7.1570212, \quad R^2 = 0.9995 \tag{16}$$

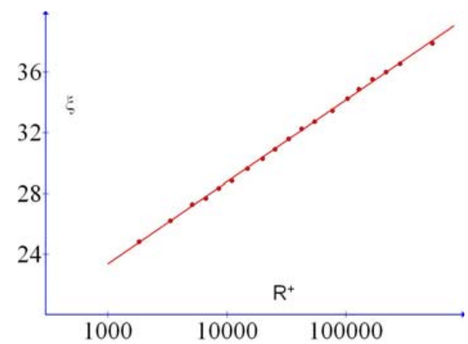

Fig. 5. Locus of ξ

It would certainly be of interest how y_1^+ and R^+ vary with Reynolds numbers, Re. The plots of measured R^+ against Re are shown in figure 6. A best fit curve through the data points is represented by

$$R^+ = 0.059454617\,\mathrm{Re}^{0.91899701}, \qquad R^2 = 0.9996 \tag{17}$$

where R^2 is the coefficient of determination.

In the same figure the values of y_1^+ calculated theoretically are also plotted against Re. A best fit line is obtained in this case as

$$y_1^+ = 0.010158002\,\mathrm{Re}^{0.91630839}, \quad R^2 = 0.9956 \tag{18}$$

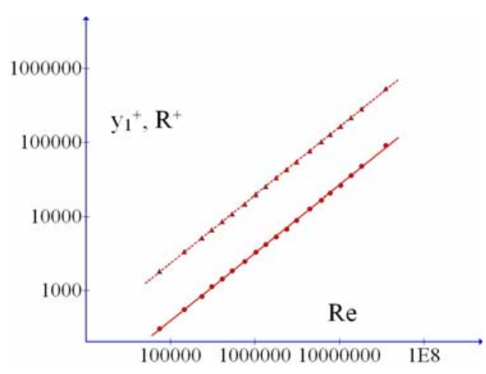

Fig. 6. y_1^+ and R^+ plotted as function Reynolds number

It can be seen that both quantities are increasing functions of Reynolds number, Re and the rate of increment is more or less the same. So, it is expected that the ratio y_1^+ / y_o^+ may be independent of the Reynolds number as it can be seen from Fig. 7. In fact, the ratio y_1^+ / y_o^+ is a measure of the non-dimensional quantity y/R from the wall below which law of the wall (4, 5) governs the flow and above which law of the wake (9) takes over. Physically, this implies that with the increase of Reynolds number, the region near to wall becomes a constant fraction of the R^+.

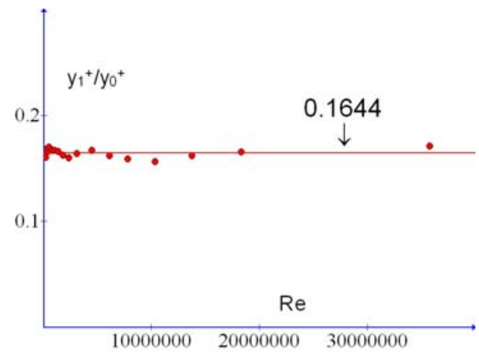

Fig. 7. y_1^+ / R^+ as function of Reynolds number

Variations of ξ and u_1^+ against Re are shown in Fig. 8. Considering ξ values and the Reynolds numbers, Re, a best fit straight line is determined as

$$\xi = 2.155108 \ln(\mathrm{Re}) + 0.5375054, \quad R^2 = 0.9995 \tag{19}$$

and u_1^+ may be computed with a high degree of accuracy as

$$u_1^+ = 2.163576 \ln(\mathrm{Re}) - 5.0838937, \quad R^2 = 0.9997 \tag{20}$$

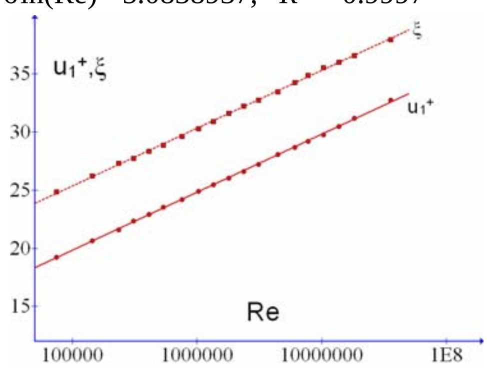

Fig. 8. u_1^+, ξ plotted as function of Re

A linear relationship between u_1^+ and ξ can be obtained from equations (19) and (20) as

$$u_1^{+} = 1.0030959\xi - 5.597233, \quad R^2 = 0.9986 \tag{21}$$

Both ξ and u_1^{+} are found to increase with Re, but the rate of increase for both cases are more or less similar. The value of ratio u_1^{+}/ξ is found to be a slowly increasing function of Re (Fig. 9).

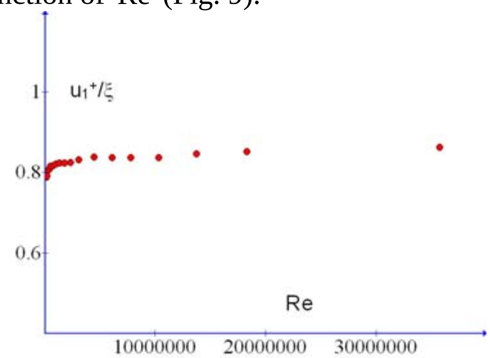

Fig. 9. Variation of u_1^{+}/ξ against Reynolds number

It is worth noticing that the ratio u_1^{+}/ξ varies within a narrow range between 0.774 and 0.863 with an average value 0.823. This indicates clearly that flow within the pipe shifts over from law of the wall to law of the wake when the velocity 'u' in the pipe reached a value within the range 0.774 U_c - 0.863U_c.

The equations (4), (16), (19), (21) and u_∞^{+} = 60 now permits evaluation of the law of the wake (9) with Reynolds number, Re as single input. This has been done the data from [9] and the result is shown in Fig. 10.

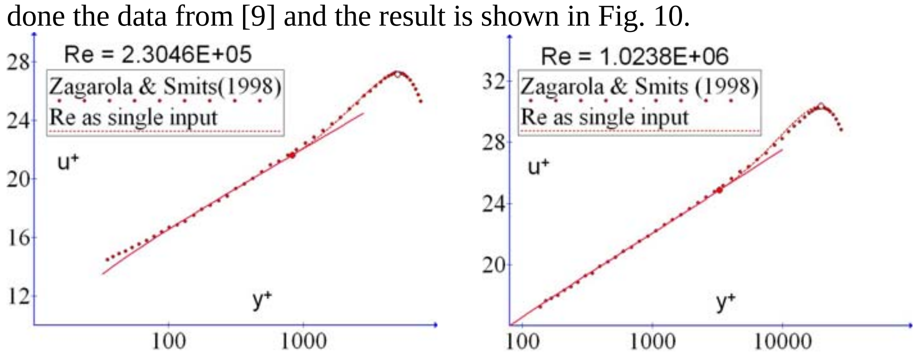

Fig.10 (a) Fig. 10 (b)

Fig. 10. The data [9] for given values Re compared with the proposed law of the wall (continous curve) and the law of the wake (dashed curve).

5. Friction factor for turbulent pipe flow

The condition of equilibrium in a horizontal pipe flow requires that the pressure force $-\Delta P \pi R^2$ acting on the end faces of a pipe of length L is equal to the shear $2\pi R \tau_w$ acting on the circumferential area, whence we obtain

$$\tau_w = \frac{-\Delta P}{L}\frac{R}{2} \tag{22}$$

where ΔP represents pressure drop, and R is the radius of pipe. The pressure drop decreases in the direction of flow so that ΔP has a negative value.
Putting $-\Delta P = \rho g h_f$, we get

$$\tau_w = \frac{\rho g h_f}{L}\frac{R}{2} \tag{23}$$

where h_f is the loss due to friction, g is the gravity. The shear stress τ_w is related to the average velocity V through the cross-section area

$$\tau_w = \frac{\mathrm{f}}{4}\frac{\rho \mathrm{V}^2}{2} \tag{24}$$

where ‘ f ’ is the Darcy-Weisbach friction factor. Introducing the energy gradient $S = h_f / L$ and $R = 2R_H$, we can get from equation (23) and (24)

$$\mathrm{f} = \frac{4gR_H S}{V^2} \tag{25}$$

where hydraulic radius is: $R_H = A$ (cross-section area)/ P (wetted perimeter).

The Reynolds number Re is defined as

$$\mathrm{Re} = \frac{VD_H}{\nu} \tag{26}$$

where the hydraulic diameter is: $D_H = 4R_H$. For the pipe $D_H = 2R$, then:

$$\mathrm{Re} = \frac{2VR}{\nu} \tag{27}$$

The average velocity through the pipe may be defined in terms of radial co-ordinate *r* or wall co-ordinate *y*, where $y = R - r$ and $dy = -dr$. The distance ‘ y ’ is measured from the wall while r is measured from the center of the pipe.

$$V = \frac{1}{\pi R^2}\int_0^R 2\pi\, r\, u\, dr = \frac{2}{R}\int_0^R udu - \frac{2}{R^2}\int_0^R uydy \tag{28}$$

Now by introducing $u^+ = u / v_*$, $y^+ = yv_* / \nu$ and $R^+ = Rv_* / \nu$ in equation (28) it becomes

$$\mathrm{Re} = \frac{2VR}{\nu} = 4\int_0^{R^+} u^+ dy^+ - \frac{4}{R^+}\int_0^{R^+} u^+ y^+ dy^+ \tag{29}$$

Equation (29) can be rewritten as

$$\mathrm{Re} = 4\left[\int_0^{u_1^+} u^+ \frac{dy^+}{du^+} du^+ + \int_{y_1^+}^{R^+} u^+ dy^+\right] - \frac{4}{R^+}\left[\int_0^{u_1^+} u^+ f(u^+)\frac{dy^+}{du^+} du^+ + \int_{y_1^+}^{R^+} u^+ y^+ dy^+\right] \tag{30}$$

or,

$$\mathrm{Re} = 4[I + II] - \frac{4}{R^+}[III + IV] \tag{31}$$

Integrands *I* and *III* are computed from law of the wall while integrands *II* and *IV* are computed from law of the wake.

Now, equation (25) can be rewritten as

$$\mathrm{f} = \frac{8v_*^{\,2}}{V^2} = 32\left[\frac{R^+}{\mathrm{Re}}\right]^2 \tag{32}$$

where $v_* = \sqrt{gR_H s}$.

The values of Re and ‘f” have been computed from equations (30) and (32) . The computed values are compared with measured data from [4, 8, 9] in Figs. 11, 12 and 13. The computed results are joined by a solid line.The agreement of the computed results with the data is excellent.

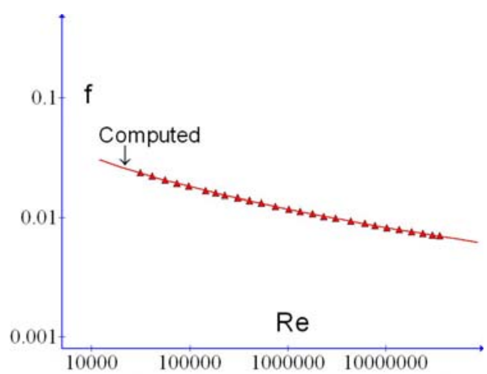

Fig. 11. The computed values of friction factor from equations (30) and (32) (continous curve) compared with [9].

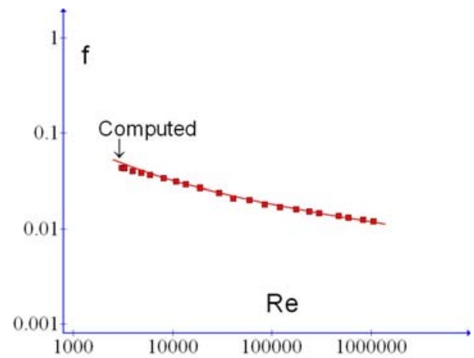

Fig. 12. The computed values of friction factor from equations (30) and (32) (fully drawn curve) compared with [8].

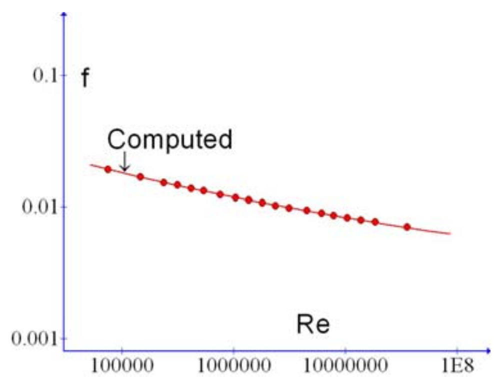

Fig. 13. The computed values of friction factor from equations (30) and (32) (fully drawn curve) compared with [4] .

6. Conclusion

From the above calculations, the following conclusions may be drawn:

i) The turbulent pipe flow profile can be divided into a region where Spalding's law is applied and a region governed by the Persen's wake law, which are quite appropriate for the description of the entire flow profile.

ii) The flow in the outer region is considered to be wake type. The wake law proposed [5], in analogy to jet flow is a unique choice as it describes the McKeon's experimental data precisely within a similarity framework.

iii) The common point (u_1^+, y_1^+) between two regions has been determined by employing the measured velocity data from both regions and satisfying a geometrical constraint.

iv) it has been shown that the co-ordinates (ξ, y_o^+) at the end of the profile layer lie on a curve, called the 'locus of ξ' e.g., equation (4.5).

v) Evaluation of the wake law (3.1) is clearly possible, taking equations (4.5), (4.8), (4.10) and (1.4) with u_∞^+ = 60 and a single input for Re.

vi) Further, it is clear that the proposed theory depends mainly on the parameters e.g., u_1^+, u_∞^+ and y_o^+ which are determined with high accuracy through mathematical procedure vis-à-vis the experimental information.

vii) Near to wall region described by law of the wall, represents a more or less constant portion of the total boundary layer as the Reynolds number increases.

vii) The friction factor computed from equation (5.9) and (5.11) is compared to experimental data from [4, 8, 9]. It is found that agreement is excellent.

A remark may finally be made that Spalding's law of the wall [6] and Persen's law of the wake, [5] are capable of explaining the structures of turbulence that prevail throughout the entire turbulent pipe flow boundary layer.

REFERENCES

[1] *D. E. Coles*, "The Law of the wake in the turbulent boundary layer", J. Fluid Mech., **vol. 1**. 1956, pp. 191-226

[2] *J. Laufer*, "The structure of turbulence in fully developed pipe flow", NACA Tech. Report, no**.** 1174**,** 1954, pp.417-434

[3] *H.P. Mazumdar and B.C. Mandal*, "On Persen theory of two dimensional turbulent boundary layer", J. Applied Mech. and Engg., **vol. 14**, no. 4, 2009, pp.1009-1028

[4] *B. J. McKeon, J. D. Li, W. Jiang, J.F. Morrison and A.J. Smits*, "Further observations on the mean velocity distribution in fully developed pipe flow", J. Fluid Mech., **vol. 501**, 2004, pp. 135-147.

[5] *L.N. Persen*, "The turbulent boundary layer and the closure problem", Proc. AGARD Conf. on Turbulent Boundary Layer – Experiments, Theory and Modelling, 1974, pp. 17-1 to 17-14.

[6] *L. Prandtl*, " Neuere ergebnisse der turbulenzforschung", Z. **vdi. 77**,1933, pp. 105-114.

[7] *D. B. Spalding*, "A single formula for the law of the wall", J. Applied Mechanics, **vol. 28**, Ser.E., 1961, pp. 455-458.

[8] *C. J. Swanson, B. Julian, G. G. Ihas, and R. J. Donnelly*, "Pipe flow measurements over a wide range of Reynolds numbers using liquid helium and various gases", J. Fluid Mech., **vol. 461,** 2002, pp. 51-60.

[9] *M. V. Zagarola, A. J. Smits*, "Mean flow scaling in turbulent pipe flow", J. Fluid Mech. , **vol. 373**, 1998, pp. 33-79.