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A NOTE ON ENRICHED CATEGORIES

Adriana Balan[1]

În această lucrare se arată că o categorie simetrică monoidală închisă bicompletă $\mathcal{V}$ cu biproduse indexate după o mulţime (mică) J are proprietatea că orice $\mathcal{V}$- categorie cu J obiecte este Morita echivalentă cu un monoid.

In this paper we show that a bicomplete symmetric monoidal closed category $\mathcal{V}$ having J-indexed biproducts, where J is small, has the property that any $\mathcal{V}$- enriched category of size J is Morita equivalent to a monoid.

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1. Introduction

In 1973, Lawvere popularized enriched categories by means of generalized metric spaces ([6]). Since this fundamental paper, various mathematical objects have been successfully coded as enrichments. The long list includes, among others, abelian categories, heavily used in commutative and non-commutative algebra, and order-enriched categories, a natural notion for domain theory and computer science. The easiest example of an enriched category is the one-object category whose arrows form a monoid in a monoidal category; the associated presheaf category is then nothing else than the category of objects on which the monoid acts.

In [8], it is shown that any small category enriched over *SupLat*, the category of sup-lattices and join-preserving maps, is Morita equivalent to a monoid, i.e. the corresponding category of presheaves is equivalent to the category of modules over a monoid. It is interesting to notice that the proof of the mentioned result relies only on the fact that *SupLat* has all small biproducts. Motivated by this, we shall extend in the present paper this result to any $\mathcal{V}$ symmetric monoidal closed category, complete and cocomplete, with zero object and having J-indexed biproducts, for a given small set J. We show that for any $\mathcal{V}$- category with J objects, the associated category of presheaves is $\mathcal{V}$- equivalent to the category of modules for a suitable monoid in $\mathcal{V}$.

[1]Lecturer, Dept. of Mathematics II, University Politehnica of Bucharest, Romania, e-mail address: `asteleanu@yahoo.com`.

2. **Preliminaires**

The main reference on enriched categories is Kelly's book ([4]). We shall use the same notation as in the quoted book.

Let $\mathscr{V} = (\mathscr{V}_0, \otimes, I, [,])$ be a symmetric monoidal closed category (short smcc), complete and cocomplete. Here $\otimes$, I, $[,]$ stand for the tensor product, unit object and internal hom respectively in $\mathscr{V}$. For simplicity, we shall write as $\mathscr{V}$ is strict, i.e. all associativity and unit isomorphisms are identities, as any monoidal category is equivalent to a strict one ([7]).

The main idea of enriched category theory is to generalize the notion of category so that, rather than having *hom-sets* of morphisms between pair of objects, one has *hom-objects* given as objects of the specified enriching category $\mathscr{V}$.

Definition 2.1. *A (small)* $\mathscr{V}$- category $\mathscr{C}$ *consists of a set* Ob$\mathscr{C}$ *of objects together with the following:*

(i) For every pair of objects X, Y, an associated hom-object $\mathscr{C}(X,Y) \in \mathscr{V}_0$*;*

(ii) An arrow in $\mathscr{V}_0$, called composition law $m_{XYZ} : (Y,Z) \otimes \mathscr{C}(X,Y) \longrightarrow \mathscr{C}(X,Z)$ *for each X,Y,Z;*

(iii) For each object X in $\mathscr{C}$, a distinguished arrow in $\mathscr{V}_0$ $j_A : I \longrightarrow \mathscr{C}(X,X)$, called identity*;*

such that for all objects X,Y,Z,U, the following diagrams commute:

$$\begin{tikzcd}
\mathscr{C}(Z,U)\otimes\mathscr{C}(Y,Z)\otimes\mathscr{C}(X,Y) \arrow[r, "Id\otimes m_{XYZ}"] \arrow[d, "m_{YZU}\otimes Id"'] & \mathscr{C}(Z,U)\otimes\mathscr{C}(X,Z) \arrow[d, "m_{XZU}"] \\
\mathscr{C}(Y,U)\otimes\mathscr{C}(X,Y) \arrow[r, "m_{XYU}"] & \mathscr{C}(X,U)
\end{tikzcd}$$

$$\begin{tikzcd}
I\otimes\mathscr{C}(X,Y) \arrow[r, "\cong"] \arrow[d, "j_Y\otimes Id"'] & \mathscr{C}(X,Y) & \mathscr{C}(X,Y)\otimes I \arrow[l, "\cong"'] \arrow[d, "Id\otimes j_X"] \\
\mathscr{C}(Y,Y)\otimes\mathscr{C}(X,Y) \arrow[ur, "m_{XYY}"'] & & \mathscr{C}(X,Y)\otimes\mathscr{C}(X,X) \arrow[ul, "m_{XXY}"]
\end{tikzcd}$$

In particular, for $\mathscr{V} = Set$, one recovers the usual notion of an ordinary (small and locally small) category. Categories like modules on a ring or vector spaces over a field are enriched in Ab, the category of abelian groups, with the tensor product as monoidal structure. The category of Hilbert spaces is enriched in that of Banach spaces, with the projective tensor product as monoidal structure. Now consider the poset $\mathbb{R}_+$ as a category. The addition of reals provides a symmetric monoidal structure on that category, with 0 as unit. Every metric space $(\mathscr{X}, d)$ can be viewed as a category $\mathscr{X}$ enriched in the monoidal category $\mathbb{R}_+$: the elements of $\mathscr{X}$ are the objects, and $\mathscr{X}(x,y) = d(x,y)$, the distance between those two points. In general, the category $\mathscr{V}$ is enriched over itself, with hom-objects given by the internal hom, $\mathscr{V}(X,Y) = [X,Y]$. In what follows, we shall write $\mathscr{V}$ when referring to it as an enriched or enriching in category, and $\mathscr{V}_0$ when we speak about the corresponding ordinary category. Every enriched category $\mathscr{C}$ has an underlying associated ordinary category $\mathscr{C}_0$, with same objects and hom-sets $\mathscr{C}_0(X,Y) = \mathscr{V}_0(I, \mathscr{C}(X,Y))$.

Definition 2.2. *For $\mathscr{V}$- categories $\mathscr{C}$ and $\mathscr{D}$, a $\mathscr{V}$-* functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ *consists of a function $F : \mathrm{Ob}\mathscr{C} \longrightarrow \mathrm{Ob}\mathscr{D}$, together with a family of arrows in $\mathscr{V}_0$, $F_{XY} : \mathscr{C}(X,Y) \longrightarrow \mathscr{D}(FX,FY)$, for each pair $X,Y \in \mathrm{Ob}\mathscr{C}$, subject to the following commutative diagrams:*

$$\begin{tikzcd}
\mathscr{C}(Y,Z)\otimes\mathscr{C}(X,Y) \arrow[r, "m_{XYZ}"] \arrow[d, "F_{YZ}\otimes F_{XY}"'] & \mathscr{C}(X,Z) \arrow[d, "F_{XZ}"] \\
\mathscr{D}(FY,FZ)\otimes\mathscr{D}(FX,FY) \arrow[r, "m_{FX,FY,FZ}"] & \mathscr{D}(FX,FZ)
\end{tikzcd}$$

$$\begin{tikzcd}
& I \arrow[dl, "j_X"'] \arrow[dr, "j_{FX}"] & \\
\mathscr{C}(X,X) \arrow[rr, "F_{XY}"] & & \mathscr{D}(FX,FX)
\end{tikzcd}$$

A $\mathscr{V}$- natural transformation between two $\mathscr{V}$- functors $F,G : \mathscr{C} \longrightarrow \mathscr{D}$ consists of a family of arrows in $\mathscr{V}_0$, $\alpha_X : I \longrightarrow \mathscr{D}(FX,GX)$ for each $X,Y \in \mathrm{Ob}\mathscr{C}$, such that the following diagram commutes

$$\begin{tikzcd}
& I\otimes\mathscr{C}(X,Y) \arrow[r, "\alpha_Y\otimes F_{XY}"] & \mathscr{D}(FY,GY)\otimes\mathscr{D}(FX,FY) \arrow[dr, "m_{FX,FY,GY}"] & \\
\mathscr{C}(X,Y) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & & \mathscr{D}(FX,GY) \\
& \mathscr{C}(X,Y)\otimes I \arrow[r, "G_{XY}\otimes\alpha_X"'] & \mathscr{D}(GX,GY)\otimes\mathscr{D}(FX,GX) \arrow[ur, "m_{FX,GX,GY}"'] &
\end{tikzcd}$$

Similarly to functor categories for ordinary functors between ordinary categories, enriched functors between two enriched categories form an enriched functor category, namely for $\mathscr{C}$ and $\mathscr{D}$ two $\mathscr{V}$- enriched categories with $\mathscr{C}$ small, there is a $\mathscr{V}$- enriched category, denoted $[\mathscr{C},\mathscr{D}]$, whose objects are the $\mathscr{V}$- functors $F : \mathscr{C} \longrightarrow \mathscr{D}$. The hom-objects $[\mathscr{C},\mathscr{D}](F,G)$ between $\mathscr{V}$- functors $F,G : \mathscr{C} \longrightarrow \mathscr{D}$ are given by the $\mathscr{V}$- enriched end, i. e. the equalizer in $\mathscr{V}_0$ (see [4], Sect. 2.1)

$$[\mathscr{C},\mathscr{D}](F,G) \rightarrowtail \prod_{X\in\mathscr{C}} \mathscr{D}(FX,GX) \rightrightarrows \prod_{X,Y\in\mathscr{C}} [\mathscr{C}(X,Y),\mathscr{D}(FX,GY)]$$

The underlying ordinary category of a functor category is the category of $\mathscr{V}$- functors from $\mathscr{C}$ to $\mathscr{D}$ and $\mathscr{V}$- natural transformations ([4], Sect.1.2) between them. For $\mathscr{V} = \mathit{Set}$, the $\mathscr{V}$- enriched functor category coincides with the ordinary functor category.

For any small $\mathscr{V}$- category $\mathscr{C}$, the $\mathscr{V}$- functor category $[\mathscr{C}^{op},\mathscr{V}]$ is usually called the *category of presheaves* over $\mathscr{C}$.

To finish this Section, we need to recall the notions of enriched limit and colimit. A *weight* is a functor $F : \mathscr{K}^{op} \longrightarrow \mathscr{V}$ with small domain. The F- weighted limit $\{F,G\}$ of a functor $G : \mathscr{K}^{op} \longrightarrow \mathscr{C}$ is defined representably by $\mathscr{C}(X,\{F,G\}) = [\mathscr{K}^{op},\mathscr{V}](F,\mathscr{C}(X,G-))$, while the F-weighted colimit $F * G$ of $G : \mathscr{K} \longrightarrow \mathscr{C}$ is defined dually by $\mathscr{C}(F * G,X) = [\mathscr{K}^{op},\mathscr{V}](F,\mathscr{C}(G-,X))$. Usual (co)limits for a $\mathscr{V}$- category $\mathscr{C}$ have a corresponding in enriched setting, called conical (co)limits. These exist, for example if $\mathscr{C} = \mathscr{V}$ ([4], Sect. 3.8).

An enriched category is (small) complete when admits all small (weighted) limits, and dually cocomplete if it has all small (weighted) colimits. In particular, $\mathscr{V}$ is complete and cocomplete as an enriched category. In an enriched functor category $[\mathscr{C},\mathscr{D}]$, with $\mathscr{C}$ small and $\mathscr{D}$ complete, limits are computed pointwise, i.e. $\{F,G\}X = \{F,(G-)X\}$ for $F : \mathscr{K}^{op} \longrightarrow \mathscr{V}$, $G : \mathscr{K}^{op} \longrightarrow [\mathscr{C},\mathscr{D}]$ and $X \in \mathscr{C}$. Similarly for colimits; in particular, the category of presheaves $[\mathscr{C}^{op},\mathscr{V}]$ is a complete and cocomplete $\mathscr{V}$- category, as $\mathscr{V}$ is so.

A $\mathscr{V}$- functor $F : \mathscr{C}^{op} \longrightarrow \mathscr{V}$ is called *small projective* if $[\mathscr{C}^{op},\mathscr{V}](F,-)$ preserves all (small) colimits. It is said to be *dense* if $\widetilde{F} : \mathscr{V} \longrightarrow [\mathscr{C},\mathscr{V}]$, $\widetilde{F}C = \mathscr{C}(F-,C)$ is fully faithful (i.e. the correspondence on arrows is an isomorphism in $\mathscr{V}_0$), and a *strong generator* if $\widetilde{F}$ is conservative (i.e.the underlying ordinary functor $\widetilde{F}_0 : \mathscr{C}_0^{op} \longrightarrow \mathscr{V}_0$) reflects isomorphisms).

The *Cauchy completion* of a $\mathscr{V}$- category $\mathscr{C}$ is the full subcategory of $[\mathscr{C}^{op},\mathscr{V}]$ determined by the small-projectives functors. Two small $\mathscr{V}$- categories $\mathscr{C},\mathscr{D}$ are called *Morita equivalent* if their Cauchy completions are equivalent, or equivalently if $[\mathscr{C}^{op},\mathscr{V}] \cong [\mathscr{D}^{op},\mathscr{V}]$ ([4], Sect. 5.5).

3. **Main result**

From now on we require that $\mathscr{V}_0$ has a zero object 0. We shall also assume that there is a small indexing set J, such that in $\mathscr{V}_0$, J-coproducts are naturally isomorphic to J-products. According to [5] or [2], the canonical natural map below has to be also an isomorphism:

$$\sum_i X_i \longrightarrow \prod_i X_i$$

where the components are $X_i \xrightarrow{Id} X_i$ and $X_i \longrightarrow 0 \longrightarrow X_j$ for $i \neq j$. In what follows, we shall denote these biproducts by $\bigoplus$. Well-known examples of such categories are *CommMon*, the category of commutative monoids, *Ab*, the category of abelian groups (both having all finite biproducts - or more generally, the category of modules over any semi-ring) and *SupLat*, the latter having all small biproducts.

We are now ready to state and prove the main result of this paper:

Theorem 3.1. *Let $\mathscr{C}$ be a small $\mathscr{V}$- category, whose underlying set of objects is indexed by J. Then $\mathscr{C}$ is Morita equivalent to a monoid in $\mathscr{V}$.*

Proof. The proof follows the same ideas as in [8]. Namely, consider the $\mathscr{V}$- functor $P:\mathscr{C}^{op}\longrightarrow\mathscr{V}$ given by $P=\bigoplus_{X\in\mathscr{C}}\mathscr{C}(-,X)$. Notice that this makes sense as biproduct of functors by the assumption on size of $\mathscr{C}$. Moreover, a similar argument as in [3] shows that is small projective, as for any small colimit $F\star G$, with $F:\mathscr{K}^{op}\longrightarrow\mathscr{V}$, $G:\mathscr{K}\longrightarrow[\mathscr{C}^{op},\mathscr{V}]$ and $\mathscr{K}$ small, we have

$$\begin{aligned}
[\mathscr{C}^{op},\mathscr{V}](\bigoplus_{X\in\mathscr{C}}\mathscr{C}(-,X),F\star G) &\cong \bigoplus_{X\in\mathscr{C}}[\mathscr{C}^{op},\mathscr{V}](\mathscr{C}(-,X),F\star G)\\
\text{(Yoneda Lemma)} &\cong \bigoplus_{X\in\mathscr{C}}(F\star G(X))\\
&\cong F\star\bigoplus_{X\in\mathscr{C}}G(X)\\
\text{(Yoneda Lemma)} &\cong F\star\bigoplus_{X\in\mathscr{C}}[\mathscr{C}^{op},\mathscr{V}](\mathscr{C}(-,X),G)\\
&\cong F\star[\mathscr{C}^{op},\mathscr{V}](\bigoplus_{X\in\mathscr{C}}\mathscr{C}(-,X),G)\\
&= F\star[\mathscr{C}^{op},\mathscr{V}](P,G)
\end{aligned}$$

Now consider the full subcategory $\mathscr{G}$ of $[\mathscr{C}^{op},\mathscr{V}]$ determinated by all the representables $\mathscr{C}(-,X)$, $X\in\mathscr{C}$. This is a dense small $\mathscr{V}$- subcategory in $[\mathscr{C}^{op},\mathscr{V}]$. As no $[\mathscr{C}^{op},\mathscr{V}]_0(\mathscr{C}(-,X),\mathscr{C}(-,X'))$ is empty ($\mathscr{V}$ has a zero object, hence all presheaves categories have same property; in particular, $[\mathscr{C}^{op},\mathscr{V}]_0(\mathscr{C}(-,X),\mathscr{C}(-,X'))$ is not empty since it contains the zero morphism), by Proposition 5.22 of [4], it follows that P is a dense $\mathscr{V}$- functor in $[\mathscr{C}^{op},\mathscr{V}]$. In particular, it is a strong generator, which is the same as saying that $[P]$, the full $\mathscr{V}$- subcategory of $[\mathscr{C}^{op},\mathscr{V}]$ generated by P, is strongly generating.

We shall need now the following result from [4] that we include for completeness:

Theorem 3.2. *(*[4]*, Th. 5.26) A necessary and sufficient condition for a $\mathscr{V}$- category $\mathscr{C}$ to be equivalent to $[\mathscr{A}^{op},\mathscr{V}]$ for some small $\mathscr{A}$, is that $\mathscr{C}$ is cocomplete and that there is a small set of small-projective objects in $\mathscr{C}$ constituting a strong generator for $\mathscr{C}$.*

Hence by the above theorem, we have an equivalence of $\mathscr{V}$- categories $[\mathscr{C}^{op},\mathscr{V}]\cong[[P]^{op},\mathscr{V}]$ given by the functor $[\mathscr{C}^{op},\mathscr{V}](P,-)$. Therefore $[\mathscr{C}^{op},\mathscr{V}]$ is equivalent to the category of modules over the $\mathscr{V}$- monoid $[\mathscr{C}^{op},\mathscr{V}](P,P)$. □

Remark 3.3. (i) A category having zero object and biproducts of size J has also J'-biproducts for any set J' of cardinality smaller than J, hence the above theorem extends to all enriched categories of size at most J.

(ii) Notice that the above proof heavily relies on the existence of the zero object: first, because biproducts are usually associated with the existence of the zero object and second, the underlying category of the presheaves category $[\mathscr{C}^{op}, \mathscr{V}]$ has at least one map between any two objects, in particular between representables. One could overcome both these issues by assuming, for example that $\mathscr{V}$ itself is enriched over pointed sets (i.e. it has zero maps), for then the *PointedSets*-enrichment will be automatically transferred to all $\mathscr{V}$- categories (as $\mathscr{C}_0(X,Y) = \mathscr{V}_0(I, \mathscr{C}(X,Y))$ for $X, Y \in \mathrm{Ob}\mathscr{C}$).

4. **Examples**

(i) Consider the case $J = \emptyset$. For any $\mathscr{V}$, it exists the empty $\mathscr{V}$- category 0 with no objects and no hom-objects. The associated presheaf category $[0^{op}, \mathscr{V}]$ is the terminal category 1; its unique object is the unique $\mathscr{V}$- functor $! : 0^{op} \longrightarrow \mathscr{V}$ and $1(!,!)$ is the terminal object 1 of $\mathscr{V}$ (which exists as $\mathscr{V}$ is assumed complete). The terminal object in $\mathscr{V}_0$ naturally carries a monoid structure. Now if the terminal object is also initial, i.e. a zero object, then it is the only module over itself: for $X \otimes 0 \cong 0$ (as $X \otimes -$ has left adjoint, hence it preserves colimits) and if X is a module over 0, then by

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[drr, equal] & X \otimes I \arrow[r] & X \otimes 0 \arrow[r, "\cong"] \arrow[d] & 0 \\
& & X &
\end{tikzcd}$$

it follows that for X, the identity morphism is also the zero map, hence X is itself the zero object. Here the horizontal arrow is the tensor product of identity with the unique morphism into the terminal object, and the vertical arrow is the module structure. So the theorem follows: $[0^{op}, \mathscr{V}]$ is the (one-object) category of modules over the zero monoid.

It is interesting to notice that for $J = \emptyset$, the converse also holds:

Proposition 4.1. *If there is a monoid M in $\mathscr{V}_0$ such that the category of modules over it, seen as an enriched category, is (isomorphic to) the terminal category above, then $0 \cong 1$ in $\mathscr{V}_0$.*

Proof. Any monoid M in a monoidal category is automatically a (left and right) module over itself. The hypothesis tells us that there is no other module; in particular, the free functor $M \otimes - : \mathscr{V}_0 \longrightarrow M - mod$ is naturally isomorphic to the constant functor at M. It follows that for any arrow $f : X \longrightarrow Y$ in $\mathscr{V}_0$, $Id_M \otimes f : M \otimes X \longrightarrow M \otimes Y$ is an isomorphism; for $0 \longrightarrow Y$ the unique map from the initial object, we get $0 \cong M \otimes 0 \cong M \otimes Y$ for any Y. Taking $Y = I$ gives us $M \cong 0$. Now, the terminal object is a module over any monoid with obvious structure maps, hence also over M. As $M - mod$ has only one object, namely M itself as noticed earlier, we must also have $M \cong 1$. $\square$

(ii) For any category, products or coproducts indexed by a singleton set are trivial, hence we recover the (obvious) fact that a $\mathscr{V}$- enriched category, for any $\mathscr{V}$ (smcc bicomplete) with one object is Morita equivalent to a monoid, being itself one.

(iii) The case of binary biproducts is well-known for long time for $\mathscr{V} = CommMon$, Ab, $R-Mod$ (for R a (semi)ring): a $\mathscr{V}$- category $\mathscr{C}$ with two objects X_1,X_2 is the same as a Morita context: two (semi)rings, respectively R-algebras $\mathscr{C}(X_1,X_1)$ and $\mathscr{C}(X_2,X_2)$ and two bimodules $\mathscr{C}(X_1,X_2)$ and $\mathscr{C}(X_2,X_1)$, connected by $\mathscr{C}(X_1,X_2)\otimes\mathscr{C}(X_2,X_1) \longrightarrow \mathscr{C}(X_1,X_1)$ and $\mathscr{C}(X_2,X_1)\otimes\mathscr{C}(X_1,X_2) \longrightarrow \mathscr{C}(X_2,X_2)$. Then the $\mathscr{V}$- monoid Morita equivalent to $\mathscr{C}$ is the matrix (semi)ring $\begin{pmatrix} \mathscr{C}(X_1,X_1) & \mathscr{C}(X_1,X_2) \\ \mathscr{C}(X_2,X_1) & \mathscr{C}(X_2,X_2) \end{pmatrix}$. Actually there is more to say in case $\mathscr{V} = Ab$: the category of presheaves for a small non-empty Ab-category is equivalent to a module category if and only if it has a finite cover ([9], Th. 8.1 for $\mathscr{A} = Ab$). In particular, any Ab-enriched (known as additive) finite category is Morita equivalent to a monoid.

(iv) For an example concerning all small biproducts, take $\mathscr{V} = SupLat$, as in [8]. A monoid in $SupLat$ is called a quantale. Equivalently, it is a poset having all joins and an associative, unital product which distributes over joins. A quantaloid is a quantale with several objects, i.e. a category enriched in $SupLat$. By [8], every small quantaloid $\mathscr{Q}$ is Morita-equivalent to a quantale, namely with $Q = Matr(\mathscr{Q})(\mathscr{Q}_0,\mathscr{Q}_0)$, where $Matr(\mathscr{Q})$ is the quantaloid of matrices with elements in $\mathscr{Q}$ (see [1]).

5. **Conclusions**

We have generalized the well-known example of Morita context of rings for any J- indexed $\mathscr{V}$- categories in case $\mathscr{V}$ has J- biproducts. It is still under consideration if the converse also holds, namely for a small bicomplete smcc $\mathscr{V}$ with zero object, if there is some small set J such that any small J- indexed $\mathscr{V}$-category is Morita equivalent to a monoid, then $\mathscr{V}$ has J- biproducts.

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