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SOLVING DIFFERENCE AND DIFFERENTIAL EQUATIONS BY DISCRETE DECONVOLUTION

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Prezentăm modul în care, folosind convoluţia şi deconvoluţia discretă, pot fi calculate valorile numerice ale soluţiilor problemelor de valori iniţiale şi la frontieră pentru ecuaţiile liniare neomogene cu coeficienţi constanţi, atât cu diferenţe cât şi diferenţiale. Sunt incluse exemple rezolvate. Metoda poate fi utilizată uşor în aplicaţii şi implementată pe calculator.

We present the way in which, using the discrete convolution and deconvolution, can be computed the numerical values of the solutions both of the initial and boundary value problems for linear nonhomogeneous difference and differential equations with constant coefficients. Worked examples are included. The method can be easily used in applications and implemented on a computer.

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1. Introduction

The convolution is a fundamental concept in mathematics and applications, its importance being in growth in the last time. The use of the convolution and others related notions, as its inverse - the deconvolution, to solve several kinds of equations is particularly of a great importance.

For that purpose, in the present paper we use the discrete convolution and deconvolution, to obtain the numerical solutions of the initial and boundary value problems for linear nonhomogeneous difference equations and differential equations with constant coefficients. In whole paper, the method will be exemplified by several worked examples.

In the section 1 of the paper, we present the notions of the convolution and deconvolution, for finite sequences of the same length and also for infinite sequences. The definitions of these notions come back to A.Cauchy and are presented, for example, in [3].

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Section 2 is concerned with the exact numerical solution of the linear nonhomogeneous difference equations (linear recurrence equations) with constant coefficients, of order n. Using the notions presented in the section 1, our main theorem gives a formula for solving such equations, if n initial values of the solution are known. It is also considered the generalized case when the n known values of the solution, are not necessary the first ones. Of course, in this case the existence or the unicity of the solution are not assured. This new method, based especially on the discrete deconvolution, gives a direct and rapid algorithm for obtain the exact numerical values of the solutions of the difference equations, instead of the laborious usual methods.In section 3, we apply the deconvolution method to obtain the approximate solution of the initial and boundary value problems, for nonhomogeneous linear differential equations with constant coefficients. This is made by replacing the unknown of the equation by the finite sequence of its values in the points of an equidistant net of values of the variable and by approximating the derivatives of the unknown function by their finite differences. Examples for second and third order equations are given.

We apply the method both for the classical initial and boundary value problems and for their generalizations. Thus, the boundary value problem is considered too in the case when the values of the unknown function and eventually of its derivatives are given not only in the extremities of the definition interval, but even for some intermediate values. The initial value problem is also considered when the values in the initial point of the unknown function and its derivatives of several orders, not necessary the firsts, are given.

In a subsequent paper [1], we shall use the present results for numerical computation of the polynomial roots. Also, in [2], the discrete convolution and deconvolution are used to solve nonhomogeneous linear differential equations.

2. Discrete convolution and deconvolution

We call *discrete convolution* (or *Cauchy product*) of two finite sequences of real or complex numbers, with the same number of terms, $a = (a_0; a_1; \ldots; a_k)$ and $b = (b_0; b_1; ...; b_k)$, the finite sequence

$$c = a * b = (c_0; c_1; \ldots; c_k) \tag{1}$$

given by the relations (see for example [2])

$$c_0 = a_0 b_0, c_1 = a_1 b_0 + a_0 b_1, \ldots, c_k = \sum_{j=0}^{k} a_{k-j} b_j \tag{2}$$

The convolution product is commutative, associative, distributive related to the addition of the sequences and has the unit $\delta = (1;0;0;...;0)$. The addition and the multiplication with scalars of the sequences are those usual.

In the case when the finite sequences a and c are known and $a_0 \neq 0$, we can determinate the finite sequence b, such that the relation (1) to be satisfied. This sequence is named the *deconvolution* (see [2]) of the sequence c by the sequence a, is denoted

$$b = c/a \tag{3}$$

and its terms are given by the relations

$$b_0 = \frac{c_0}{a_0}, b_1 = \frac{1}{a_0}(c_1 - a_1 b_0), \ldots, b_k = \frac{1}{a_0}\left(c_k - \sum_{j=0}^{k-1} a_{k-j} b_j\right) \quad , \tag{4}$$

hence can be computed by the algorithm

$$\begin{array}{cccc|cccc}
c_0 & c_1 & \cdots & c_k & a_0 & a_1 & \cdots & a_k \\ \cline{5-6}
c_0 & a_1 b_0 & \cdots & a_k b_0 & b_0 = \dfrac{c_0}{a_0} & b_1 = \dfrac{c_1 - a_1 b_0}{a_0} & \cdots & b_k \\ \cline{1-3}
 & / & c_1 - a_1 b_0 & \cdots \; c_k - a_k b_0 & & & & \\
 & & c_1 - a_1 b_0 & \cdots \; a_k b_1 & & & & \\ \cline{1-3}
 & / & \cdots & & & & &
\end{array} \tag{5}$$

Denoting $a^{-1} = \delta/a$, the inverse of the finite sequence a ,we have

$$c/a = c * a^{-1} \; . \tag{6}$$

Although we will not used here, we mention the matrix possibility for calculus of the convolution and deconvolution, given by the relation

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \cdot \begin{pmatrix} b_0 & b_1 & b_2 & \cdots \\ 0 & b_0 & b_1 & \cdots \\ 0 & 0 & b_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots \\ 0 & c_0 & c_1 & \cdots \\ 0 & 0 & c_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad ,$$

where a , b and $c = a * b$ are the finite sequences considered above.

The notions of convolution and deconvolution can also be considered, with the same definitions and notations, in the case of the infinite sequences. In this case the natural number k from the above definitions is arbitrary, hence

$$(a_0; a_1; \ldots; a_k; \ldots) * (b_0; b_1; \ldots; b_k; \ldots) =$$

$$=\left(\left(a_0;a_1;\dots;a_k\right)*\left(b_0;b_1;\dots;b_k\right):k=0,1,2,\dots\right)$$

If we multiply two absolute convergent power series having the coefficients a_k and b_k, then the result absolute convergent power series has the components of the convolution product $c=a*b$ as coefficients (Cauchy theorem). Consequently there results that the deconvolution of the infinite sequences corresponds to the division of the power series. Contrary with the situation at infinite sequences, in the case of finite sequences (of the same length), both the convolution and the deconvolution are not exactly the same with the multiplication and respectively the division of polynomials. This observation must be take into account, the more so as in MathLab, for example, the instructions "conv" and "deconv" means the multiplication and the division of the polynomials.

If a is a finite sequence and b a finite or infinite sequence, we denote $(a;b)$ the sequence obtained by adjoining the two sequences.

3. Linear difference equations with constant coefficients

3.1. The main result

We consider the nonhomogeneous linear difference equation

$$\sum_{j=0}^{n} a_{n-j}u_{j+k}=b_k\,,\ k=0,1,\dots\ , \tag{7}$$

with the coefficients $a_0\neq 0,\ a_1,\dots,a_n$ and right terms $b_0,b_1,\dots,b_k,\dots$.

We denote

$$a=\left(a_0;a_1;\dots;a_n;0;0;\dots\right)=\left(a_0;a_1;\dots;a_k;\dots\right)\qquad, \tag{8}$$

with the convention that $a_k=0$,if $k>n$,and

$$b=\left(b_0;b_1;\dots;b_k;\dots\right)\quad . \tag{9}$$

THEOREM . *The unique solution* $u=\left(u_0;u_1;\dots;u_k;\dots\right)$ *of the equation* (7) *with the known initial values* $u_0,u_1,\dots,u_{n-1}$, *is given by the formula*

$$u=\left(a_0u_0;a_1u_0+a_0u_1;\sum_{j=0}^{n-1}a_{n-1-j}u_j;b_0;b_1;\dots\right)\Big/\left(a_0;\dots;a_n;0;0;\dots\right)=$$

$$=\left(\left(a_0;a_1;\dots;a_{n-1}\right)*\left(u_0;u_1;\dots;u_{n-1}\right);b\right)*a^{-1}= \tag{10}$$

$$=(\underbrace{0;\dots;0}_{n};b*a^{-1})+u_0\cdot(a_0;a_1;\dots;a_{n-1};0;0;\dots)*a^{-1}+$$

$$+u_1\cdot(0;a_0;\dots;a_{n-2};0;0;\dots)*a^{-1}+\dots+u_{n-1}\cdot(\underbrace{0;\dots;0}_{n-1};a_0;0;0;\dots)*a^{-1}\quad .$$

Proof. Let denote

$$c = (c_0; c_1; \ldots; c_k; \ldots) = a * u \quad , \tag{11}$$

From the definition (2) of the convolution product, it results that

$$c_0 = a_0 u_0, c_1 = a_1 u_0 + a_0 u_1, \ldots, c_{n-1} = \sum_{j=0}^{n-1} a_{n-1-j} u_j \quad ,$$

hence

$$(c_0; c_1; \ldots; c_{n-1}) = (a_0; a_1; \ldots; a_{n-1}) * (u_0; u_1; \ldots; u_{n-1}) \tag{12}$$

If u is solution of the equation (7), making the index change $j = i + k$ and taking into account that $a_{n+k} = 0, k = 1,2,\ldots$, it results

$$c_{n+k} = \sum_{j=0}^{n+k} a_{n+k-j} u_j = \sum_{i=-k}^{n} a_{n-i} u_{i+k} = \sum_{i=0}^{n} a_{n-i} u_{i+k} = b_k \ , \tag{13}$$

$$k = 0,1,2,\ldots$$

From the relations (12) and (13), it results that the sequence c given by the formula (11) has the form

$$c = ((a_0; \ldots; a_{n-1}) * (u_0; \ldots; u_{n-1}); b) =$$

$$= \left(a_0 u_0; a_1 u_0 + a_0 u_1; \ldots; \sum_{j=0}^{n-1} a_{n-1-j} u_j; b_0; b_1; \ldots; b_k; \ldots\right) \tag{14}$$

Finally, from (11) and (14) , it results that the solution u of the equation (7) with the known initial values $u_0, u_1, \ldots, u_{n-1}$ is given by the formula (10).

Conversely, if the sequence u is given by the formula (10), then in conformity with the notation (11), we have

$$c = (c_0; \ldots; c_{n-1}; c_n; \ldots; c_{n+k}; \ldots) = a * u = ((a_0; \ldots; a_{n-1}) * (u_0; \ldots; u_{n-1}); b) \ ,$$

from where it results the relation (12) and

$$c_{n+k} = b_k \ , \ k = 0,1,\ldots \ . \tag{15}$$

Computing the deconvolution

$u = c / a = \left(a_0 u_0; a_1 u_0 + a_0 u_1; \ldots; \sum_{j=0}^{n-1} a_{n-1-j} u_j; c_n; \ldots; c_{n+k}; \ldots\right) / a$, we see that the first n components of the sequence u are the numbers $u_0, u_1, \ldots, u_{n-1}$ Making the index change $i = j + k$, taking into account that $a_{n+k} = 0, k = 1,2,\ldots$ and the formulae (2), (11) and (15), it results $\sum_{j=0}^{n} a_{n-j} u_{j+k} = \sum_{i=k}^{n+k} a_{n+k-i} u_i = \sum_{i=0}^{n+k} a_{n+k-i} u_i = c_{n+k} = b_k \ , k = 0,1,\ldots$, hence the sequence u given by the formula (10), satisfies the equation (7).

Observation. The last form of u in (10), that obviously results from the first one, express the fact that the general solution of the nonhomogeneous difference equation (7) is the sum between its particular solution $(\underbrace{0;\cdots;0}_{n};b*a^{-1})$ and the general solution of the homogeneous equation

$$\sum_{j=0}^{n} a_{n-j}u_{j+k} = 0\ ,\ \ k = 0,1,\ldots\ , \tag{16}$$

associated to (7), that is the linear combination with arbitrary coefficients $u_0,\ldots,u_{n-1}$ of the elements $(a_0;\cdots;a_{n-1};0;0;\cdots)*a^{-1}$, $(0;a_0;\cdots;a_{n-2};0;0;\cdots)*a^{-1}$ $,\ldots,$ $(\underbrace{0;\ldots;0}_{n-1};a_0;0;0;\ldots)*a^{-1}$ of the base of the n- dimensional vector space of all solutions of the homogeneous equation.

3.2. Initial value (Cauchy) problem

If we known the first n components $u_0,u_1,\ldots,u_{n-1}$, named *initial values*, of the solution u of the equation (7), the formula (10) gives the possibility to compute how much components of u are required.

Example 1. The initial value problem formed by the linear difference equation $u_{k+2} - 2u_{k+1} - 3u_k = k,\quad k = 0,1,2,...$ and the initial conditions $u_0 = u_1 = 1$, has $n = 3$, $a = (1;-2;-3;0;0;\ldots), b = (0;1;2;\ldots)$,

$$c = ((a_0;a_1)*(u_0;u_1);b) = ((1;-2)*(1;1);b) = (1;-1;0;1;2;3;\ldots).$$

The deconvolution algorithm

```
1 -1  0  1  2 ... | 1 -2 -3  0  0 ...
1 -2 -3  0  0 ... | 1  1  5 14 45 ...
-----------------
   1  3  1  2 ...
   1 -2 -3  0 ...
   --------------
      5  4  2 ...
      5 -10 -15 ...
      ----------
        14 17 ...
        14 -28 ...
        -------
           45 ...
           45 ...
           ------
              ...
```

gives the sequence $u = c/a = (1;1;5;14;45;...)$ as solution of the Cauchy problem.

3.3. Boundary value problem

We consider now the problem of finding the solution $u=(u_0;u_1;\ldots;u_k;\ldots)$ of the difference equation (7) for which n components $u_{k_1},u_{k_2},\ldots,u_{k_n}$, named *generalized boundary values*, are known. Making equals these values with the corresponding components of the solution of the equation (7), given by the last form of the formula (10), we obtain a linear algebraic system of order n, from which it is eventually possible to compute the initial values $u_0,u_1,\ldots,u_{n-1}$. Replacing these values in the formula (10), we obtain the solution of the considered boundary value problem. As will be see in the example 3 below, such a generalized boundary value problem can to have an unique solution, an infinity of solutions or can not have any solution.

Example 2 . We consider the same equation as in **example 1**, but now with the boundary conditions $u_5=45, u_7=409$. Because

$$a^{-1}=(1;-2;-3;0;0;\ldots)^{-1}=(1;0;0;\ldots)/(1;-2;-3;0;0;\ldots)=$$
$$=(1;2;7;20;61;182;547;1640;\ldots)\ ,$$

(here and in the following examples, the deconvolution algorithms not be effectively presented), we have

$$u=(0;0;b*a^{-1})+u_0\cdot(a_0;a_1;0;0;\ldots)*a^{-1}+u_1\cdot(0;a_0;0;0;\ldots)*a^{-1}=$$
$$=(0;0;b*a^{-1})+u_0\cdot(1;-2;0;0;\ldots)*a^{-1}+u_1\cdot(0;1;0;0;\ldots)*a^{-1}=$$
$$=(0;0;0;1;4;14;44;135;\ldots)+u_0\cdot(1;0;3;6;21;60;183;546;\ldots)+$$
$$+u_1\cdot(0;1;2;7;20;61;182;547;\ldots).$$

The boundary value conditions $u_5=4+21u_0+20u_1=45$, $u_7=44+183u_0+182u_1=409$ gives $u_0=u_1=1$. Replacing these values in the above formula for u , we obtain the same value for u as in **example 1**, namely $u=(1;1;5;14;45;135;409;1228;\ldots)$.

Example 3. The difference equation $u_{k+2}+u_k=b_k$, $k=0,1,2,\ldots$, with $b=(b_0;b_1;\ldots;b_k;\ldots)=(2;0;-2;0;2;0;-2;0;2;\ldots)$, has $a=(1;0;1;0;0;\ldots)$, and $a^{-1}=(1;0;-1;0;1;0;-1;0;\ldots)$, hence

$$u=(0;0;b*a^{-1})+u_0\cdot(1;0;0;\ldots)*a^{-1}+u_1\cdot(0;1;0;0;\ldots)*a^{-1}=$$
$$=(0;0;2;0;-4;0;6;0;\ldots)+u_0\cdot(1;0;-1;0;1;0;-1;0;1;\ldots)+u_1\cdot(0;1;0;-1;0;1;0;-1;\ldots)$$

a) If $u_3=0,u_4=-3$, it results $u_1=-u_3=0$ and $u_4=-4+u_0=-3$ hence $u_0=1$. The boundary value problem has the unique solution $u=(1;0;1;0;-3;0;5;0;\ldots)$.

b) If $u_2=1,u_4=-3$, from $u_2=2-u_0=1$ and $u_4=-4+u_0=-3$, it results only $u_0=1$, so the boundary value problem has an infinity of solutions given by the relation $u=(1;0;1;0;-3;0;5;0;\dots)+u_1\cdot(0;1;0;-1;0;1;-1;0;\dots)$, where u_1 is an arbitrary parameter.

c) If $u_2\neq 1, u_4=-3$, it results both $u_0\neq 1$ and $u_0=1$, contradiction, the considered boundary value problem having not solution.

4. Linear differential equations with constant coefficients

4.1. Discreetization of a differential equation

Let be the linear differential equation with constant coefficients

$$\sum_{j=0}^{n}\alpha_{n-j}u^{(j)}(x)=f(x) \quad , \tag{17}$$

where x is a real variable. We consider a net $x_k=x_0+kh, k=0,1,2,\dots,m$, where x_0 is a fixed real number, $h>0$ (or $h<0$) is the step of the net and $m>n$. We denote

$$u_k=u(x_k) \quad , \quad f_k=f(x_k) \quad , \quad k=0,1,2,\dots \tag{18}$$

If approximate the derivatives of the unknown $u(x)$ for $x=x_k$ by the usual formulas

$$u'(x_k)\cong\frac{u_{k+1}-u_k}{h}, u''(x_k)\cong\frac{u_{k+2}-2u_{k+1}+u_k}{h^2},$$
$$u'''(x_k)\cong\frac{u_{k+3}-3u_{k+2}+3u_{k+1}-u_k}{h^3} ,\dots,$$
$$u^{(j)}(x_k)\cong\frac{(-1)^j}{h^j}\sum_{i=0}^{j}(-1)^i\binom{j}{i}u_{k+i} , \quad j,k=0,1,2,\dots \quad , \tag{19}$$

the numerical values u_k of the solution u of the differential equation (17) in the net points x_k will be the solutions of a difference equation of form (7), where

$$a_j=\sum_{i=0}^{j}(-1)^i\binom{n-j+i}{n-j}h^{j-i}\alpha_{j-i} \quad , \quad b_k=h^n f_k \ , \tag{20}$$
$$j=0,1,\dots,n \quad , k=0,1,2,\dots$$

Example 4. The differential equation of second order

$$\alpha_0u''(x)+\alpha_1u'(x)+\alpha_2u(x)=f(x) \tag{21}$$

is reduced to the difference equation

$$\alpha_0 u_{k+2} + (h\alpha_1 - 2\alpha_0)u_{k+1} + (\alpha_0 - h\alpha_1 + h^2\alpha_2)u_k = h^2 f(x_k), \qquad (22)$$
$$k = 0,1,2,\dots$$

Example 5. The differential equation of third order

$$\alpha_0 u'''(x) + \alpha_1 u''(x) + \alpha_2 u'(x) + \alpha_3 u(x) = f(x) \qquad (23)$$

is reduced to the difference equation

$$\alpha_0 u_{k+3} - (3\alpha_0 - h\alpha_1)u_{k+2} + (3\alpha_0 - 2h\alpha_1 + h^2\alpha_2)u_{k+1} -$$

$$-(\alpha_0 - h\alpha_1 + h^2\alpha_2 - h^3\alpha_3)u_k = h^3 f_k, \quad k = 0,1,2,\dots \,. \qquad (24)$$

4.2. Boundary value problem

In conformity with those who was mentioned in the section 3.1, the differential equation (17), with the unknown $u(x)$, is reduced to the difference equation (7) with the coefficients given by the formula (20) and the unknown $u = (u_0; u_1; \dots; u_k; \dots)$, where $u_k = u(x_k)$ and $x_k = x_0 + kh$, $k = 0,1,2,\dots,m$.

We suppose that the unknown function $u(x)$ and eventually some of its derivatives (see **example 8** below) in n different values x_k are given. From these *generalized boundary conditions* can be eventually deduced n components of the sequence $u = (u_0; u_1; \dots; u_k; \dots)$. Using the deconvolution method presented in the section 2, we can determine other components u_k of u, these being the approximate values of the unknown $u(x)$ of the differential equations in the considered values of the variable.

Example 6. We consider the differential equation $u''(x) - u(x) = 0$ .For $t_0 = 0$ and $h = 0.1$, hence for the net $x_k = 0.1 \cdot k$, $k = 0,1,\dots$, in conformity with the formula (21), the differential equation is reduce to the linear difference equation $u_{k+2} - 2 \cdot u_{k+1} + 0.99 \cdot u_k = 0$ with the unknowns $u_k = u(0.1 \cdot k), k = 0,1,\dots$. We have $a = (1;-2;0.99;0;0;\dots)$, $b = 0$, $a^{-1} = (1;2;3.01;4.04;5.1;6.2;7.35;8.57;9.85;11.225;16.7;\dots)$, hence

$u = u_0 \cdot (1;-2;0;0;\dots) * a^{-1} + u_1 \cdot (0;a^{-1}) = u_0 \cdot (1;0;-0.99;-1.98;\ -2.98;-4;$
$-5.05;-6.14;-7.28;-8.48;-9.75;\dots) + u_1 \cdot (0;1;2;3.01;4.04;5.1;$
$6.2;7.35;8.57;9.85;11.225;\dots)$.

We consider several boundary value problems :

a) $u(0) = 0, u(1) = 1$; then $u_0 = 0$,
$u(1) = u_{10} = 11.225 \cdot u_1 - 9.75 \cdot u_0 = 11.225 \cdot u_1 = 1$, whence $u_1 \cong 0.089$,
$u = (0;0.089;0.178;0.27;0.36;0.45;0.55;0.65;\ \ 0.76;0.87;0.999;\dots)$, the last giving the approximate values of the exact solution $u(x) = sh(x)/sh(1)$ for $x = 0.1k$.

b) $u(0) = 0, u(0.1) = 1$; using directly (10) we obtain $u = ((1;-2) * (0;1);0;0;\dots) * a^{-1} = (0;1;0;0;\dots) * a^{-1} = (0;a^{-1}) =$
$= (0;2;3.01;\ 4.04;5.1;6.2;7.35;8.57;9.85;11.225;\dots)$, the approximate values of the exact solution $u(x) = sh(x)/sh(0.1)$;

Example 7. The differential equation $u'''(x) - u''(x) - u'(x) + u(x) = 4e^x$, with the same net x_k as in example 6, is reduce in conformity with the relation (24) to difference equation $u_{k+3} - 3.1 \cdot u_{k+2} + \ 3.19 \cdot u_{k+1} - 1.089 \cdot u_k = 0.004 \cdot e^{0.1k}$, $k = 0,1,\dots$. Then $a = (1;-3.1;3.19;-1.089;0;0;\dots)$,

$a^{-1} = (1;3.1;6.42;11.1;17.31;25.24;35.12;47.2;\ 61.77;79.17;99.787;0;0;\dots)$,

$b = 10^{-4}(40;44;49;54;60;66;73;80;90;98;109;\dots)$, $u = (0;0;0;b * a^{-1}) +$
$+ u_0 \cdot (1;-3.1;3.19;0;0;\dots) * a^{-1} + u_1 \cdot (0;1;-3.1;0;0;\dots) * a^{-1} + u_2 \cdot (0;0;a^{-1}) =$
$= (0;0;0;0.004;0.017;0.044;0.093;0.17;0.29;0.46;0.7;\dots) +$
$+ u_0 \cdot (1;0;0;1.087;3.38;7;12.1;18.84;27.48;38.25;51.4;\dots) +$
$+ u_1 \cdot (0;1;0;-3.2;-8.8;-17.1;-28.42;-43.12;-61.67;-84.55;-112.317;\dots) +$
$+ u_2 \cdot (0;0;1;3.1;6.42;11.1;17.31;25.24;35.12;47.2;61.77;\dots)$.

We consider the following boundary problems:

a) $u(0) = 1$, $u(0.1) = 1.558$, $u(1) = 16.3$; we have $u_0 = 1$, $u_1 = 1.558$ and $u_{10} = u(1) = 0.7 + 51.4 \cdot u_0 - 112.317 \cdot u_1 + 61.77 \cdot u_2 = 16.3$, hence $u_2 = 2.25$. Then we obtain the solution u replacing these values of u_0, u_1, u_2 in the above expression of u or by the computation
$u = ((1;-3.1;3.19) * (1;1.558;2.25);b)/a = (1;-1.542;0.61;b)/a =$
$= (1;1.558;2.25;3.1;4.13;5.4;6.85;8.6;10.7;13.2;16.1;\dots)$.

b) $u(0) = 1, u(0.9) = 13.3, u(1) = 16.3$; we have $u_0 = 1$, $u_9 = u(0.9) =$
$= 0.46 + 38.25 \cdot u_0 - 84.55 \cdot u_1 + 47.2 \cdot u_2 = 13.3$. From the algebraic system

composed from these relations and that obtained to the point a) from the expression of u_{10}, it results $u_1 = 1.52$, $u_2 = 2.2$, hence $u = ((1;-3.1;3.19)*(1;1.53;2.2);b)/a = (1;-1.57;0.647;b)/a =$
$= (1;1.53;2.2;3.03;4.05;5.3;6.78;8.56;10.68;13.19;16.15;\ldots)$;

c) $u(0)=1, u(0.5)=5.36, u(1)=16.3$; we have $u_0 = 1$, $u_5 = u(0.5) =$
$= 0.044 + 7\cdot u_0 - 17.1\cdot u_1 + 11.1\cdot u_2 = 5.36$. From these relations and that obtained at the point a) from u_{10} ,we get the values $u_1 = 1.54$, $u_2 = 2.22$, hence $u = ((1;-3.1;3.19)*(1;1.54;2.22);b)/a =$
$= (1;1.54;2.22;3.06;4.1;5.34;6.84;8.638;10.77;13.3;16.26;\ldots)$.

The values obtained at the points a), b), c) above, are approximate values of the exact solution $u(x) = (x^2 + 4x + 1)\cdot e^x$ in the points of the considered net.

4.3. Initial value problem.

We denote $u^{(k)} = u^{(k)}(x_0), k = 0,1,2,\ldots$. If we known the initial values $u^{(0)} = u_0$, $u^{(1)},\ldots,u^{(n-1)}$ of the solution $u(x)$ of the differential equation (17), from the relations

$$u^{(1)} \cong \frac{u_1 - u_0}{h} \ , \ u^{(2)} \cong \frac{u_2 - 2u_1 + u_0}{h^2} \ , \ u^{(3)} \cong \frac{u_3 - 3u_2 + 3u_1 - u_0}{h^3}, \ldots,$$

$$u^{(k)} \cong \frac{(-1)^k}{h^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}u_j \ , \ k = 1,2,\ldots \ , \qquad (25)$$

we can determinate the approximate values $u_0, u_1, \ldots, u_{n-1}$ of the solution $u(x)$ in the points $x_k = x_0 + hk, k = 0,1,\ldots,n-1$, of the above considered net. Taking these numbers as initial values for the unknown $u = (u_0;u_1;\ldots;u_k;\ldots)$ of the difference equation (7) to which is reduce the differential equation (17) as was indicated in the **section 3.2**, we can compute the approximate values u_k of the solution in another points of the net by the deconvolution method given in the **section 2**. We can use the same method in the case in which we known values of solution and same of its derivatives in certain points of the net.

Example 8. For the differential equation from example 6, we consider the classical initial value problem from the point a) below, but several other generalized situations are given at the other points .

a) $u(0)=1, u'(0)=2$; then $u_0 = 1$ and $(u_1 - u_0)/0.1 = 2$ that gives $u_1 = 1.2$, hence $u = (1;1.2;1.4;1.63;\ 1.9;2.12;2.4;2.7;33.34;3.72;\ldots)$, these being the approximate values of the exact solution $u(x)=\left(3e^{x}-e^{-x}\right)/2$ in the points $x_k=0.1\cdot k$ of the considered net.

b) $u(0)=0, u'(1)=1$; we have $u_0=0$, $u'(1)=(u_{10}-u_9)/0.1=$ $=13.75u_1-12.7u_0=13.75u_1=1$, which give $u_1=0.07$,hence $u=(0;0.07;0.14;0.21;0.28;0.36;0.43;0.51;0.6;0.69;0.79;\ldots)$, the approximate values of the exact solution $u(x)=sh(x)/ch(1)$;

c) $u'(0)=0, u(1)=1$; from $u'(0)\cong(u_1-u_0)/0.1=0$ it results $u_0=u_1$, $u(1)=(11.225-9.75)u_0=1.475u_0=1, u_0=0.68$,
hence $u=(0.68;0.68;0.687;0.7;0.72;0.75;0.78;0.82;0.88;0.93;1;\ldots)$, the approximate values of the exact solution $u(x)=ch(x)/ch(1)$.

d) $u'(0)=0$, $u'(1)=1$; we have $u_0=u_1$. In conformity with the computation making at the point b), $u'(1)=(13.75-12.74)u_0=1.01u_0=1$, $u_0=0.99$, hence $u=(0.99;0.99;1;1.02;1.05;1.09;1.14;1.2;1.27;1.36;1.46;\ldots)$ the approximate values of the exact solution $u(x)=ch(x)/sh(1)$.

Example 9. Let be the differential equation considered in example 7, now with the initial values $u(0)=1, u'(0)=5, u''(0)=11$. With the same notations we have $u_0=u^{(o)}=u(0)=1$, $u^{(1)}=\dfrac{u_1-u_0}{0.1}=u'(0)=5$, that gives $u_1=1.5$ and $u^{(2)}=\dfrac{u_2-2u_1+u_0}{0.01}=u''(0)=11$, that gives $u_2=2.11$. Replacing these values in formula of u obtained in example 7), we have $u=(u_0;u_1;\ldots;u_k;\ldots)$ $=(u(0);u(0.1);\ldots;u(0.1k);\ldots)=$

$$=(0;0;0;0.004;0.017;0.044;0.093;0.17;0.29;0.46;0.7;\ldots)+$$
$$+(1;0;0;1.087;3.38;7;12.1;18.84;27.48;38.25;51.4;\ldots)+$$
$$+1.5\cdot(0;1;0;-3.2;-8.8;-17.1;-28.42;-43.12;-61.67;-84.55;-112.317;\ldots)+$$
$$+2.11\cdot(0;0;1;3.1;6.42;11.1;17.31;25.24;35.12;47.2;61.77;\ldots)= .$$
$$=(1;1.5;2.11;4.37;4.96;6.22;7.66;9.35;11.28;14.56;\ldots) .$$

4.4. Determination of the initial values

Applying the differentiation of order k to the equation (17) and taking $x=x_0$,it results that the sequence $\left(u_0;u^{(1)};\ldots;u^{(k)};\ldots\right)=$

$= (u(x_0); u'(x_0); \ldots; u^{(k)}(x_0); \ldots)$ of the initial values of the unknown $u(x)$ of the differential equation (17) is solution of the difference equation

$$\sum_{j=0}^{n} \alpha_{n-j} u^{(j+k)} = f^{(k)}(x_0) \ , \ k = 0,1,2,\ldots, \tag{26}$$

hence is the difference equation of form (7) with $a_k = \alpha_k$ and $b_k = f^{(k)}(x_0)$, $k = 0,1,2,\ldots$ Using the deconvolution method given in the **section 2**, from the difference equation (26) we can compute as much as we like of the values $u^{(k)}(x_0)$, if we known n of them. Similar to the difference equations case, the problem considered in this section for differential equations can to have an unique or an infinity of solutions or can have not any solution, as can we see in the following example, based on the **example 3** above.

Example 10. Denoting $u_k = u^{(k)}(\pi/2)$, $k = 0,1,\ldots,$ the differential equation $u''(t) + u(t) = 2\sin t = f(t)$, is reduced to the difference equation $u^{(k+2)} + u^{(k)} = b_k$,where $b_k = f^{(k)}(\pi/2) = 2 \cdot (-1)^k$, $k = 0,1,2,\ldots$ that was considered in example 3).

If we consider the initial conditions:

a) $u^{(3)}(\pi/2) = 0, u^{(4)}(\pi/2) = -3$; b) $u''(\pi/2) = 1$, $u^{(4)}(\pi/2) = -3$;

c) $u''(\pi/2) \neq 1, u^{(4)}(\pi/2) = -3$, we obtain the same situations and solutions as in **example 3**, these solutions u_k being now the initial values $u^{(k)}(\pi/2), k = 0,1,2,\ldots$.

Example 11. As it results from the second Newton' law, the mathematical model for a mass-spring system is governed by the linear differential equation $u''(t) + pu'(t) + qu(t) = 0$, where $q = k/m$, where m is the weight of the mass, k the elasticity constant of the spring given by the Hooke law and p the constant of proportionality between the air resistance and velocity. Using the notation $u^{(k)} = u^{(k)}(t_0)$, the differential equation is reduced to the difference equation $u^{(k+2)} + pu^{(k+1)} + qu^{(k)} = 0$, $k = 0,1,\ldots$. If are known the initial position $u(t_0) = u^{(0)}$ of the mass and the initial velocity $u'(t_0) = u^{(1)}$, then the initial acceleration can be obtained by the deconvolution between the sequences $c = ((u^{(0)}; u^{(1)}) * (1; p); 0; \ldots) = (u^{(0)}; pu^{(0)} + u^{(1)}; 0; \ldots)$ and $a = (1; p; q; 0; 0; \ldots)$, namely $u''(t_0) = u^{(2)} = -(qu^{(0)} + pu^{(1)})$.

The solution by deconvolution for the problem considered here will be also used in the following point.

4.5. Generalized initial value problem

Let suppose that we known n initial values $u^{(j_j)}(x_0) = u^{(j_j)}$, $i = 1,2,\ldots,n$, of the unknown $u(x)$ of the linear differential equation (17) and we want to determine the values $u(x_k) = u_k$, in the net points $x_k = x_0 + k \cdot h$, $k = 0,1,2,\ldots,m$, with $h > 0$ and $m > n$.

For this, we shall combine the methods given in the **sections** 4**.4.** and **4.3**. Namely, we will determinate by formula (10) the first n initial values $u^{(0)} = u_0, u^{(1)},\ldots,u^{(n)}$ by the method given at the **section 4.4.** for the difference equation (26). In conformity with the method given in the **section 4.3.**, from these initial values we shall compute using the relations (25), the values $u_0 = u^{(0)}, u_1,\ldots,u_n$ of the function u in the first n points of the net by the relations (25). Using again the formula (10), now for the difference equation (7), we obtain the desired values $u_k, k = 0,1,\ldots,m$, of the solution $u(x)$ of the differential equation (17) in the points of the net.

Example 12. We consider the differential equation from the **examples 7** and **9**, $u'''(x) - u''(x) - u'(x) + u(x) = 4 \cdot e^x$, with the same net $x_k = 0.1 \cdot k, k = 0,1,\ldots,10$ and the initial values $u'''(0) = 19$, $u^{(5)}(0) = 41$, $u^{(6)}(0) = 55$. Denoting $u^{(j)} = u^{(j)}(0), j = 0,1,\ldots,$ differentiating of j times the equation and taking $x = 0$, we obtain the difference equation $u^{(j+3)} - u^{(j+2)} - u^{(j+1)} + u^{(j)} = 4$, for $j = 0,1,\ldots$. With the notations $a = (1;-1;-1;1;0;0;\ldots)$ and $b = (4;4;\ldots)$, applying the formula (10) it results that the solution of the difference equation has the form $u = \left(0;0;0;b * a^{-1}\right) + u^{(0)} \cdot (1;-1;-1;0;0;\ldots) * a^{-1} + u^{(1)} \cdot (0;1;-1;0;0;\ldots) * a^{-1} + u^{(2)} \cdot \left(0;0;a^{-1}\right) = (0;0;0;4;8;16;24;36;\ldots) + u^{(0)} \cdot (1;0;0;-1;-1-2;-2;-3;\ldots) + u^{(1)} \cdot (0;1;0;1;0;1;0;1;\ldots) + u^{(2)} \cdot (0;0;1;1;2;2;3;3;\ldots)$. Using the given initial values, we obtain the relations $u^{(3)} = 4 - u^{(0)} + u^{(1)} + u^{(2)} = 19$, $u^{(5)} = 16 - 2 \cdot u^{(0)} + u^{(1)} + 2 \cdot u^{(2)} = 41$, $u^{(6)} = 24 - 2 \cdot u^{(0)} + 3 \cdot u^{(2)} = 55$, from which it results $u^{(0)} = 1, u^{(1)} = 5, u^{(2)} = 11$ and the calculus continues as in **example 9**.

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