

KINEMATICS AND DYNAMICS OF A PLANETARY GEAR TRAIN FOR ROBOTICS

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Lucrarea prezintă stabilește unele relații matriceale recurente privind analiza geometrică, cinematică și dinamică a unui tren de roți dințate planetare, utilizat în robotică. Prototipul acestui mecanism este un sistem mecanic cu trei grade de libertate, care este constituit din șapte elemente și patru angrenaje conice. Controlate de motoare electrice, trei elemente active ale robotului au trei rotații independente. Presupunând că poziția și mișcarea de rotație a platformei sunt cunoscute, se determină relații matriceale și grafice pentru momentele celor trei cupluri active.

Some recursive matrix relations concerning the geometric analysis, kinematics and dynamics of a Bendix wrist planetary bevel-gear train for robotics are established in this paper. The prototype of this mechanism is a three-degree-of-freedom system with seven links and four bevel gear pairs. Controlled by electric motors, three active elements of the robot have three independent rotations. Supposing that the position and the rotational motion of the platform are known, an inverse dynamics problem is developed using the virtual powers approach. Finally, some recursive matrix relations and some graphs for the torques of the actuators are determined.

Keywords: dynamics, gear, matrix, platform, robotics

Introduction

Generally, a parallel manipulator needs at least six degrees of freedom to freely manipulate an object in space. The first three moving links are used primarily for manipulating the position, while the second mechanism is used to control the orientation of the end-effectors. The subassembly associated with its last moving links is called the wrist, and their joint axes are often designed to intersect at a common point called the wrist centre.

Planetary bevel-gear trains with three degrees of freedom are adopted as the design concept for robotic wrist (Hsieh and Sheu, 1995). Bevel-gear wrist mechanisms have been incorporated in most industrial robots because they are simple and compact in size compared to others and can be sealed in a metallic housing that keeps the gear trains free of contamination. Furthermore, using bevel

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gear trains for power transmissions, actuators can be mounted remotely on the forearm, thereby reducing the weight and inertia of a robot manipulator.

In the present paper some recursive matrix relations for the kinematics and dynamics of a Bendix wrist planetary bevel-gear train for robotics are established.

1. Inverse geometric model

Since a robot wrist must rotate about three axes, it is a mechanism with three degrees of freedom. Further, we present a matrix methodology for the geometric analysis and the kinematics using the concept of fundamental circuit of an open-loop chain (Tsai, 1988, 1999). This method involves the identification of an open-loop chain and the derivation of the geometric relationship between the orientation of the end-effector and the joint angles of the chain, including the input actuator displacement.

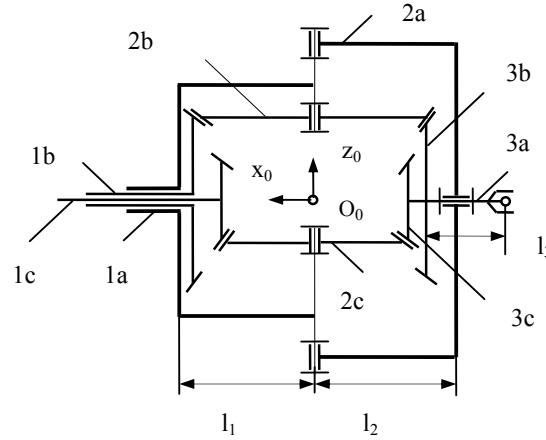


Fig. 1 Bendix wrist mechanism

Let $Ox_0y_0z_0(T_0)$ be a fixed Cartesian frame, about which the mechanism moves. The wrist architecture consists of seven links, seven turning pairs and four bevel gear pairs (Fig. 1). Therefore, the wrist is a 3-dof spherical mechanism, which has a limited rotational range about the second joint axis.

In the Bendix wrist, link $1a$, of l_1 in length, masse m_1^A and tensor of inertia \hat{J}_1^A , one of the three driving parts of the robot, serves as carrier for the $1b-2b$ and $1c-2c$ bevel gear pairs, while link $2a$, of l_2 in length, masse m_2^A and tensor of inertia \hat{J}_2^A , serves as carrier for the $2b-3b$ and $2c-3c$ bevel gear pairs. The gears $1b$ and $1c$ are sun gear, and gears $2b, 2c$ are planet gears adjacent to carrier $1a$ when

gear $3a$ is planet gear adjacent to carrier $2a$. Three coaxial members numbered $1a$, $1b$, $1c$ are supported by bearings housed in the forearm.

Bevel gear pairs $1b-2b-3b$ and $1c-2c-3c$ transmit the rotations of the coaxial input links to the end-effector. This moving platform of length l_3 , masse m_3^A and tensor of inertia \hat{J}_3^A is attached to the link $3a = 3b = 3c$, which is housed in the carrier $2a$ and is free to arbitrarily undergo three concurrent rotations with respect to the centre O_0

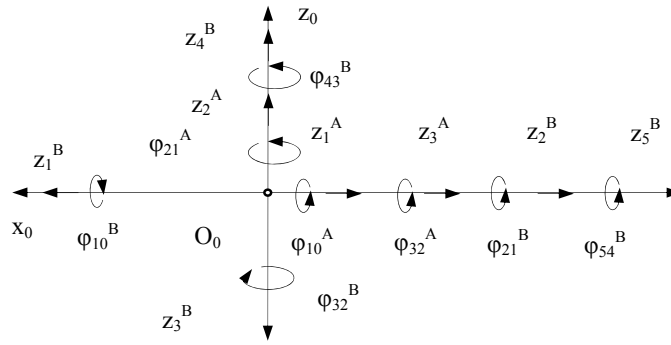


Fig. 2 Kinematical scheme of the wrist

Let us consider the rotation angles $\varphi_{10}^A, \varphi_{10}^B, \varphi_{10}^C$ of the three actuators A_1, B_1, C_1 as variables giving the instantaneous position of the mechanism (Fig. 2). Pursuing the circuits A , B and C we obtain the successive transformation matrices (Staicu, 1998):

$$\begin{aligned}
 a_{10} &= a_{10}^\varphi \theta_2, a_{21} = a_{21}^\varphi \theta_1, a_{32} = a_{32}^\varphi \theta_2 \\
 b_{10} &= b_{10}^\varphi \theta_1, b_{21} = b_{21}^\varphi \theta_3, b_{32} = b_{32}^\varphi \theta_2 \\
 b_{43} &= b_{43}^\varphi \theta_3, b_{54} = b_{54}^\varphi \theta_2 \\
 c_{10} &= c_{10}^\varphi \theta_1, c_{21} = c_{21}^\varphi \theta_3, c_{32} = c_{32}^\varphi \theta_2 \\
 c_{43} &= c_{43}^\varphi \theta_3, c_{54} = c_{54}^\varphi \theta_2
 \end{aligned} \tag{1}$$

where one denoted

$$\theta_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \theta_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \theta_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2}$$

$$a_{k,k-1}^{\varphi} = \begin{bmatrix} \cos \varphi_{k,k-1}^A & \sin \varphi_{k,k-1}^A & 0 \\ -\sin \varphi_{k,k-1}^A & \cos \varphi_{k,k-1}^A & 0 \\ 0 & 0 & 1 \end{bmatrix}, a_{k0} = \prod_{j=1}^k a_{k-j+1,k-j} \quad (k = 1, 2, \dots, 3).$$

Further on, we introduce a matrix approach, which utilizes the theory of fundamental circuits. There exists a real or fictitious carrier for every gear pair in a planetary gear train and a fundamental matrix equation for each loop can be written as

$$a_{k+1,k-1}^{\varphi} = a_{k+1,k}^{\varphi} a_y^{\delta_k} a_{k,k-1}^{\varphi} \quad (3)$$

$$\varphi_{k,k-1} = N_{k+1,k-1} \varphi_{k+1,k}, \quad \delta_k = \alpha_{k-1} + \alpha_{k+1},$$

where $\varphi_{k,k-1}$ and $\varphi_{k+1,k}$ denote the relative angular displacements of carrier T_k and planet gear T_{k+1} , respectively, and α_{k-1} , α_{k+1} are the angles of the gear T_{k-1} and T_{k+1} . The gear ratios of a gear pair is defined as $N_{k+1,k-1} = r_{k+1} / r_{k-1} = z_{k+1} / z_{k-1}$, where r_{k-1} , r_{k+1} and z_{k-1} , z_{k+1} denote the radius and the number of teeth of two gears, respectively (Fig.3).

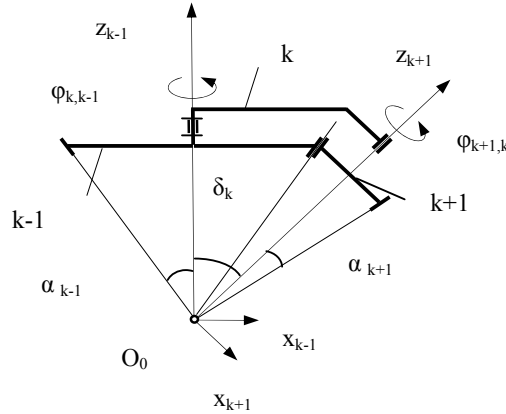


Fig. 3 Gear fundamental circuit

Let us suppose that the absolute motion of the platform attached to planet gear 3a is a general rotation around the centre O_0 . In the inverse geometric problem, however, the orientation of the end-effector is known by intermediate of the three Euler's angles α_1 , α_2 , α_3 expressed by the functions

$$\alpha_l = \alpha_l^* [1 - \cos(\frac{\pi}{3} t)], \quad (l = 1, 2, 3). \quad (4)$$

Representing the orientation of the platform in the fixed frame, the product of successive known matrices of rotation $a_1 = a_x^T(\alpha_1)$, $a_2 = a_z(\alpha_2)$, $a_3 = a_x^T(\alpha_3)$

leads to the general rotation matrix

$$a = a_3 a_2 a_1 \quad (5)$$

Geometric conditions of rotation for the platform are given by the following identities

$$a_{30}^{\circ T} a_{30} = b_{50}^{\circ T} b_{50} = c_{50}^{\circ T} c_{50} = a \quad (6)$$

where

$$a_{30}^{\circ} = \theta_2 \theta_1 \theta_2, b_{50}^{\circ} = c_{50}^{\circ} = \theta_2 \theta_3 \theta_2 \theta_3 \theta_1. \quad (7)$$

From these equations, we obtain the real-time evolution of all characteristic joints: $\varphi_{10}^A, \varphi_{21}^A, \varphi_{32}^A, \varphi_{10}^B, \varphi_{21}^B, \varphi_{32}^B, \varphi_{43}^B, \varphi_{54}^B, \varphi_{10}^C, \varphi_{21}^C, \varphi_{32}^C, \varphi_{43}^C, \varphi_{54}^C$.

2. Kinematics of robotic wrist mechanism

In the design of power transmission mechanisms, it is often necessary to analyse the speed ratios between their input and output members and angular velocities or angular accelerations of the intermediate members.

The analysis of the kinematics of bevel-gear wrist mechanisms of gyroscopic structure is very complex, due to the fact that the carriers and planet gears may have simultaneous angular velocities about nonparallel axes. The conventional tabular or analytical method, which concentrates on planar epicyclic gear trains, is no longer applicable. To overcome this difficulty, Freudenstein, Longman and Chen (1984) applied the dual relative velocity and dual matrix of transformation for the analysis of epicyclic bevel-gear trains. Chang and Tsai (1989) and Hedman (1993) showed that the kinematical analysis of geared robotic mechanisms can be accomplished by applying the theory of fundamental circuits.

Since a kinematical chain is an assemblage of links and joints, these can be symbolized in a more abstract form known as equivalent graph representation (Fig. 4). For the reason that will be clear later, we use the associated graph to represent the topology of the mechanism: vertices denote the links and edges denote the joints (Yang and Hsieh, 1991). Two small concentric circles label the vertex denoting the fixed link 0. To distinguish the differences between pairs connections, gear pairs $1b-2b, 2b-3b, 1c-2c, 2c-3c$ are denoted by thick edges and revolute joints are denoted by thin edges. The three edged paths, which start from the base link 0 and end at the end-effector link $3a$, consist of vertices $1a, 2a, 3a, 1b, 2b, 3b$ and $1c, 2c, 3c$. There are four independent loops and we identify four fundamental circuits.

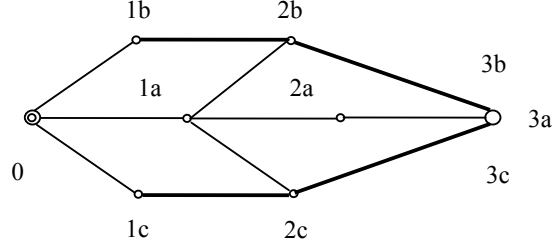


Fig. 4 Associated graph of the wrist

The kinematics of an element for each circuit (for example the circuit A) are characterized by skew-symmetric matrices given by the recurrence relations (Staicu, 2000):

$$\tilde{\omega}_{k0}^A = a_{k,k-1} \tilde{\omega}_{k-1,0}^A a_{k,k-1}^T + \omega_{k,k-1}^A \tilde{u}_3 \quad (8)$$

These matrices are associated to the absolute angular velocities

$$\vec{\omega}_{k0}^A = a_{k,k-1} \vec{\omega}_{k-1,0}^A + \omega_{k,k-1}^A \vec{u}_3, \quad \omega_{k,k-1}^A = \dot{\phi}_{k,k-1}^A \quad (9)$$

Knowing the rotation motion of the platform by the relations (4), one develops the inverse kinematical problem and determines the velocities $\vec{v}_{k0}^A, \vec{\omega}_{k0}^A$ and accelerations $\vec{\gamma}_{k0}^A, \vec{\varepsilon}_{k0}^A$ of each of the moving links. The following *matrix relations of connectivity* constitute the inverse kinematical model

$$\begin{aligned} & \omega_{10}^A \vec{u}_i^T a_{10}^T \vec{u}_3 + \omega_{21}^A \vec{u}_i^T a_{20}^T \vec{u}_3 + \omega_{32}^A \vec{u}_i^T a_{30}^T \vec{u}_3 = \\ & = \vec{u}_i^T \{ \dot{\alpha}_1 a_1^T \vec{u}_1 + \dot{\alpha}_2 a_2^T a_2^T \vec{u}_2 + \dot{\alpha}_3 a_3^T a_2^T a_3^T \vec{u}_3 \}, (i=1,2,3) \end{aligned} \quad (10)$$

where $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ are three skew-symmetric matrices associated with the orthogonal unit vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. The method is very straightforward and can be implemented on a computer for the automated analysis of a planetary gear train. This results in a system of linear equations that can be solved for angular velocities of all the links. These relations give the relative angular velocities $\omega_{10}^A, \omega_{21}^A, \omega_{32}^A$ as a function of the angular velocities $\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3$ of the end-effector. For the other two circuits B, C of wrist, analogous relations can then be obtained with important remark

$$\omega_{k,k-1} = N_{k+1,k-1} \omega_{k+1,k} \quad (11)$$

Starting from (10), a complete expression of the Jacobian of the mechanism is easily written in an invariant form. This square invertible matrix is an essential element for the analysis of singularity loci into robot workspace.

Let us assume now that the mechanism has successively three independent virtual motions defined by the angular velocities $\omega_{10a}^{Av}=1, \omega_{10a}^{Bv}=0, \omega_{10a}^{Cv}=0, \omega_{10b}^{Bv}=1, \omega_{10b}^{Cv}=0, \omega_{10b}^{Av}=0$ and $\omega_{10c}^{Cv}=1, \omega_{10c}^{Av}=0, \omega_{10c}^{Bv}=0$.

Characteristic virtual velocities expressed as functions of robot's position are given by the above conditions of connectivity concerning the relative velocities of three circuits:

$$\vec{u}_i^T a_{30}^T \vec{\omega}_{30a}^{Av} = \vec{u}_i^T b_{50}^T \vec{\omega}_{50a}^{Bv} = \vec{u}_i^T c_{50}^T \omega_{50a}^{Cv} = \vec{u}_i^T \vec{\omega}_{0a}^{Av} \quad (12)$$

Concerning the relative angular accelerations $\varepsilon_{10}^A, \varepsilon_{21}^A, \varepsilon_{32}^A$ of the elements of circuit A , these are given by some other *conditions of connectivity*, obtained by deriving the relations (10); it results:

$$\begin{aligned} & \varepsilon_{10}^A \vec{u}_i^T a_{10}^T \vec{u}_3 + \varepsilon_{21}^A \vec{u}_i^T a_{20}^T \vec{u}_3 + \varepsilon_{32}^A \vec{u}_i^T a_{30}^T \vec{u}_3 = \\ & = \vec{u}_i^T \{ \ddot{\alpha}_1 a_1^T \vec{u}_1 + \ddot{\alpha}_2 a_1^T a_2^T \vec{u}_2 + \ddot{\alpha}_3 a_1^T a_2^T a_3^T \vec{u}_3 + \\ & + \dot{\alpha}_1 \dot{\alpha}_2 a_1^T \vec{u}_1 a_2^T \vec{u}_2 + \dot{\alpha}_2 \dot{\alpha}_3 a_1^T a_2^T \vec{u}_2 a_3^T \vec{u}_3 + \\ & + \dot{\alpha}_3 \dot{\alpha}_1 a_1^T \vec{u}_1 a_2^T a_3^T \vec{u}_3 - \omega_{10}^A \omega_{21}^A a_{10}^T \vec{u}_3 a_{21}^T \vec{u}_3 - \\ & - \omega_{21}^A \omega_{32}^A a_{20}^T \vec{u}_3 a_{32}^T \vec{u}_3 - \omega_{32}^A \omega_{10}^A a_{10}^T \vec{u}_3 a_{21}^T a_{32}^T \vec{u}_3 \}. \end{aligned} \quad (13)$$

The angular accelerations ε_{k0}^A and the matrices $\tilde{\omega}_{k0}^A \tilde{\omega}_{k0}^A + \tilde{\varepsilon}_{k0}^A$ are easily calculated with the recurrence relations

$$\begin{aligned} \tilde{\varepsilon}_{k0}^A &= a_{k,k-1} \tilde{\varepsilon}_{k-1,0}^A + \varepsilon_{k,k-1}^A \vec{u}_3 + \omega_{k,k-1}^A a_{k,k-1} \tilde{\omega}_{k-1,0}^A a_{k,k-1}^T \vec{u}_3 \\ \tilde{\omega}_{k0}^A \tilde{\omega}_{k0}^A + \tilde{\varepsilon}_{k0}^A &= a_{k,k-1} \left(\tilde{\omega}_{k-1,0}^A \tilde{\omega}_{k-1,0}^A + \tilde{\varepsilon}_{k-1,0}^A \right) a_{k,k-1}^T + \\ &+ \omega_{k,k-1}^A \omega_{k,k-1}^A \tilde{u}_3 \tilde{u}_3 + \varepsilon_{k,k-1}^A \tilde{u}_3 + 2\omega_{k,k-1}^A a_{k,k-1} \tilde{\omega}_{k-1,0}^A a_{k,k-1}^T \tilde{u}_3. \end{aligned} \quad (14)$$

3. Equations of motion

Such bevel-gear wrist is often built with direct drive actuators and hence, the dynamics of this robot has a very important effect on the actuator torques. Therefore, the derivation of a dynamic model is desirable for the design of an efficient controller.

The motion of the Bendix wrist is controlled by three electric motors, A_1, B_1, C_1 which generate three moments $\vec{m}_{10}^A = m_{10}^A \vec{u}_3, \vec{m}_{10}^B = m_{10}^B \vec{u}_3, \vec{m}_{10}^C = m_{10}^C \vec{u}_3$ having the directions of the coaxial axes $O_0 z_1^A, O_0 z_1^B, O_0 z_1^C$. Considering that the mobile platform motion is given, the position, angular velocity, angular acceleration as well as the velocity and acceleration of the centre of mass are known of each

element. The inertia force and the resultant moment of inertia forces acting at rigid body T_k are also evaluated with respect to the wrist centre O_0 . On the other hand, the characteristic vectors \vec{f}_k^* , \vec{m}_k^* designate the action of the weight $m_k \vec{g}$ and of any other external and internal applied forces at the same element of the mechanism.

In the context of the real-time control, neglecting the frictional forces and considering the gravitational effect, the relevant objective of a dynamic model is to determine the input torques, which must be exerted by the actuators in order to produce a given trajectory of the end-effector.

In the inverse dynamic problem, in the present paper one applies the principle of virtual power in order to establish some recursive matrix relations for the torques of the three active couples. This fundamental principle states that a mechanism is under dynamic equilibrium if and only if the virtual power developed by all external, internal and inertia forces vanishes during any general virtual displacement, which is compatible with the constraints imposed on the mechanism. Applying *the fundamental equations of parallel robots dynamics* obtained in a matrix compact form by Ștefan Staicu (2000), the following matrix relation results

$$m_{10}^A = \vec{u}_3^T \{ \vec{M}_1^A + \omega_{21a}^{Av} \vec{M}_2^A + \omega_{32a}^{Av} \vec{M}_3^A + \omega_{21a}^{Bv} \vec{M}_2^B + \omega_{32a}^{Bv} \vec{M}_3^B + \omega_{21a}^{Cv} \vec{M}_2^C + \omega_{32a}^{Cv} \vec{M}_3^C \} \quad (15)$$

where are denoted:

$$\begin{aligned} \vec{F}_{k0}^A &= m_k^A [\vec{\gamma}_{k0}^A + (\tilde{\omega}_{k0}^A \tilde{\omega}_{k0}^A + \tilde{\varepsilon}_{k0}^A) \vec{r}_k^{CA}] - \vec{f}_k^{*A} \\ \vec{M}_{k0}^A &= m_k^A \tilde{r}_k^{CA} \vec{\gamma}_{k0}^A + \hat{J}_k^A \tilde{\varepsilon}_{k0}^A + \tilde{\omega}_{k0}^A \hat{J}_k^A \tilde{\omega}_{k0}^A - \vec{m}_k^{*A} \\ \vec{F}_k^A &= \vec{F}_{k0}^A + a_{k+1,k}^T \vec{F}_{k+1}, \quad (k=1, 2, 3) \\ \vec{M}_k^A &= \vec{M}_{k0}^A + a_{k+1,k}^T \vec{M}_{k+1} + \tilde{r}_{k+1,k}^T a_{k+1,k}^T \vec{F}_{k+1} \\ \vec{f}_k^{A*} &= 9.81 m_k^A a_{k0} \vec{u}_3, \quad \vec{m}_k^{A*} = 9.81 m_k^A \tilde{r}_k^{CA} a_{k0} \vec{u}_3. \end{aligned} \quad (16)$$

The relations (15) and (16) represent the *inverse dynamic model* of the planetary gear trains. We can obtain analogous expressions for the torques m_{10}^B, m_{10}^C exerted by the other two B_1, C_1 actuators.

The procedure developed above leads to very good estimates of the actuators torques for given displacement of end-effector, provided that the inertial properties of the gears are known with sufficient accuracy and that friction is not significant. It is also remarked that, depending on the masses and inertias of the bodies, the present matrix dynamic model leads to interesting and useful results for purposes of control. The new dynamic approach developed here is completely general and can be used for any gyroscopic bevel-gear train with revolute actuators.

As application let us consider the commonly known Bendix wrist which has the following characteristics:

$$l_1 = l_2 = 0.05 \text{ m}, l_3 = 0.075 \text{ m}, r_1^B = 0.02 \text{ m}, r_2^B = 0.025 \text{ m}, r_3^B = 0.02 \text{ m},$$

$$r_1^C = 0.015 \text{ m}, r_2^C = 0.02 \text{ m}, r_3^C = 0.02 \text{ m}, m_1^A = 0.25 \text{ kg}, m_2^A = 0.35 \text{ kg},$$

$$m_3^A = m_3^B = m_3^C = 1.15 \text{ kg}, m_1^B = 0.15 \text{ kg}, m_2^B = 0.20 \text{ kg}, m_1^C = 0.10 \text{ kg}$$

$$m_2^C = 0.15 \text{ kg}, \alpha_1^* = \frac{\pi}{3}, \alpha_2^* = \frac{\pi}{4}, \alpha_3^* = \pi, \Delta t = 6 \text{ s}.$$

Finally, the graphs of the torques m_{10}^A (Fig. 5), m_{10}^B (Fig. 6), m_{10}^C (Fig. 7) of three actuators are obtained.

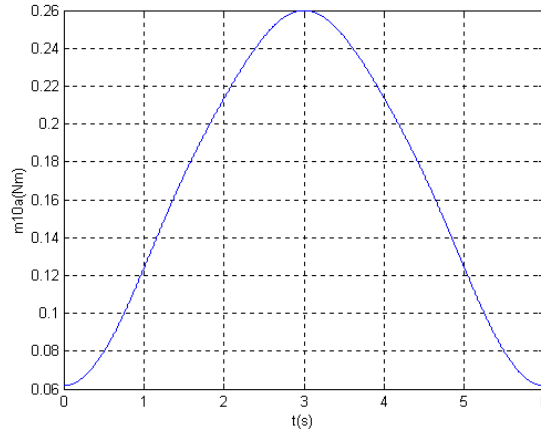


Fig. 5 Torque m_{10}^A of first actuator

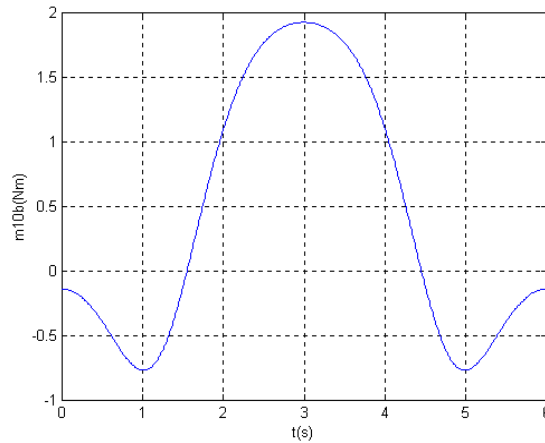


Fig. 6 Torque m_{10}^B of second actuator

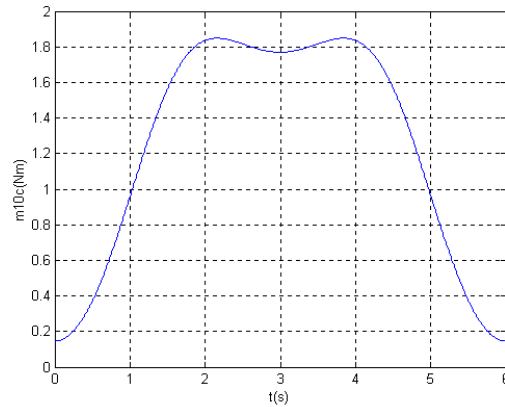


Fig. 7 Torque m_{10}^C of third actuator

5. Conclusion

Based on the principle of virtual work, the new approach described above is very efficient and establishes a direct recursive determination of the variation in real-time of the torques of the actuators. The dynamical approach can be transformed into a model for automatic command of a bevel-gear wrist mechanism.

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