EXTENDING CLOSED ANALYTICAL FORMULAS TO CASCADE AERODYNAMICS BY USING CONFORMAL MAPPING AND FAST FOURIER TRANSFORM

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Se extinde transformarea conformă a rețelelor de plăci la rețele de profile groase, de formă dată. Expresiile analitice închise ale potențialului complex al vitezelor sunt, de asemenea, menținute. În acest scop, se utilizează interpretarea directă, dată de autori, efectelor vitezei introduse în amonte, ca singularități ale curgerii în planele transformate. De asemenea, se defineşte o re[illegible]ea de plăci ataşată de cea reală. Pentru obținerea conturului circular, se utilizează Transformata Fourier Rapidă în variabilă complexă, în combinație cu transformarea conformă Joukowsky, pentru accelerarea convergenței. Metoda este validată. Se prezintă aplicații la distribuțiile de viteze şi presiuni pe profile de turbină.

The conformal mapping for cascade blades is extended to cascade of thick profiles of given form. The closed analytical expressions for the complex potential of the velocity field are maintained. To this aim the author's direct interpretation of the upstream velocity effects as flow singularities in the transformed planes is used. A row of plates attached, to the real one is defined. To obtain the circle contour the Fast Fourier Transform in complex variable is applied, combined with Joukowsky transform, to accelerate the convergence. The method is validated. Applications to velocity and pressure distributions for turbine profiles are given.

Keywords: turbine profile; Fast Fourier Transform; complex velocity potential, conformal mapping.

1. Introduction

The determination of the velocity field in incompressible flow by using the conformal mapping has several advantages: a direct view of the flow geometry, more compact formulas and a straightforward implementation of the Joukowsky condition in order to determine the circulation. Although the numerical have been strongly developed, a combination between the analytical and numerical calculation could be the optimal strategy, as one point out in this paper. The incompressible flow is met in several applications like: hydraulic turbines [1], [2], [3], [4], wind turbines [5]. It can be as well a possible comparison and an initial

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approximation for subsonic compressible flows [3], [6]. Of course, the compressor cascades are simpler to solve because the profiles are thinner.

Unlike the other papers where particular profiles are obtained from transformation of particular contours [7], here more general profiles of given form are considered. In comparison with [8], a more general interpretation and an improvement of the method is achieved.

2. The interpretation of the upstream velocity effects as singularities in the circle plan.

Now let us consider a parallel stream of constant complex velocity at infinity in the z - plan, w_∞ (Fig.1), i_∞ being the angle of attack.

The simplest problem for cascade flow is to determine the distribution of speeds and pressure on linear cascade of plates in incompressible stationary flow.

One starts from a closed analytical formula for the conformal mapping of a row of blades on a circle of radius one [1], [2], [3], namely:

$$z_p(\zeta) = \frac{t}{2\pi}\left(e^{i\lambda} \ln\frac{\zeta + R}{\zeta - R} + e^{-i\lambda} \ln\frac{\zeta R + 1}{\zeta R - 1} + B \right) \ , \tag{2.1}$$

where t is the cascade pitch, λ- the angle of installation ($\lambda > 0$ for turbine and $\lambda < 0$ for compressor), ζ - complex variable in the circle plan, z - complex variable in the profile plan; $R > 1$ is a parameter to be determined and B an adjustment constant.

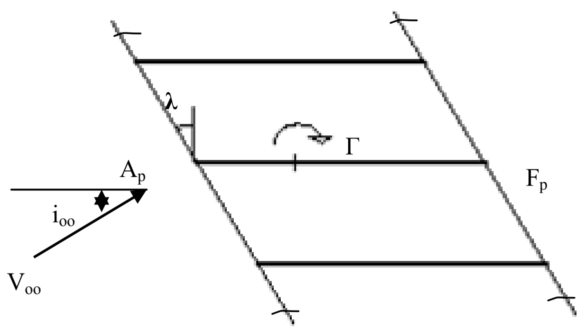

Fig. 1. Linear cascade of plates

The complex velocity is written under the form:

$$w_\infty = V_\infty \cdot e^{-i \cdot i_\infty} \tag{2.2}$$

where V_∞ is the velocity modulus.

If i_∞ is different from zero, a circulation Γ will occur around every plate from the cascade.

One proceeds in two steps: first one determines the effect of w_∞; then one expresses the effect of Γ.

2.1. Determination of the velocity potential without circulation

The relation between velocities in the two plans is:

$$w = \frac{W}{\frac{dz_p}{d\zeta}} \tag{2.3}$$

where W is the complex velocity in the plan ζ.

From the expression (2.1) one notes that, for $z \to \infty$ correspond two points outside the circle, namely $\zeta = \pm R$. For the corresponding velocities in the plan ζ one obtains:

$$(W_1)_{\zeta \to \pm R} \to \pm \frac{W_\infty \cdot t}{2\pi} \cdot \frac{e^{i\lambda}}{\zeta \pm R}\ , \tag{2.4}$$

i.e. in points $\zeta = \pm R$ one should have combinations of sources and vortices of the following intensities:

$$I_1 = \pm t \cdot V_\infty \cdot e^{i \cdot \varphi_1} \quad ; \quad \varphi_1 = \lambda - i_\infty\ . \tag{2.5}$$

In order to conserve the circle as streamline, at points $\zeta = \pm 1/R$ combinations of sources and vortices of complex-conjugate intensities have to be placed, as follows:

$$\bar{I}_1 = \pm t \cdot V_\infty \cdot e^{-i \cdot \varphi_1} \tag{2.6}$$

One obtains the complex potential of the flow without circulation:

$$F_1(\zeta) = \frac{t \cdot V_\infty}{2\pi} \cdot \left(e^{i \cdot \varphi_1} \cdot \ln\left(\frac{\zeta + R}{\zeta - R}\right) + e^{-i \cdot \varphi_1} \cdot \ln\left(\frac{R\zeta + 1}{R\zeta - 1}\right)\right) + const. \tag{2.7}$$

2.2. Determination of the velocity potential due to circulation

The row of vortices placed on plates of intensity Γ disturbs the velocity at infinity, according to Joukowsky theorem, with a velocity $\frac{\Gamma}{2t}$ parallel to the front of the cascade (Fig.1).

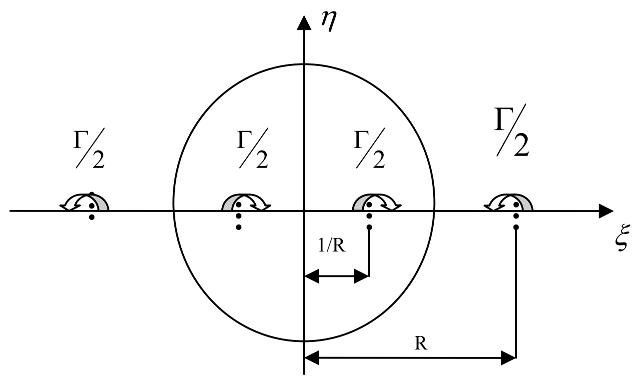

Fig. 3. Sources and vortices

The additional velocity due to circulation at infinity, *w'*, is:

$$w' = \frac{i\Gamma}{2t} \cdot e^{-i\cdot\lambda} \ , \qquad z \to \infty \tag{2.8}$$

By introducing (2.1) in the expression (2.6) one obtains:

$$\left(W'\right)_{\zeta \to \pm R} \to \pm \frac{i\Gamma}{4\pi} \cdot \frac{1}{\zeta \pm R}, \tag{2.9}$$

that is vortices of intensity $\frac{\Gamma}{2}$ should be placed in points $\zeta = \pm R$, as well as other vortices to maintain the circle as a streamline (Fig.3).

The complex potential of the flow due to circulation is then:

$$F_2(\zeta) = -\frac{i\Gamma}{4\pi} \cdot \ln \frac{\zeta^2 - R^2}{R^2 \cdot \zeta^2 - 1} + const. \tag{2.8}$$

The total complex potential of the flow in the plan ζ is the sum:

$$F(\zeta) = F_1(\zeta) + F_2(\zeta) \ .$$

(2.10)

The Joukowsky condition is a null velocity at the trailing edge i.e.:

$$\left(\frac{dF}{d\zeta}\right)_{\zeta_{F'}} = 0, \qquad (\rho_{F'} = 1 \quad \theta_{F'} = -\theta_0 \ ), \tag{2.11}$$

Where from one obtains the circulation:

$$\Gamma = \frac{4R}{R^2 + 1} \frac{\cos(\theta_0)}{\cos \lambda} \cdot t \cdot V_\infty \cdot \sin(i_\infty) \tag{2.12}$$

and the velocity, *v*, on the profile (plate):

$$\frac{-v}{V_\infty} = \frac{R^2 \cdot \sin(\varphi_1 + \theta) - \sin(\varphi_1 - \theta)}{R^2 \cdot \sin(\lambda + \theta) - \sin(\lambda - \theta)} + \frac{\Gamma}{4 \cdot \pi \cdot V_\infty} \cdot \frac{R^4 - 1}{R^2 \cdot \sin(\lambda + \theta) - \sin(\lambda - \theta)} \tag{2.13}$$

The angle θ_0 and the parameter R are determined from the system of two equations given bellow:

$$\frac{b}{t}=\frac{1}{\pi}\cdot\left(\cos\lambda\ln\frac{1+2R\cos\theta_0+R^2}{1-2R\cos\theta_0+R^2}+2\sin\lambda\tan^{-1}\frac{2R\sin\theta_0}{R^2-1}\right)$$

$$\tan\theta_0=\frac{R^2-1}{R^2+1}\tan\lambda \qquad (2.14)$$

Remarks.1. According to Carafoli theory [2], for profiles with round trailing edge, the Joukowsky condition is maintained in the form (2.11), leading to a null velocity on the profile trailing edge.

2. The intensity of singularities (sources and vortices) is conserved by conformal mapping, but these singularities are placed in changed positions with respect to the circle (points *M, N* - Fig.6).

3. The cascade of profiles of given shape

If the blade shape, thickness and angle of installation are specified, one will determine the function that transforms the outside of the unit circle to outside the cascade profiles.

First one defines an ***attached cascade of plates*** (Fig.4**)** by choosing a chord plate attached to the given profile, satisfying the condition:

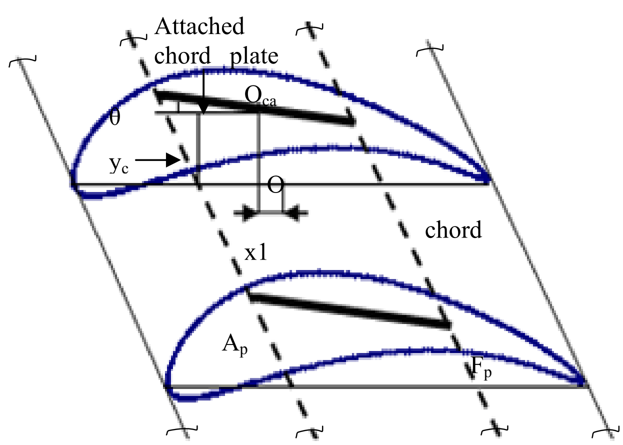

Fig.4. The attached cascade of plates

$$|y|<\frac{t}{2}\cos(\lambda), \qquad (3.1)$$

leading to:

$$\left|(x-x_c)\sin\theta+(y-y_c)\cos\theta\right|<\frac{t}{2}\cos(\lambda+\theta) \tag{3.2}$$

where t - cascade pitch, λ – cascade profiles installation angle, (x_c, y_c) is the center of the attached plate chord and θ its inclination angle (Fig.4).

The total transformation consists of the relation (2.1) ***applied to the attached cascade of plates*** which transforms the cascade of profiles into a shape E (z_2 plane) and the transformation g witch is leading this shape on the unit circle (Fig. 5 - plane K) [4], [5]:

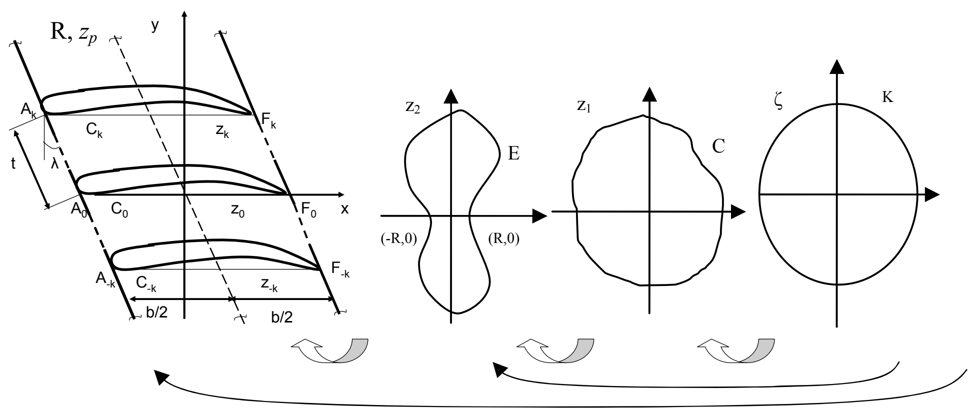

Fig. 5. Successive transformations

To determine the transformation, one will first calculate the contour E coordinates using Newton-Raphson method.

Then one will determine a ***closest to the contour E ellipse*** of equation:

$$ax_2^{\ 2}+by_2^{\ 2}+cx_2y_2+dx_2+ey_2+1=0 \tag{3.3}$$

The contour coordinates (x_{2i}, y_{2i}) verify the equation (3.2) with approximation ε_i.

$$ax_{2i}^{\ 2}+by_{2i}^{\ 2}+cx_{2i}y_{2i}+dx_{2i}+ey_{2i}+1=\varepsilon_i, i=1,2,\ldots,n \tag{3.4}$$

One defines ***closest*** an ellipse whose coefficients (a, b, c, d, e) verify the condition of minimum amount with the weights $\mathrm{w_i}$:

$$S=\sum_{i=1}^{n}\varepsilon_i^2 w_i=\min \tag{3.5}$$

Then one will apply the Joukowsky transform to obtain a circle from ellipse:

$$g=\left(z_1+\frac{c^2}{z_1^2}\right)e^{i\tau}+z_0, \quad c^2=\frac{b^2-a^2}{4}, \tag{3.6}$$

where: z_0 is centre, τ the axial inclination and a and b are the semi-axes of the closest ellipse.

The advantage is that the transformed contour *C* (Fig.5) is close to a circle and the transformation to a final circle *K* is accelerated. The five parameters (*a, b, c, d, e*) are obtained by applying the least squares method.

If z_1 is the function that transforms the unit circle in the contour *C*, for finding this transform one uses the Fast Fourier Transform (FFT). A FFT is a way to compute the result more quickly because one obtains the result in only $O(N\log N)$ instead of $O(N^2)$ operations. Then one uses the development:

$$z_1 = \sum_{j=-1}^{n-2} u_{-j}\zeta^{-j}, \quad n = 2^s,\ s \in \mathrm{N}. \tag{3.7}$$

The above truncated Laurent series takes into account that in a flow outside a contour, the points at infinity have to correspond to each other by conformal mapping [9]. One takes equally spaced points on circle and one writes:

$$\zeta_k = \exp(ik\delta);\ \delta = 2\pi/n;\ k = \overline{0;(n-1)};\ i = \sqrt{-1}$$

$$u_{-j} = \frac{1}{n}\sum_{k=0}^{n-1}\exp(ijk\delta);\ j = \overline{-1;(n-2)} \tag{3.8}$$

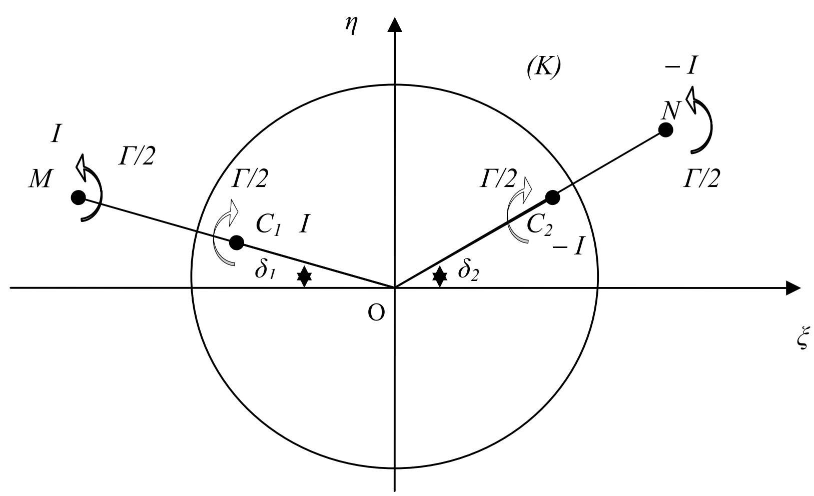

Fig. 6. The position of singularities

The coefficients u_{-j} are determined so that the coordinates (x_i,y_i) are close enough to the curve *C*. The velocity potential is easily written as before (see also (2.7)), the outer singularities being placed in the new points *M, N* (Fig.6).One obtains the total potential in a ***closed analytical formula***:

$$F(\varsigma) = \frac{tV_\infty}{2\pi}\cdot\left(e^{i\varphi}\ln\frac{\varsigma-\varsigma_M}{\varsigma-\varsigma_N} + e^{-i\varphi}\ln\frac{\varsigma-\varsigma_{C_1}}{\varsigma-\varsigma_{C_2}}\right) - \frac{i\Gamma}{4\pi}\ln\frac{(\varsigma-\varsigma_M)(\varsigma-\varsigma_N)}{(\varsigma-\varsigma_{C_1})(\varsigma-\varsigma_{C_2})} + const. \tag{3.9}$$

The Joukowsky condition is imposed at the point *F'*, the correspondent of the trailing edge point F_0 of the given profile.

The velocity distribution results from the relation:

$$V = \left|\frac{dF}{d\zeta}\right| / \left|\frac{dz_p}{d\zeta}\right| \quad (3.11)$$

Then one calculates the pressure coefficient, C_p:

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2 \quad (3.12)$$

4. Validation and applications

4.1. Test and validation. To validate and test the proposed method, especially as regard the FFT, one remarks that, if the first obtained contour (*E*-Fig.5) would be exact an ellipse, the calculation is completely analytical. We proceed as follows:

a. one takes an ellipse of semi-axis s = 1.4, l = 1.1 with the centre at (-0,0675; 0,2923) and τ = 103˚ in the plan z_2, transformed in a cascade of profiles (b = 50; t/b = 1.4; λ = 25˚; i_∞ = 10˚; s = 1.4; l = 1.1) in plan z_p (Fig. 7);

b. then one chooses an attached cascade of plates as in Fig. 7 and applies the general procedure. The results are compared in Fig. 8 and Fig. 9. One obtains a very good agreement and thus the method is validated (mainlyregarding the truncated Laurent series and FFT).

Remark. If the chord C_a of the attached cascade is even the chord AF one obtains an exact solution. However, in general, the chord AF of the given profile is not the best selection.

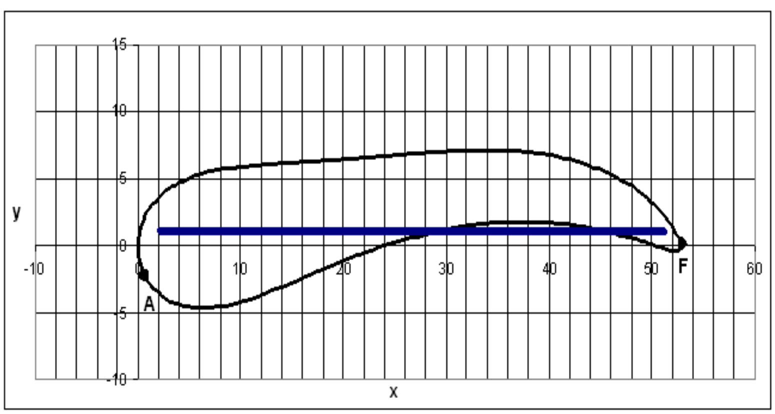

Fig. 7 Turbine cascade: Profile b=50; t/b=1.4; λ=25˚; i_∞=10˚; s=1.4; l=1.1

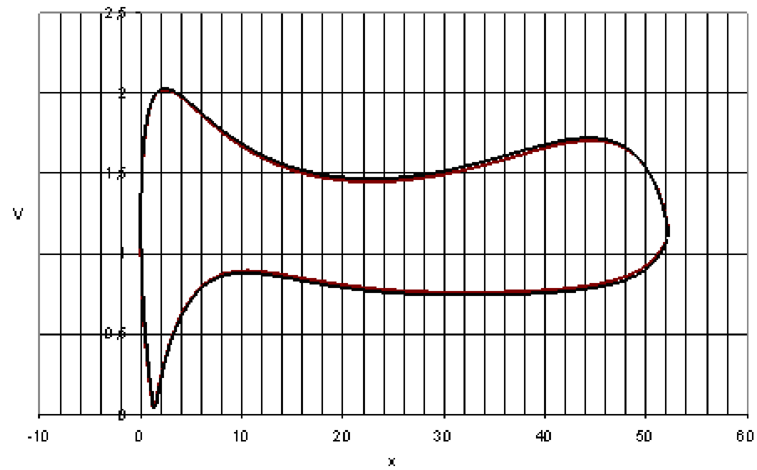

Fig. 8. The distribution of speeds

analytical
combined analytical -FFT

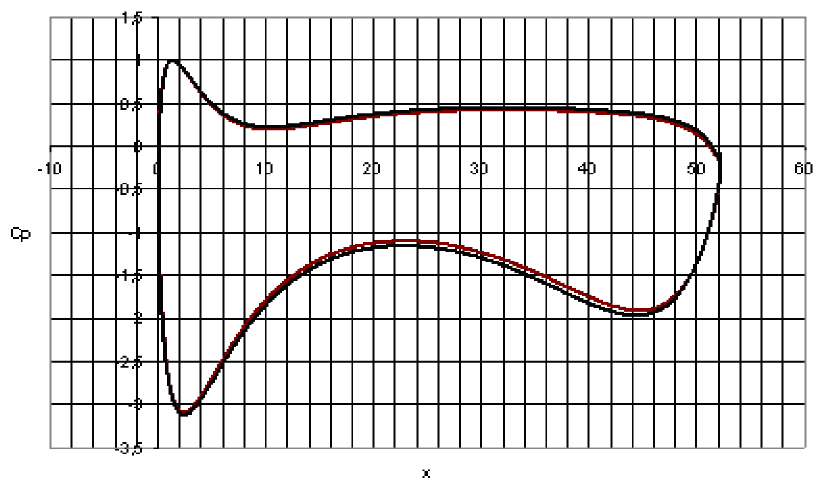

Fig. 9. The distribution of pressure

4.2. Applications

The following parameters were varied: 1) *t/b* - the relative cascade pitch; λ – cascade installation angle. From Table 1 one observes:

- when the installation angle λ is constant and the relative pitch decreases, *R* increases and the contour *E* is deformed very much;

- when the relative pitch is constant, the installation angle increases, *R* decreases.

Table 1

Variation of cascade profiles parameters

$\|\lambda\|$	**5**	**10**	**20**	**30**	**40**	**45**
t/b	***R***	***R***	***R***	***R***	***R***	***R***
0,80	1,32512	1,32078	1,30346	1,27480	1,23522	1,21154
0,90	1,42146	1,41675	1,39795	1,36672	1,32322	1,29695
0,95	1,47175	1,46691	1,44757	1,41542	1,37057	1,34344
1,00	1,52322	1,51827	1,49851	1,46566	1,41982	1,39207
1,10	1,62915	1,62406	1,60376	1,57004	1,52311	1,49477
1,20	1,73834	1,73320	1,71266	1,67865	1,63154	1,60326
1,30	1,85013	1,84498	1,82444	1,79056	1,74392	1,71613
1,40	1,96401	1,95889	1,93852	1,90506	1,85931	1,83227
1,50	2,07959	2,07453	2,05446	2,02160	1,97702	1,95087
1,60	2,19657	2,19160	2,17190	2,13978	2,09652	2,07134
1,70	2,31472	2,30985	2,29057	2,25928	2,21740	2,19321
1,80	2,43386	2,42910	2,41028	2,37985	2,33938	2,31615
1,90	2,55384	2,54919	2,53084	2,50129	2,46222	2,43993
2,00	2,67453	2,66999	2,65214	2,62346	2,58575	2,56435

One proposes a given airfoil design (Fig.7-9) the simulation is performed for the attached chord plate (C_a) by varying its inclination angle θ and center position (Table 2).

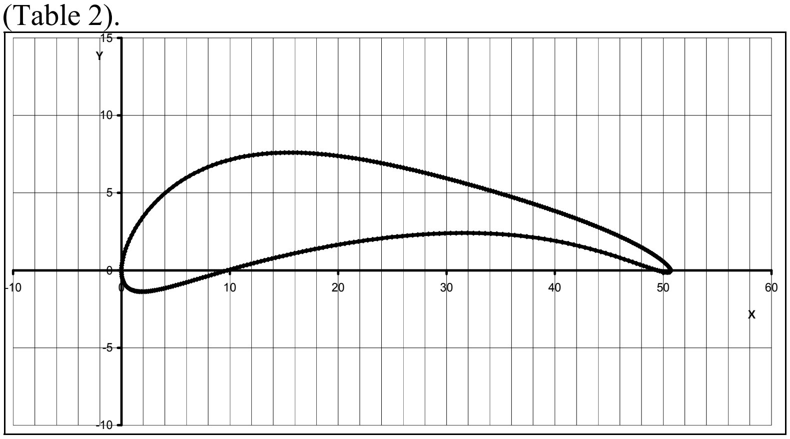

Fig. 10. Turbine profile: Profile b=50_t/b=1.5_λ=30

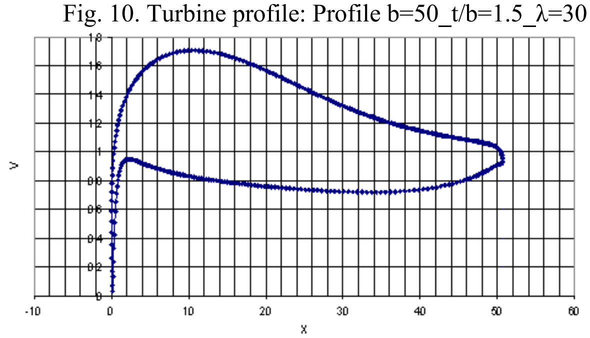

Fig. 11. The distribution of speeds

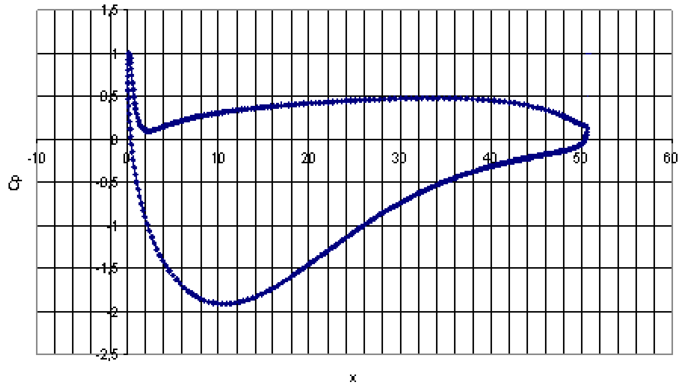

Fig.12. The distribution of pressure

Table 2

Effects of attached chord

Nr. crt.	b	θ [˚]	λ' [˚]	R	t/b_1	b_1	C_h/C_a
1	50,51224	2	37	1,449303	1,00068	50,47784	1,01420
2		4	39	1,451820	1,00131	50,44593	1,02881
3		5	40	1,478071	1,00200	50,41165	1,05708
4		7	42	1,503385	1,00284	50,36901	1,09170
5		9	44	1,553692	1,00385	50,31845	1,14942
6		11	46	1,574308	1,00442	50,29016	1,17924
7		-1	34	1,459136	1,00008	50,50818	1,01010
8		-3	32	1,482677	1,00006	50,50915	1,02459
9		-5	30	1,515388	1,00027	50,49815	1,04821

b - initial plate chord, C_h-cascade chord, t/b_1- the new relative pitch, b_1 – new chord plate, C_a-attached chord plate.

For positive angles θ one achieves an increase of angle of installation, a decrease of density, but also an increase of R (Table 3).

Table 3

Effects of θ angle

Nr. crt.	θ [˚]	λ' [˚]	R	C_h/C_a	b_1	t/b_1
1	1	36	1,590844	1,14416	50,37777	1,00266
2	3	38	1,580900		50,36384	1,00294
3	5	40	1,570441		50,34968	1,00323
4	7	42	1,559470		50,33516	1,00351
5	8	43	1,553794		50,32991	1,00362
6	-2	33	1,604785		50,40204	1,00218
7	-3	32	1,609171		50,40491	1,00212
8	-4	31	1,613425		50,41532	1,00192

5. Conclusions

The proposed method is stable, fast convergent and determines a variety of profiles, with velocities and pressure distributions in the full spectrum of motion.

The main advantages of the conformal mapping are preserved and analytical expressions for velocity potentials are obtained.

In order to apply the FFT to conformal mappings, the Laurent series relating the flows in two complex plans was conveniently applied, in combination with Joukowsky transform.

The proposed method allows a detailed study of the influence of the cascades parameters with reduced computational effort, as done in this paper.

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R E F E R E N C E S

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